XC-algebras and quantum knot invariants

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Slides available at bit.ly/jbecerra

Based on...

Jorge Becerra Garrido.
Universal quantum knot invariants.
PhD thesis, University of Groningen, 2024.

Jorge Becerra.
On Bar-Natan–van der Veen's perturbed Gaussians.
Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 118(2):Paper No. 46, 58, 2024.

Jorge Becerra.
A refined functorial universal tangle invariant.
arXiv:2501.17668.

Jorge Becerra.
XC-tangles and universal invariants.
arXiv:2511.08045.

Jorge Becerra and Kevin van Helden.
Minimal generating sets of rotational Reidemeister moves.
arXiv:2506.15628.

What are *quantum* knot invariants?

These are knot invariants whose construction is not intrinsic to the knot, but rather they have the extra data of some algebraic structure (a certain category, a TQFT, an algebra, a power series...). In some cases they admit extensions to quantum invariants of 3-manifolds.

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Some examples:

- The Jones polynomial,
- Khovanov homology,
- Khovanov-Rozansky / Lee homology,
- The Kontsevich invariant,
- $\vdots$

Today: the universal invariant

Let (H, R, v) be a *ribbon Hopf algebra*: this is a Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$ over some ring $\Bbbk$ with

- a *universal R-matrix* $R \in H \otimes H$, i.e. an invertible element with

$$(\Delta \otimes \mathrm{Id})R = R_{13} \cdot R_{23} \quad , \quad (\mathrm{Id} \otimes \Delta)R = R_{13} \cdot R_{12} \quad , \quad \Delta^{op} = R \cdot \Delta(-) \cdot R^{-1}$$

- a *ribbon element* $v \in H$,

$$v \in \mathcal{Z}(A) \ , \ v^2 = uS(u) \ , \ \Delta(v) = (R_{21}R)^{-1}(v \otimes v) \ , \ \epsilon(v) = 1 \ , \ S(v) = v$$

with $u = \mu^{\mathrm{op}}(\mathrm{Id} \otimes S)(R)$ the Drinfeld element. The ribbon element is automatically invertible.

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Theorem (Lawrence, Lee, Ohtsuki, Habiro,...)

Given a ribbon Hopf algebra (H, R, v), one can obtain an invariant $\mathfrak{Z}_H(K) \in H$ of framed, oriented (long) knots.

The construction

Write

$$R = \sum_i \alpha_i \otimes \beta_i \quad , \quad R^{-1} = \bar{\alpha}_i \otimes \bar{\beta}_i \quad , \quad \kappa := uv^{-1}.$$

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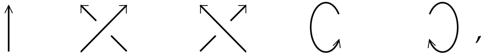
Consider a knot diagram in *rotational form*, i.e. formed by stitching on the plane the following elementary blocks (twisting them is NOT allowed !)



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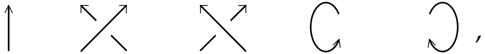
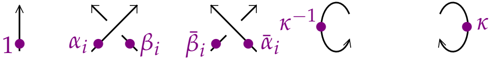
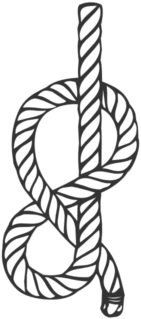


this is always possible.
Decorate these pieces with copies of $R^{\pm 1}$ and $\kappa^{\pm 1}$ as follows,



and then define $\mathfrak{z}_H(K)$ as the element resulting from multiplying these beads from right to left following the orientation of the knot.

Example



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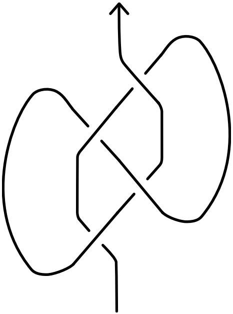

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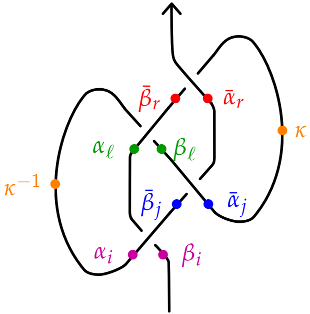
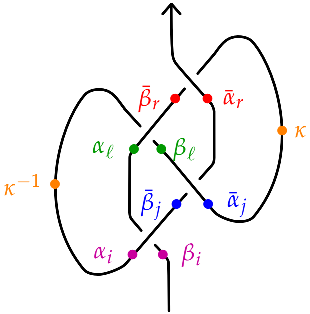


$\bar{\beta}_r$
$\bar{\alpha}_r$
κ
α_ℓ
β_ℓ
κ^{-1}
$\bar{\beta}_j$
$\bar{\alpha}_j$
α_i
β_i

Example



$$\mathfrak{Z}_H(4_1) = \sum_{i,j,\ell,r} \bar{\alpha}_r \, \bar{\beta}_j \, \alpha_i \, \kappa^{-1} \, \beta_\ell \, \bar{\alpha}_j \, \kappa \, \bar{\beta}_r \, \alpha_\ell \, \beta_i \in H$$

Why *universal*?

