A categorical approach to the universal tangle invariant

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DIAMANT Symposium

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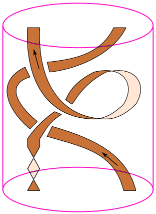
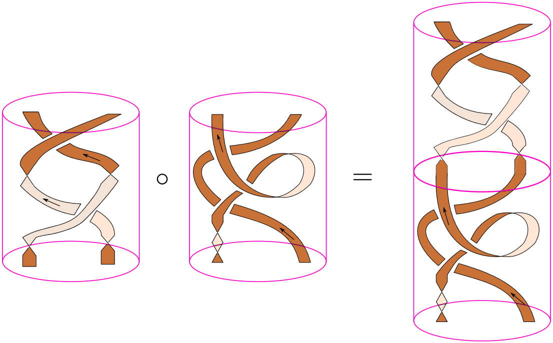
Classical tangles

Classically, (open, oriented, framed) tangles are viewed in a vertical position as (locally flat) embeddings

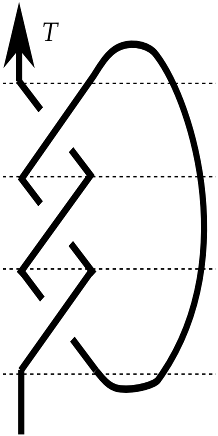
$$T : \amalg D^1 \times D^1 \hookrightarrow D^2 \times D^1$$



Tangles form a (ribbon) category Tangle where the composite is given by stacking:



This gives a functorial invariant $F_{\mathcal{C}} : \mathsf{Tangle}_{\mathcal{C}} \to \mathcal{C}$, the *Reshetikhin–Turaev invariant*. For the tangle T the map $f_T : V \to V$ does not depend on the diagram.



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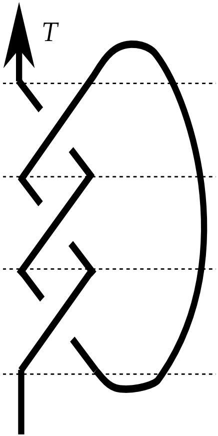


$\mathcal{C} = \mathsf{Mod}_A$, where A is a Hopf algebra that contains two preferred invertible elements

$$R = \sum_i \alpha_i \otimes \beta_i \in A \otimes A \quad , \quad \kappa \in A.$$

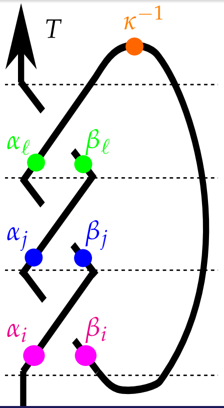
There is a way of obtaining a tangle invariant directly from A:

Theorem (Lawrence-Ohtsuki 90's)

Given a (ribbon) Hopf algebra A and a tangle T, placing copies of R on the positive crossings and κ^{-1} on the left-to-right cap and multiplying the elements along the diagram gives an element

$$Z_A(T) \in A^{\otimes(\#\text{ components of } T)}$$

that does not depend on the tangle diagram.



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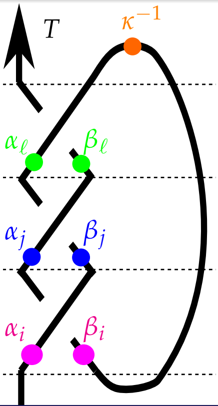

$$Z_A(T) = \sum_{i,j,\ell} \alpha_i \, \beta_j \, \alpha_\ell \, \kappa^{-1} \, \beta_i \, \alpha_j \, \beta_\ell$$

is an invariant of T.

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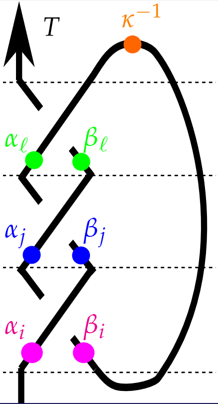

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For any $(V, \rho_V) \in \mathrm{Mod}_A$, we have

$$\rho_V(Z_A(T)) = f_T.$$

$Z_A(T)$ is called the *universal invariant* of T associated to A.

