A categorical approach to the universal tangle invariant

Jorge Becerra (RuG)*

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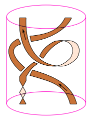
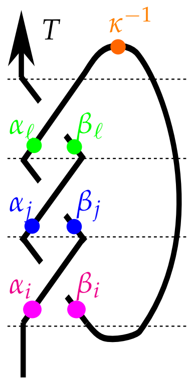
Usually one depicts tangles  as sliced diagrams on the plane. If A is a Hopf algebra with two preferred, invertible elements

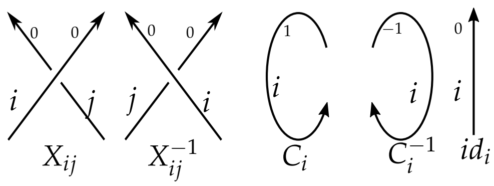
$$R = \sum_i \alpha_i \otimes \beta_i \in A \otimes A \quad , \quad \kappa \in A$$

satisfying some relations (this algebra is called *ribbon*), then one can cook up a tangle invariant $Z_A(T) \in A^{\otimes(\#\text{ components of } T)}$ that encodes the *Reshetikhin–Turaev invariant* coming from the representation theory of A. For instance

$$Z_A(T) = \sum_{i,j,\ell} \alpha_i\, \beta_j\, \alpha_\ell\, \kappa^{-1}\, \beta_i\, \alpha_j\, \beta_\ell .$$



This point of view is not suitable if one wishes to study natural operations on tangles, e.g. merging only two strands, strand doubling or strand removal. A more convenient setup for this purpose is to study 1- and 4-valent graphs whose edges have a number attached (*rotation number*) and strands are labelled by some finite set I. We also distinguish between over- and under-strand around the 4-valent vertices. We call these *rv-tangles*. They are obtained by merging copies of the following building blocks:



Write

$$\mathcal{T}_I := \frac{\{\text{rv-tangles labelled by } I\}}{\text{rv-Reidemeister moves}}.$$

Moreover, let $\mathcal{H}$ be the symmetric monoidal category generated by a Hopf algebra with objects finite sets.

Theorem (B. 2020) *There is a lax monoidal functor*

$$\mathcal{T} : (\mathcal{H}, \amalg) \longrightarrow (\mathsf{Set}, \times)$$

sending I to $\mathcal{T}_I$. Furthermore, if given a ribbon Hopf algebra A, we write $A : \mathcal{H} \longrightarrow \mathsf{Set}$ for the canonical functor sending I to (the underlying set of) $A^{\otimes I}$, then there is a monoidal natural transformation

$$Z_A : \mathcal{T} \Longrightarrow A$$

such that

$$(Z_A)_{i,j}(X_{ij}^{\pm 1}) = R_{ij}^{\pm 1} \in A^{\otimes\{i,j\}} \qquad , \qquad (Z_A)_i(C_i^{\pm 1}) = \kappa_i^{\pm 1} \in A^{\otimes\{i\}}.$$

*Slides and handout available at `sites.google.com/view/becerra/talks`