UNIVERSAL INVARIANTS, PERTURBED GAUSSIANS AND THE 2-LOOP POLYNOMIAL

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In this talk I will give account of some new results around Bar-Natan - van der Veen's novel approach to the universal tangle invariant via Gaussian calculus [BV21].

In the following we let k be a commutative ring and A a topological ribbon Hopf algebra over $k[[h]]$ with universal R-matrix and balancing element

$$R = \sum_i \alpha_i \hat{\otimes} \beta_i \in A \hat{\otimes} A \qquad , \qquad \kappa \in A.$$

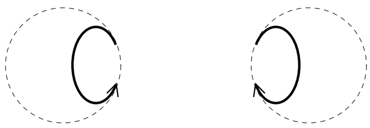
We assume the reader is familiar with the universal tangle invariant subject to A, which will be denoted by Z_A.

This handout is available online at bit.do/jbecerra.

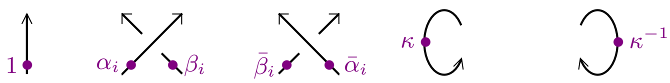
1. **Upwards tangles.** Let $\mathcal{T}$ be the strict monoidal category of framed, oriented tangles in a cube. Its objects are the free monoid $\mathrm{Mon}(+,-)$ generated by $\{+,-\}$, and its arrows are isotopy classes of framed, oriented tangles rel. endpoints. This is the "free strict ribbon category generated by one object", and hence it is particularly useful to study tangle invariants $\mathcal{T} \longrightarrow \mathcal{C}$ induced by a ribbon category $\mathcal{C}$ (aka Reshetikhin-Turaev invariants). However, if one wants to study the universal tangle invariant subject to some ribbon Hopf algebra, it is most convenient to restrict oneself to a certain subcategory.

Definition 1. The category of *upwards tangles* is the subcategory $\mathcal{T}^{\mathrm{up}} \subset \mathcal{T}$ on the objects $\mathrm{Mon}(+)$ and arrows tangles without closed components.

Upwards tangles admit diagrams in *rotational form*, that is, diagrams in which all crossings point up and all maxima and minima appear in pairs of the following two forms:



Computing the universal invariant for a upwards tangle in rotational form is particularly easy as there are only five building blocks to take into account:



2. **Functoriality and naturality of Z_A.** Let A denote the strict monoidal category whose objects are non-negative integers and

$$\mathrm{Hom}_{\mathsf{A}}(n,m) = \begin{cases} \emptyset, & n \neq m, \\ U(A^{\hat{\otimes} n} \times \Sigma_n), & n = m, \end{cases}$$

where $U : \mathsf{Mod}_{k[[h]]} \times \mathsf{Grp} \longrightarrow \mathsf{Set}$ denotes the canonical forgetful functor. The composite law is defined as follows: if $f = (u, \sigma)$ and $g = (v, \tau)$ are arrows of A, then

$$g \circ f := (u \cdot \sigma_*^{-1} v \ , \ \tau \circ \sigma),$$

where $\sigma_*^{-1} : A^{\hat{\otimes} n} \longrightarrow A^{\hat{\otimes} n}$ is the map that permutes the factors induced by σ^{-1} and the symmetric braiding of $\mathsf{Mod}_{k[[h]]}$, and $\tau \circ \sigma$ denotes the composite of bijections of elements of the symmetric group. The monoidal product is given by addition of integers and tensor product of elements of the n-fold tensor product of the ribbon Hopf algebra and block sum of permutations.

Proposition 2 ([Bec23b])**.** *The universal tangle invariant induces a strong monoidal functor*

$$Z_A : \mathcal{T}^{\mathrm{up}} \longrightarrow \mathsf{A}.$$

The most interesting feature of the universal invariant is nevertheless the naturality with respect to the Hopf algebra structure maps. To state this, it is most convenient to pass to a extended class of not necessarily planar tangles called *rotational virtual tangles*.