Ribbon Hopf algebras

A **ribbon Hopf algebra** is a Hopf algebra $(A, \mu, \eta, \Delta, \varepsilon, S)$ together with two preferred invertible elements

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satisfying the following relations:

- $(\Delta \otimes \mathrm{Id})R = R_{13}R_{23}$,
- $(\mathrm{Id} \otimes \Delta)R = R_{13}R_{12}$,
- $(\tau \circ \Delta)(x) = R\Delta(x)R^{-1}$, $\forall x \in A$,
- $\Delta(\kappa) = \kappa \otimes \kappa$,
- $\varepsilon(\kappa) = 1$,
- $S^2(x) = \kappa x \kappa^{-1}$, $\forall x \in A$,
- $\sum_i \alpha_i \kappa^{-1} \beta_i = \sum_i \beta_i \kappa \alpha_i$.

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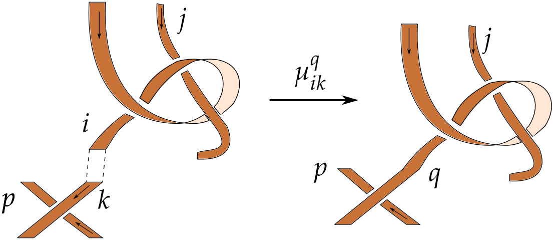
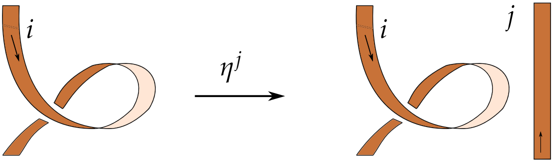
- $S^2(x) = \kappa x \kappa^{-1}$, $\forall x \in A$,
- $\sum_i \alpha_i \kappa^{-1} \beta_i = \sum_i \beta_i \kappa \alpha_i$.

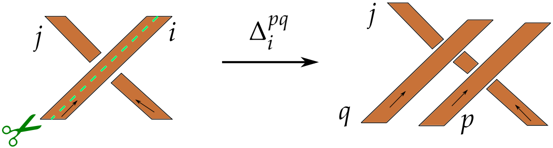
Examples

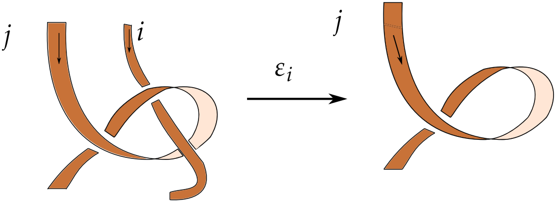
1. For a finite group G and a field k, $k(G) \rtimes kG$ is a ribbon Hopf algebra.
2. Many instances using the Drinfeld double construction, Drinfeld - Jimbo's quantum enveloping algebras, etc.

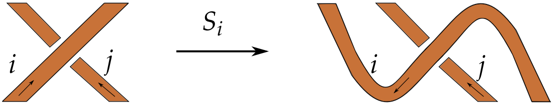
However, this vertical realisation is rather coarse if one wants to study natural operations on tangles:






j
i
Δ_i^{pq}
j
q
p

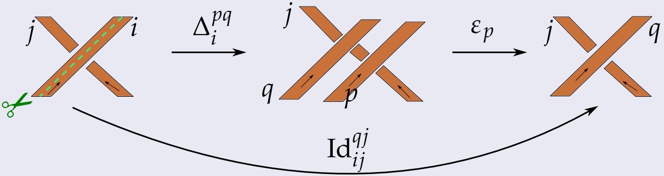

j
i
ε_i
j


i
j
S_i
i
j

It turns out that tangles mimic the Hopf algebra axioms:

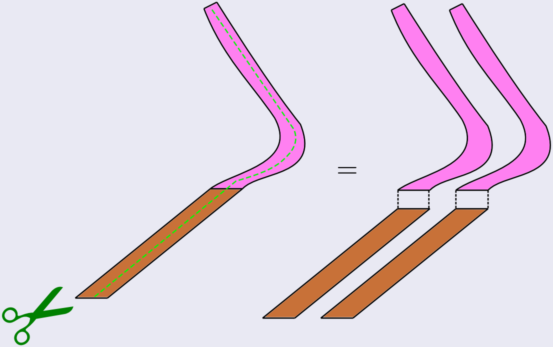
Example

The coalgebra axiom $(\varepsilon \otimes \mathrm{Id})\Delta = \mathrm{Id}$ becomes in tangles