For any finite-dimensional representation (V, ρ) of H, the map

$$\rho(\mathfrak{Z}_H(K)) : V \longrightarrow V$$

equals the celebrated *Reshetikhin-Turaev invariant* $RT_V(K)$ obtained from the ribbon category H-mod of finite-dimensional H-modules.

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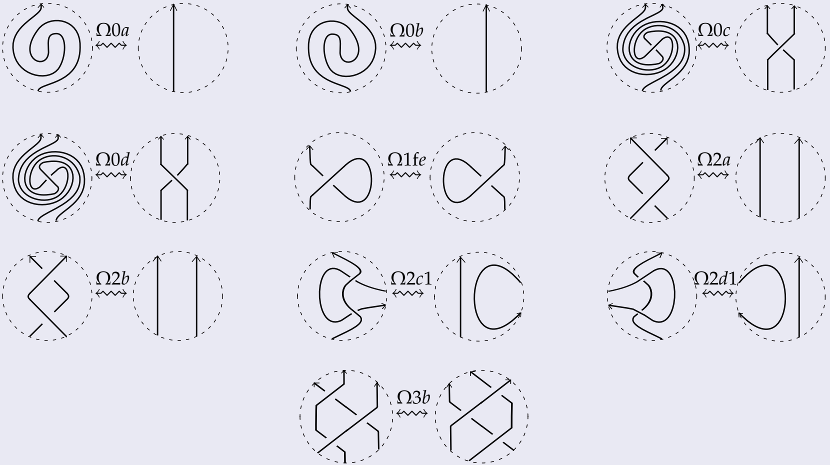
Key observation

The comultiplication, counit and the antipode of H are not used at all to construct $\mathfrak{Z}_H$!

Actually, to construct a knot invariant from this construction, we only need an algebra A with a couple of elements R, κ such that the Reidemeister moves are preserved!

Theorem (B.-van Helden 2025)

The following is a (non-minimal) generating set of rotational Reidemeister moves for rotational knot diagrams:



XC-algebras

Definition

Let A be a $\Bbbk$-algebra. An *XC-structure* on A is the choice of two invertible elements

$$R \in A \otimes A \qquad , \qquad \kappa \in A$$

satisfying

(XC0) $R^{\pm 1} = (\kappa \otimes \kappa) \cdot R^{\pm 1} \cdot (\kappa^{-1} \otimes \kappa^{-1})$,

(XC1f) $\sum_i \beta_i \kappa \alpha_i = \sum_i \alpha_i \kappa^{-1} \beta_i$,

(XC2c) $1 \otimes \kappa^{-1} = \sum_{i,j} \alpha_i \bar{\alpha}_j \otimes \bar{\beta}_j \kappa^{-1} \beta_i$,

(XC2d) $\kappa \otimes 1 = \sum_{i,j} \bar{\alpha}_i \kappa \alpha_j \otimes \beta_j \bar{\beta}_i$,

(XC3) $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$.

The triple (A, R, κ) is called an *XC-algebra*.

An XC-algebra is the minimum algebraic structure needed to define a framed, oriented knot invariant.

Proposition

If (A, R, κ) is an XC-algebra, then the same construction of $\mathfrak{Z}_A(K) \in A$ from above gives rise to a well-defined knot invariant.

A few examples of XC-algebras

1. Any ribbon Hopf algebra (H, R, v) has an underlying XC-structure

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$$(\mathrm{End}_{\Bbbk}(V),\ (\rho \otimes \rho)(R),\ \rho(\kappa)).$$

3. There are XC-structures that do not have a ribbon Hopf-algebraic origin: on the *Sweedler algebra* $SW = \langle s, w | s^2 = 1, w^2 = 0 \rangle$
$$R := 1 \otimes 1 + (1 + s + w + sw) \otimes (s + w + sw) \quad , \quad \kappa := -s - w - sw.$$

   We have
$$\mathfrak{Z}_{SW}(\quad$$

$$) = -1 - 2(s + w + sw) \notin \mathcal{Z}(SW) = \Bbbk 1.$$

A few honest examples

Set $\Bbbk := \mathbb{Z}[q, q^{-1}]$ and $A := \mathrm{End}_{\Bbbk}(\Bbbk^2) \cong \mathcal{M}_2(\Bbbk)$.

Proposition

The elements

$$R := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} + (q - q^{-3}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in A^{\otimes 2}$$

and

$$\kappa := \begin{pmatrix} q^2 & 0 \\ 0 & q^{-2} \end{pmatrix} \in A$$

define a (traced) XC-structure on A.
Furthermore, for a 0-framed knot K we have that

$$\mathfrak{Z}_A(K) = J_2(K)_{|q^2 = -t^{-1/2}} \cdot Id_{\Bbbk^2}.$$

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Furthermore, for a 0-framed knot K we have that

$$\mathfrak{Z}_A(K) = \Delta(K)_{|q^2 = t^{1/2}} \cdot Id_{\Bbbk^2}.$$

The *Dilbert algebra* is

$$DLB = \langle d, l, b | dl = d,\ db = 1 - l,\ lb = b,\ l^2 = l = bd,\ \text{others} = 0\rangle.$$