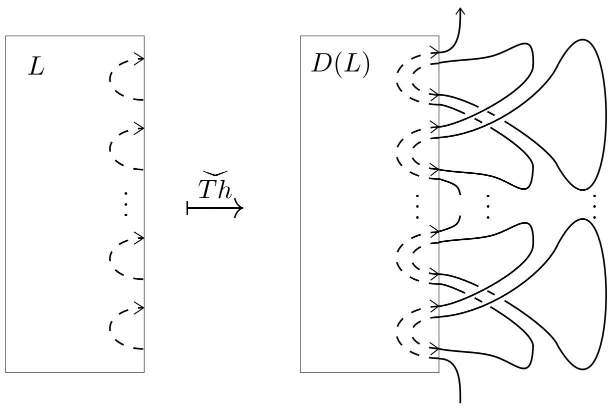
Let H be the universal category with respect to Hopf algebra objects.

Theorem 3 ([Bec23c]). *The universal tangle invariant gives rise to a monoidal natural transformation*

$$Z_A : rvT \Longrightarrow A^{\hat{\otimes}-}$$

between certain lax monoidal functors $rvT, A^{\hat{\otimes}-} : \mathsf{H} \longrightarrow \mathsf{Set}$.

3. **The thickening map.** Let L be a *vertical bottom tangle*, that is, the image of a bottom tangle (à la Habiro) under a $(-\pi/2)$ rotation about the y axis. The *thickening* of L is the following 0-framed knot $\widetilde{Th}(L)$:



where $D(L)$ is the result of doubling every component of L and reversing the orientation of the inner one for each pair of doubled component, and the dashed components of L indicate that they are possibly knotted.

The *cross map* Cr and the *band map* B are defined as follows:

$$Cr : A \hat{\otimes} A \longrightarrow A \hat{\otimes} A \qquad , \qquad Cr(x \hat{\otimes} y) := \sum_i x\alpha_i \hat{\otimes} \beta_i y$$

and

$$B : A \hat{\otimes} A \longrightarrow A \qquad , \qquad B(x \hat{\otimes} y) := \sum_{(x),(y)} x_{(2)} S(y_{(1)}) \kappa S(x_{(1)}) \kappa y_{(2)}.$$

For $g \geq 1$, the *algebraic thickening map* Th is the composite

$$A^{\hat{\otimes}2g} \xrightarrow{Cr^{\hat{\otimes}g}} A^{\hat{\otimes}2g} \xrightarrow{B^{\hat{\otimes}g}} A^{\hat{\otimes}g} \xrightarrow{\mu^{[g]}} A$$

where $\mu^{[g]}$ denotes the g-fold multiplication map.

Proposition 4. *Let* $g \geq 1$. *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathsf{vBT}_{2g} & \xrightarrow{\widetilde{Th}} & \left\{ \begin{array}{c} \text{genus} \leq g \\ \text{0-framed knots} \end{array} \right\} \\ {\scriptstyle Z_A}\big\downarrow & & \big\downarrow{\scriptstyle Z_A} \\ A^{\hat{\otimes}2g} & \xrightarrow{\quad Th \quad} & A \end{array}$$

4. **Gaussian calculus.** From now on let us fix $k = \mathbb{Q}_\varepsilon := \mathbb{Q}[\varepsilon]$ and $A := \mathbb{D}$, and recall that $\mathbb{D} \cong \mathbb{Q}_\varepsilon[y, t, a, x][[h]]$ as topological $\mathbb{Q}_\varepsilon[[h]]$-modules. If I, J are finite ordered sets with $\#I = n$ and $\#J = m$, the previous isomorphism induces a bijection

$$\mathrm{Hom}_{\mathbb{Q}_\varepsilon[[h]]}(\mathbb{D}^{\hat{\otimes}n}, \mathbb{D}^{\hat{\otimes}m}) \xrightarrow{\cong} \mathbb{Q}_\varepsilon[y_J, t_J, a_J, x_J][[\eta_I, \tau_I, \alpha_I, \xi_I]][[h]]$$

where η, τ, α, ξ are the dual variables of y, t, a, x, respectively. The image of a $\mathbb{Q}_\varepsilon[[h]]$-linear map f will be denoted f_I^J and called the *generating series* of f. It is shown in [BV21] that the generating series of all ribbon Hopf algebra structure maps of $\mathbb{D}$ are "two-step perturbed Gaussians"

$$e^G(P_0 + P_1\varepsilon + P_2\varepsilon^2 + \cdots).$$

5. **A normal form for Z_A.** Consider the central element

$$w := yA^{-1}x + \frac{qA^{-1} + TA - \frac{1}{2}(q+1)(T+1)}{h(q-1)} \quad , \quad q := e^{h\varepsilon},\ T := e^{-ht},\ A := e^{-h\varepsilon a}.$$

