Groupe de travail on (non)-semisimple TQFTs

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The goal of this *groupe de travail* is to learn different semisimple and non-semisimple TQFTs and quantum invariants derived from them, to explain the usual constructions and to learn the relations between them.

A guide: https://sites.google.com/site/psafronov/notes/non-semisimple-tqfts

Survey: https://arxiv.org/pdf/2401.10587

1 Basics on TQFTs

Motivation, definition and basic properties. Classification of 1-d and 2-d TQFTs. A very good reference is [CR18].

2 The BHMV TQFT

[BHMV95]

3 Categorical background

Monoidal, braided categories, rigidity, pivotal structures, twists, ribbon structures. Spherical structures, the Drinfeld center, modular categories.

Fusion categories, modular fusion categories. S-matrix. Relation with $SL(S, \mathbb{Z})$.

4 Examples: quantum groups

Ribbon Hopf algebras give rise to ribbon categories.

The Drinfeld double. If A is a finite dimensional semisimple Hopf algebra, then $D(A)$-mod is a modular fusion category [EG97].

The quantum group $U_q(sl_2)$, for generic q and q a root of unity. The finite-dimensional small quantum groups $u_q(sl_2)$. A comment about other semisimple Lie algebras.

The topological ribbon Hopf algebra $U_h(sl_2)$. The unrolled quantum group $\bar{U}_\zeta^H(sl_2)$.

5 The RT 3d TQFT

Turaev's book, Bakalov-Kirillov

6 The TV TQFT

6j symbols,... Patureau-Mirand - Geer's book, "The Classical and Quantum 6j-symbol" by Carter, Flath and Saito.

7 Relation between RT and TV

If $\mathcal{C}$ is spherical fusion, then $Z_{\mathcal{C}}^{TV} \simeq Z_{Z(\mathcal{C})}^{RT}$.

8 Habiro's unified WRT quantum invariant

"On the quantum sl2 invariants of knots and integral homology spheres" and "A UNIFIED WITTEN-RESHETIKHIN-TURAEV INVARIANT FOR INTEGRAL HOMOLOGY SPHERES" for sl2 (Habiro)

"UNIFIED QUANTUM INVARIANTS FOR INTEGRAL HOMOLOGY SPHERES ASSOCIATED WITH SIMPLE LIE ALGEBRAS" (Habiro-Le) for general simple Lie algebras

9 Crane-Yetter invertible 4d-TQFT

"State-sum invariants of 4-manifolds". L. Crane, L. H. Kauffman, and D. N. Yetter.

10 Extended TQFTs

Thesis Marco de Renzi

11 Hennings-Kauffman-Radford invariants

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See also Habiro's bottom tangles paper

12 Kuperberg invariants

13 Relation between TQFTs and the Hennings and Kuperberg invariants

$$Z_{Kup}(X;H) = Z_{TV}(X;Rep(H))$$

for H semisimple; this was by Barrett and Westbury.

$$Z_{HKR}(X;H) = Z_{WRT}(X;Rep(H))$$

H semisimple and modular (Kerler)

All together with $Z_{\mathcal{C}}^{TV} \simeq Z_{Z(\mathcal{C})}^{RT}$ implies

$$Z_{Kup}(X;H) = Z_{HKR}(X;D(H))$$

when H is semisimple.

In "On Two Invariants of Three Manifolds from Hopf Algebras", Chang Cui show a variation

$$Z_{Kup}(X;b,H) = Z_{Hen}(X;\phi,D(H))$$

once fixed a framing and a 2-framing b and ϕ respectively.

14 Virelizier invariant from Kirby elements

KIRBY ELEMENTS AND QUANTUM INVARIANTS

It recovers the 3-manifolds invariants of Reshetikhin-Turaev [RT91, Tur94], of Hennings-Kauffman-Radford [KR95, Hen96], and of Lyubashenko [Lyu95a]

15 Luybashenko TQFT

Non-ss. The Lyubashenko invariant does not form a TQFT in the usual sense because it does not behave well under the disjoint union operation (in particular, when the category is not semisimple, it vanishes on all closed 3-manifolds with positive first Betti number). However, the Lyubashenko invariant forms an extended TQFT in a weaker sense (by considering cobordisms with corners between connected surfaces and using the connected sum as monoidal product), see [KL]. It DOES form a modular functor.

16 Modified traces

17 TQFTs constructed from modified traces

18 TQFTs with defects

19 HQFTs

Bring potentially Matthew Jackson/Cellot from Lille

20 Modular functors

Maybe naïve approach, [BK].

21 Factorisation homology

22 Skein categories and relation with factorisation homology

Cooke.

23 Classification of anomalous free, extended 3d TQFTs

BDSV MODULAR CATEGORIES AS REPRESENTATIONS OF THE 3-DIMENSIONAL BORDISM 2-CATEGORY, https://arxiv.org/pdf/1509.06811

24 The finiteness conjecture

GJS and BD

References

[BHMV95] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology **34** (1995), no. 4, 883–927.

[CR18] Nils Carqueville and Ingo Runkel, *Introductory lectures on topological quantum field theory*, Banach Center Publications **114** (2018), 9–47.

[EG97] Pavel Etingof and Shlomo Gelaki, *Some properties of finite-dimensional semisimple hopf algebras*, 1997.