Quantum invariants via perturbed Gaussian expressions

Jorge Becerra* (RuG), joint with Roland van der Veen (RuG)

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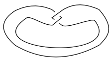
Given a knot $K \subset S^3$, its *Whitehead double* WK is the satellite knot of K with pattern the knot  (lying inside a standard torus). It is well-known that the Alexander polynomial of a Whitehead double is $\Delta_{WK} = 1$. However there are many knots with trivial Alexander out there. How can we detect Whitehead doubles?

Theorem 1 (van der Veen, B.) *There exist knot polynomial invariants*

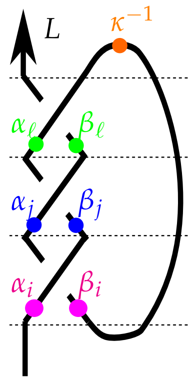
$$\rho_{1,0}(K), \rho_{2,0}(K) \in \mathbb{Z}[T, T^{-1}]$$

such that for a Whitehead double WK we have

$$\rho_{1,0}(WK) = -4v_2(T^{-1} - 2 + T) \qquad , \qquad \rho_{2,0}(K) = \lambda_0 + \lambda_1(T + T^{-1}) + \lambda_2(T^2 + T^{-2})$$

where $v_2, \lambda_i \in \mathbb{Z}$ are completely determined by K.

These polynomials arise as an instance of the so-called *universal knot invariant*: if A is a Hopf algebra with two preferred, invertible elements $R = \sum_i \alpha_i \otimes \beta_i \in A \otimes A$ and $\kappa \in A$ (this algebra is called *ribbon*), then one can cook up a (long) knot invariant $Z_A(K) \in A$. For instance



$$Z_A(L) = \sum_{i,j,\ell} \alpha_i \, \beta_j \, \alpha_\ell \, \kappa^{-1} \, \beta_i \, \alpha_j \, \beta_\ell.$$

The main drawback is that in practice this is very hard to compute, as A usually has infinite rank over $\mathbb{Q}[\![h]\!]$. In [♣][1], the authors devised a getaway by considering an algebra over $\mathbb{K} := \mathbb{Q}[\varepsilon][\![h]\!]$ and dealing only with finite expressions mod ε^n, as follows.

Suppose $A \cong \mathbb{K}[z]$, $z = (z_1, \ldots, z_p)$, and let I, J be some sets of labels. In computing $Z_A(K)$ one has to consider elements in

$$\mathrm{Hom}_{\mathbb{K}}(A^{\otimes I}, A^{\otimes J}) \cong \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[z_I], \mathbb{K}[z_J]) \cong \mathbb{K}[z_J][\![\zeta_I]\!]$$

and the trick consists in looking only at a class $\mathcal{PG}(I, J)$ of power series of the form

$$e^Q(P_0 + P_1\varepsilon + P_2\varepsilon^2 + \cdots)$$

where Q is a particular finite quadratic expression and P_i is a finite polynomial. For a particular algebra $\mathbb{D} \cong \mathbb{K}[y, t, a, x]$, the class $\mathcal{PG}$ is closed under composition.

In [♣] it is shown that for a 0-framed knot K,

$$Z_{\mathbb{D}}(K) = \frac{1}{\Delta_K(T)} \mod \varepsilon \quad , \quad T := e^{-ht},$$

and more generally, $Z_{\mathbb{D}}(K) \mod \varepsilon^n$ is completely determined by a collection of polynomials $\rho_{k,j} \in \mathbb{Z}[T, T^{-1}]$, $k < n$, $0 \leq j \leq 2k$. These polynomials can be computed efficiently (in polynomial time).

The invariant $\rho_{k,0}$ is conjectured to coincide with Rozansky's $(k+1)$-loop invariant.

*Handout available at `sites.google.com/view/becerra/talks`

[1][♣] D. Bar-Natan, R. van der Veen, *Perturbed Gaussian generating functions for universal knot invariants*, arXiv:2109.02057.