Theorem 5 ([BV21])**.** *For any 0-framed knot K, there exist knot polynomial invariants*

$$\rho_K^{i,j} \in \mathbb{Q}[t, t^{-1}] \quad , \quad i \geq 0\ ,\ 0 \leq j \leq 2i,$$

such that

$$Z_{\mathbb{D}}(K) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{2i} h^{i+j} \frac{\rho_K^{i,j}(T)}{\Delta_K^{2i+1-j}(T)} w^j \right) \varepsilon^i.$$

Moreover $\rho_K^{0,0} = 1$.

Therefore, studying the universal invariant $Z_{\mathbb{D}}(K)$ amounts to studying the collection of Bar-Natan - van der Veen polynomials $\rho_K^{i,j}$ for $i \geq 0$, $0 \leq j \leq 2i$. The natural question then is to ask how strong these knot polynomial invariants are. The polynomials $\rho_K^{i,0}$ are expected to be separate a large number of knots. For $j > 0$, we expect these polynomials to be either trivial or consequence of the ones for $j = 0$ (and the Alexander polynomial). In fact,

Theorem 6 ([Bec23b], a Conjecture in [BV21])**.** *For any 0-framed knot K,*

$$\rho_K^{i,j}(t) = 0$$

for $j > i > 0$.

Theorem 7 ([Bec23b])**.** *For any 0-framed knot K we have*

(1) $\rho_K^{1,1}(t) = \dfrac{2t}{1-t}\Delta'_K(t)$,

(2) $\rho_K^{2,1}(t) = \dfrac{2t\Delta_K^4(t)}{t-1}\left(\dfrac{\rho_K^{1,0}}{\Delta_K^3}\right)'(t)$,

(3) $\rho_K^{2,2}(t) = t\left(\dfrac{\Delta_K(t)}{1-t}\right)^3 \left((3-t)(\Delta_K^{-1})'(t) + 2t(1-t)(\Delta_K^{-1})''(t)\right)$.

These two results together imply that for a 0-framed knot K, the value of $Z_{\mathbb{D}}(K) \pmod{\varepsilon^3}$ is completely determined by the triple $(\Delta_K, \rho_K^{1,0}, \rho_K^{2,0})$.

6. **Connected sums.** The Alexander polynomial is multiplicative with respect to connected sums,

$$\Delta_{K\#K'} = \Delta_K \Delta_{K'}.$$

There are also (uglier) formulas for the polynomials $\rho_K^{i,j}$:

Theorem 8 ([Bec23b])**.** *Let K, K' be 0-framed knots, and let $n \geq 0$ and $0 \leq r \leq n$ be integers. Then*

$$\rho_{K\#K'}^{n,r} = \sum_{i=0}^{n} \sum_{j=\max(0,r+i-n)}^{\min(i,r)} \rho_K^{i,j} \rho_{K'}^{n-i,r-j} \Delta_K^{2(n-i)-(r-j)} \Delta_{K'}^{2i-j}.$$

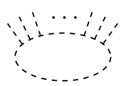
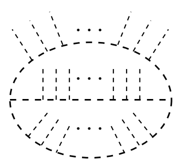
In particular,

$$\rho_{K\#K'}^{n,0} = \sum_{i=0}^{n} \rho_K^{i,0} \rho_{K'}^{n-i,0} \Delta_K^{2(n-i)} \Delta_{K'}^{2i}.$$

7. **Connection to the 2-loop polynomial.** Fix $\varphi \in \mathbb{Q}\langle\langle X, Y\rangle\rangle$ a rational Drinfeld associator. The Kontsevich invariant

$$Z = Z_\varphi : \mathcal{T}_q \longrightarrow \hat{\mathcal{A}}$$

admits a *loop expansion*

$\log_{\amalg}(Z(K)) = \sum_i \lambda_i$  $+ \sum_i \mu_i$  $+ \left(\text{$n$-loop terms, } n > 2 \right).$