Example

The bialgebra axiom $\Delta(xy) = \Delta(x)\Delta(y)$ becomes in tangles



Rough dictionary

Operation	Hopf algebra	Tangles
μ	multiplication	strand merging
η	unit	trivial strand
Δ	comultiplication	strand doubling
ε	counit	strand removal
S	antipode	"tweaked" strand orientation reversal

Rough dictionary

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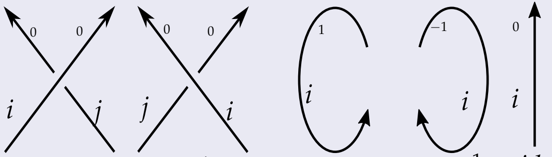
We would like to find a suitable setup for tangles to make the previous argument precise.

Rotational virtual tangles

Sketch definition (van der Veen)

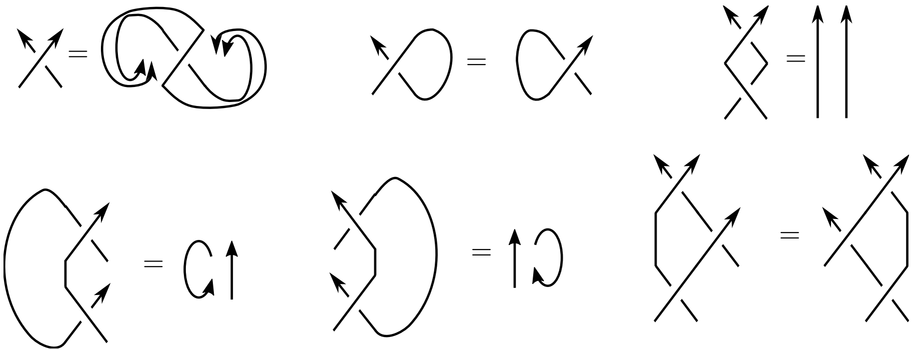
Let I be a finite set. An *rv-tangle labelled by I* is a finite, oriented graph such that

- There are only four-valent and univalent vertices.
- Each edge has two labels: (1) an element of I, (2) an integer called the *rotation number*.
- Locally they look like



X_{ij} X_{ij}^{-1} C_i C_i^{-1} id_i

If we are to represent tangles, we must mod out by the "rv-Reidemeister moves":



For a finite set I, let us denote

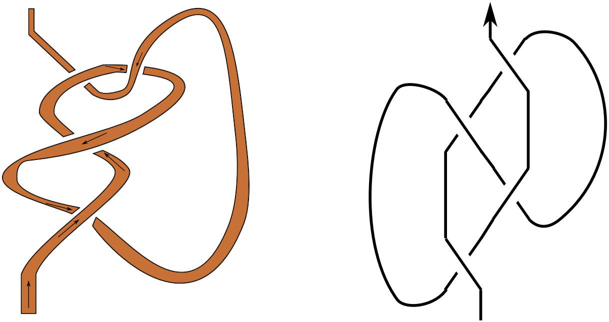
$$\mathcal{T}_I := \frac{\{\text{rv-tangles labelled by } I\}}{\text{rv-Reidemeister moves}}.$$

The benefit of considering rv-tangles is twofold:

- There is an injective realisation map

$$\left(\begin{array}{c} \text{Framed oriented} \\ \text{tangles in } D^2 \times D^1 \\ \text{labelled with } I \end{array} \right) \hookrightarrow \mathcal{T}_I$$

so any classical tangle can be viewed as a rv-tangle.