The first summand of the above expression (the *1-loop part*) can be shown to be tantamount to the Alexander polynomial Δ_K of K. The second summand (the *2-loop part*) is equivalent to a two-variable polynomial

$$\Theta_K(t_1, t_2) \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}]$$

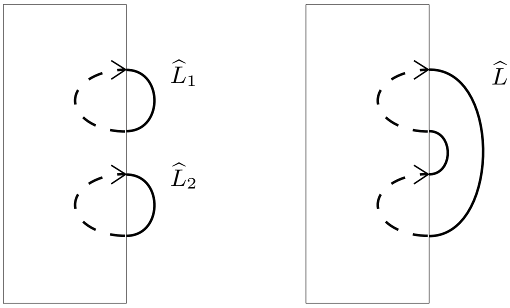
called the *2-loop polynomial* of K. Let us put

$$\hat{\Theta}_K(t) := \Theta_K(t, 1) \in \mathbb{Q}[t, t^{-1}].$$

Conjecture 9 ([BV21]). *For any 0-framed knot K,*

$$\rho_K^{1,0} = \hat{\Theta}_K.$$

Let $L = L_1 \cup L_2$ be a 2-component vertical bottom tangle, let $K = \breve{Th}(L)$ and let $\widehat{L}_1, \widehat{L}_2$ and $\widehat{L}$ denote the following knots:



In a future publication, we will show the following results:

Theorem 10 ([Bec23a]). *Let L be a 2-component tangle as above. Then the value $Z_{\mathbb{D}}(L)_{|t=0} \pmod{\varepsilon^2}$, and therefore $Z_{\mathbb{D}}(K) \pmod{\varepsilon^2}$, only depends on Vassiliev invariants of L of degree less or equal than three.*

More precisely, let

$$n := \mathrm{framing}(\widehat{L}_1) \qquad , \qquad m := \mathrm{framing}(\widehat{L}_2) \qquad , \qquad k := lk(\widehat{L}_1, \widehat{L}_2)$$

and let

$$\nabla_{\widehat{L}_1} = 1 + v_2^{11} z^2 \pmod{z^4}, \qquad \nabla_{\widehat{L}_2} = 1 + v_2^{22} z^2 \pmod{z^4}$$
$$\nabla_{\widehat{L}} = 1 + (v_2^{11} + v_2^{22} + v_2^{12}) z^2 \pmod{z^4}, \qquad J^2_{\widehat{L}}(q = e^h) = \cdots + 12 v_3 h^3 \pmod{h^4}$$

where ∇_K denotes the Conway polynomial of K, that is, $\nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t)$. Then

$$\begin{aligned}
\Big[Z_{\mathbb{D}}(L)_{|t=0}\Big]^{1,2} &= \exp\left[\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} n & k+1 \\ k & m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right] \Big[1 + \Big(n \cdot (a_1 + a_1^2) + m \cdot (a_2 + a_2^2) + (2k+1) \cdot a_1 a_2 \\
&\quad - 4 v_2^{11} \cdot y_1 x_1 - 4 v_2^{22} \cdot y_2 x_2 + (k(k+1) - 2 v_2^{12}) \cdot (y_1 x_2 + y_2 x_1) - n(n-1) \cdot a_1 y_1 x_1 \\
&\quad - k(k+1) \cdot (a_1 y_1 x_2 + a_2 y_1 x_2) - k^2 \cdot (a_1 y_2 x_2 + a_2 y_2 x_1) + (k(k+1) - (2k+1)m) \cdot a_1 y_2 x_2 \\
&\quad + (k(k+1) - (2k+1)n) \cdot a_2 y_1 x_1 - m(m-1) \cdot a_2 y_2 x_2 + \tfrac{4n^3 - 6n^2 - n}{12} y_1^2 x_1^2 \\
&\quad + \tfrac{4m^3 - 6m^2 - m}{12} y_2^2 x_2^2 + \Big(\tfrac{8k^3 + 12k^2 + 6k - 3}{12} - 2 v_3\Big) \cdot y_1^2 x_2^2 + \Big(\tfrac{8k^3 + k}{12} - 2 v_3\Big) \cdot y_2^2 x_1^2 \\
&\quad + \Big(k(k+1)n - \tfrac{2k^3 + 3k^2 + k}{6}\Big) \cdot y_1^2 x_1 x_2 + \Big(k(k+1)m - \tfrac{2k^3 + 3k^2 + k}{6}\Big) \cdot y_1 y_2 x_2^2 \\
&\quad + \Big(k^2 n - \tfrac{2k^3 + 3k^2 + k}{6}\Big) \cdot y_1 y_2 x_1^2 + \Big(k^2 m - \tfrac{2k^3 + 3k^2 + k}{6}\Big) \cdot y_2^2 x_1 x_2 \\
&\quad + \Big(\tfrac{2k^3 + 3k^2 + 1}{3} - k(k+1)(n+m) + (2k+1)nm + 4 v_3\Big) y_1 y_2 x_1 x_2 \Big) \varepsilon \Big] \pmod{\varepsilon^2}.
\end{aligned}$$