- Let $\mathcal{H}$ be the symmetric monoidal category monoidally generated by a Hopf algebra, that is $\mathcal{H}$ has finite sets as objects and the morphisms are monoidally generated by

$$\mu^k_{i,j} : \{i,j\} \to \{k\} \quad , \quad \eta^k : \varnothing \to \{k\} \quad , \quad \Delta^{j,k}_i : \{i\} \to \{j,k\},$$

$$\varepsilon_i : \{i\} \to \varnothing \quad , \quad S_i : \{i\} \to \{i\} \quad , \quad \mathrm{Id}^j_i : \{i\} \to \{j\}.$$

subject to the Hopf algebra axioms. A Hopf algebra is the same data as a strong monoidal functor $\mathcal{H} \to \mathsf{Vect}_k$.

Proposition (van der Veen, B.)

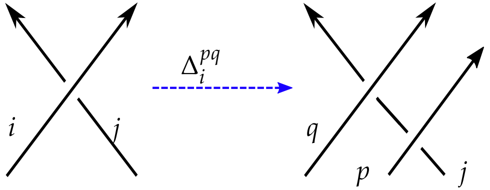
There is a lax monoidal functor

$$\mathcal{T} : (\mathcal{H}, \amalg) \to (\mathsf{Set}, \times)$$

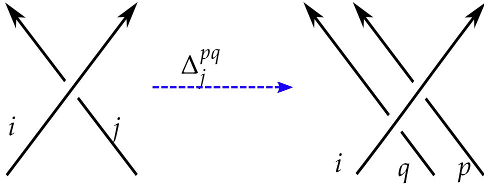
sending I to $\mathcal{T}_I$.

On generators:

$$\Delta_i^{pq}(X_{ij}) = \mu_{ab}^{j}(X_{pa} \amalg X_{qb})$$



$$\Delta_j^{pq}(X_{ij}) = \mu_{ab}^{i}(X_{aq} \amalg X_{bp})$$



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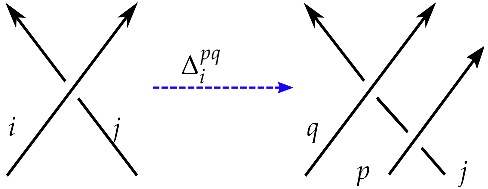

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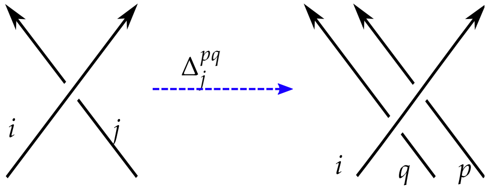
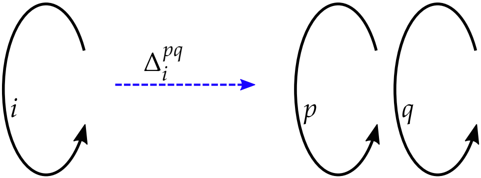


These relations already “appeared” in the axioms for a ribbon Hopf algebra (!!)

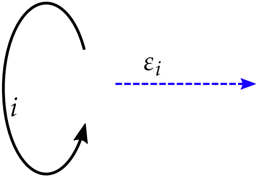
$$(\Delta \otimes \mathrm{Id})R = R_{13}R_{23} \qquad , \qquad (\mathrm{Id} \otimes \Delta)R = R_{13}R_{12}.$$

For the "spinner" C_i:

$\Delta_i^{pq}(C_i) = C_p \amalg C_q$



$\varepsilon_i(C_i) = \emptyset$



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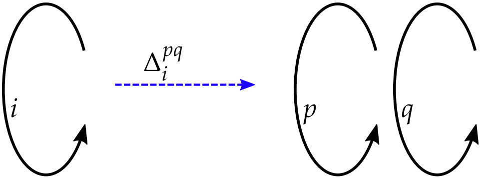

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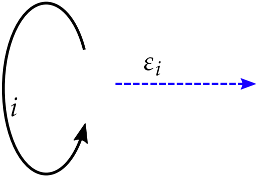

These relations already “appeared” as well as

$$\Delta(\kappa) = \kappa \otimes \kappa \qquad , \qquad \varepsilon(\kappa) = 1.$$

Rough dictionary

	Ribbon Hopf algebra	RV-tangles
μ	multiplication	strand merging
η	unit	trivial strand
Δ	comultiplication	strand doubling
ε	counit	strand removal
S	antipode	"tweaked" strand orientation reversal
	$R^{\pm 1}$	$X_{ij}^{\pm 1}$
	$\kappa^{\pm 1}$	$C_i^{\pm 1}$
	$\otimes$	$\amalg$

Upshot

For a finite set I, one can define an I-fold *unordered tensor product*

$$A^{\otimes I} \cong A^{\otimes \# I}$$

as the colimit of a certain functor.