Using Proposition 4, we can further show the following key result.

Theorem 11 ([Bec23a])**.** *The previous Conjecture holds for knots of genus less or equal to one.*

More precisely, for non-negative integers n, m, let $\mathcal{S}_{n,m}$ be the following symmetric Laurent polynomial,

$$\mathcal{S}_{n,m}(t) := 2(t^n + t^m + t^{-n} + t^{-m} + t^{n-m} + t^{m-n} - 6) \in \mathbb{Z}[t + t^{-1}].$$

Then, for any 0-framed knot K of genus ≤ 1 and L a 2-component vertical bottom tangle such that $K = \widetilde{Th}(L)$ we have

$$\begin{aligned}\rho_K^{1,0}(t) = &\Big((n+m)(d - \tfrac{nm}{2}) - k(k+\tfrac{1}{2})(k+1) + 12v_3\Big)\Big(-d(d-1)\mathcal{S}_{1,0} - \tfrac{1}{2}d(d+1)\mathcal{S}_{2,0} + \tfrac{3d^2+d-1}{3}\mathcal{S}_{2,1}\Big)\\ &+ 12\Big(mv_2^{11} + nv_2^{22} - (k+\tfrac{1}{2})v_2^{12} + 3v_3\Big)\Big(\tfrac{6d^2-6d+1}{6}\mathcal{S}_{1,0} + \tfrac{1}{2}d^2\mathcal{S}_{2,0} - (d^2 - \tfrac{1}{3}d)\mathcal{S}_{2,1}\Big).\end{aligned}$$

where $d := nm - k(k+1)$ and $n, m, k, v_2^{11}, v_2^{12}, v_2^{22}, v_3$ are as above. In particular, the palindromicity of $\rho_K^{1,0}$ follows.

8. **An application to Whitehead doubles.** For any integer $q \in \mathbb{Z}$ and any q-framed knot K, let us write $Wh_q^+(K)$ for the 0-framed positively clasped q-twisted Whitehead double of K. It is well-known that

$$\Delta_{Wh_q^+(K)} = -q(t + t^{-1}) + 2q + 1.$$

Corollary 12 ([Bec23a])**.** *The invariant $\rho^{1,0}$ of $Wh_q^+(K)$ is given by*

$$\begin{aligned}\rho_{Wh_q^+(K)}^{1,0}(t) = (t^{-1} - 2 + t)\Big(&\big(\tfrac{1}{6}(1-q)q(2q-1) + 4q \cdot v_2(K)\big)(t + t^{-1})\\ &+ \tfrac{2}{3}(q-1)q(q+1) - (8q+4)v_2(K)\Big),\end{aligned}$$

where $\nabla_K = 1 + v_2(K)z^2 \pmod{z^4}$.

References

[Bec23a] J. Becerra. A Hopf algebraic construction of the 2-loop polynomial of genus one knots. In preparation, 2023+.

[Bec23b] J. Becerra. On Bar-Natan - van der Veen's perturbed gaussians. In preparation, 2023+.

[Bec23c] J. Becerra. On the naturality of the universal tangle invariant. In preparation, 2023+.

[BV21] D. Bar-Natan and R. van der Veen. Perturbed gaussian generating functions for universal knot invariants, 2021. doi:10.48550/ARXIV.2109.02057.