This defines a lax monoidal functor

$$A : \mathcal{H} \to \mathsf{Set}$$

sending I to (the underlying set of) $A^{\otimes I}$.

Theorem (B.)

Given a ribbon Hopf algebra A, the universal invariant gives rise to a monoidal natural transformation of lax monoidal functors

$$Z_A : \mathcal{T} \Longrightarrow A$$

This means that for any finite set I there are maps

$$(Z_A)_I : \mathcal{T}_I \to A^{\otimes I}$$

mapping

$$(Z_A)_{\{i,j\}}(X_{ij}^{\pm 1}) = R_{ij}^{\pm 1} \in A^{\otimes \{i,j\}} \qquad , \qquad (Z_A)_{\{i\}}(C_i^{\pm 1}) = \kappa_i^{\pm 1} \in A^{\otimes \{i\}}$$

which are tangle invariants and that are natural with respect to all Hopf operations.

This point of view is powerful as it realises the universal tangle invariant as a map in the presheaf category $\mathsf{hSet} := \mathsf{Set}^{\mathcal{H}}$, the category of Hopf sets. This is similar to realising simplicial sets as the presheaf category $\mathsf{sSet} := \mathsf{Set}^{\Delta^{op}}$.

Actually both categories can be related:

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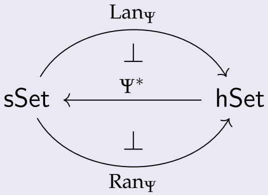
Actually both categories can be related:

Proposition (B.)

There is a functor $\Psi : \Delta^{op} \to \mathcal{H}$ which gives rise to adjuctions



Thank you for your attention.

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- Each edge carries an element of I and an integer called the *rotation number*.
- Edges around every four-valent vertex are cyclically ordered and pairs of opposite edges are labelled with the same element of I and are marked as the overpass or underpass.
- Edges labelled with the same element of I form connected oriented paths with distinct endpoints, called *strands*.

Upshot

For a finite set I, one can define an I-fold unordered tensor product

$$A^{\otimes I} \cong A^{\otimes \# I}.$$

Set

- Bij = category of ordinals and bijections,
- $\mathsf{fSet}^{\cong}$ = category of finite sets and bijections.

If $U : \mathsf{Bij} \to \mathsf{fSet}^{\cong}$ is the forgetful, for a finite set I, we write $U \downarrow I$ for the corresponding comma category.

Given a collection $(A^i : i \in I)$ of copies of a ribbon Hopf algebra A indexed by I, the *unordered tensor product* of A is the colimit

$$A^{\otimes I} := \operatorname*{colim}_{U \downarrow I} A^{\sigma(1)} \otimes \cdots \otimes A^{\sigma(n)},$$

where $\sigma : [n] \xrightarrow{\cong} I$ runs through the elements of $U \downarrow I$.

The functor $\mathcal{T} : (\mathcal{H}, \amalg) \to (\mathsf{Set}, \times)$ being lax monoidal means that there is a map

$$\mathcal{T}_I \times \mathcal{T}_J \to \mathcal{T}_{I \amalg J}$$

natural on I and J.

The natural transformation $Z_A : \mathcal{T} \Longrightarrow A$ being monoidal means that the following square commutes:

$$\begin{array}{ccc} \mathcal{T}_I \times \mathcal{T}_J & \longrightarrow & \mathcal{T}_{I \amalg J} \\ {\scriptstyle (Z_A)_I \times (Z_A)_J}\downarrow & & \downarrow{\scriptstyle (Z_A)_{I \amalg J}} \\ A^{\otimes I} \times A^{\otimes J} & \longrightarrow & A^{\otimes I \amalg J} \end{array}$$