Proposition

The elements

$$R := 1 \otimes 1 - 2(1-l) \otimes l + 2b \otimes d \qquad , \qquad \kappa := \mathbf{i}(1-2l)$$

define a (traced) XC-algebra structure on DLB.
Furthermore, for a 0-framed knot K we have that

$$\mathfrak{Z}_{DLB}(K) = \Delta(K)(-1).$$

Theorem (B., to appear hopefully next week)

Any *XC-algebra structure on the Sweedler algebra SW produces a framed knot invariant that only depends on the framing.*
In particular, such an invariant is trivial for any 0-framed knot:

$$\mathfrak{Z}_{SW}(K) = 1.$$

Categorical framework

Most of the constructions in quantum topology are categorical/functorial (TQFTs, RT invariant, lasagna skein modules, Kh/Lee homology,...). The universal invariant did not have one even in the ribbon Hopf algebra setting.

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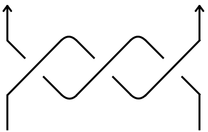

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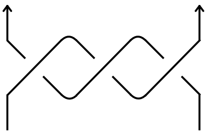
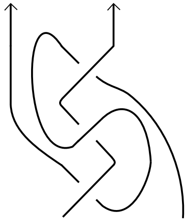
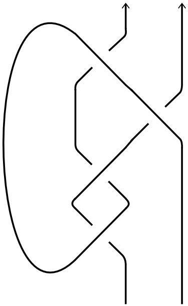




Given an XC-algebra (A, R, κ), the universal invariant can be defined similarly for upwards tangles:

T



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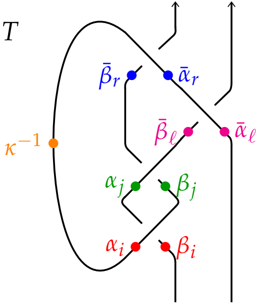

$$\mathfrak{Z}_A(T) = \sum_{i,j,\ell,r} \bar{\beta}_\ell \alpha_j \beta_i \otimes \bar{\beta}_r \beta_j \alpha_i \kappa^{-1} \bar{\alpha}_r \bar{\alpha}_\ell \in A \otimes A,$$

Theorem (B. 2024)

Let (A, R, κ) be an XC-algebra. There exists a monoidal category $\mathcal{E}(A)$ and a strict monoidal full functor

$$Z_A : \mathcal{T}^{\mathrm{up}} \longrightarrow \mathcal{E}(A)$$

which encodes the universal invariant $\mathfrak{Z}_A$:

$$Z_A(T) = (\mathfrak{Z}_A(T), \sigma_T).$$

Furthermore, this functor in fact arises in a canonical way – from a universal property.

If A is equipped with a trace, then one can in fact extend the construction to a functor

$$Z_A : \mathcal{T}^{+} \longrightarrow \mathcal{E}^{tr}(A)$$

encoding $\mathfrak{Z}_A$ also arising canonically.

This setting allows to a functorial comparison with the Reshetikhin-Turaev invariant $RT_V : \mathcal{T}^{\mathrm{up}} \longrightarrow H\text{-}\mathsf{mod}$.

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Theorem (B. 2024)

Let H be a ribbon Hopf algebra and let V be a finite-free H-module. Then the Reshetikhin-Turaev invariant RT_V factors through Z_H:

$$\begin{array}{ccc} \mathcal{T}^{\mathrm{up}} & \xrightarrow{RT_V} & H\text{-mod} \\ {\scriptstyle Z_H}\big\downarrow & \nearrow{\scriptstyle \rho_V} & \\ \mathcal{E}(H) & & \end{array}$$

That is, $RT_V(T) = \rho_V(Z_H(T))$ for any upwards tangle T. In other words, this diagram categorifies the equality

$$RT_V(K) = \rho(\mathfrak{Z}_H(K))$$

seen before.

If H is ribbon and V an H-module, we can produce two functors:

$$RT_V : \mathcal{T}^+ \longrightarrow H\text{-}\mathsf{mod} \qquad , \qquad Z_{\mathrm{End}_{\Bbbk}(V)} : \mathcal{T}^+ \longrightarrow \mathcal{E}(\mathrm{End}_{\Bbbk}(V)).$$

It turns out that these two invariants are essentially the same.

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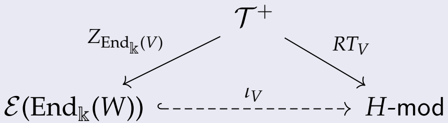
Let H be a ribbon Hopf algebra and let V be a finite-free H-module. Then we have the following commutative diagram:



with ι_V a monoidal embedding. That is, viewing $\mathcal{E}(\mathrm{End}(V))$ as a traced monoidal subcategory of H-mod, the functors $Z_{\mathrm{End}_{\Bbbk}(V)}$ and RT_V coincide.

Extension to virtual tangles

A virtual knot is a knot diagram with positive, negative and *virtual* crossings –the latter are really not there!– modulo appropriate Reidemeister moves.

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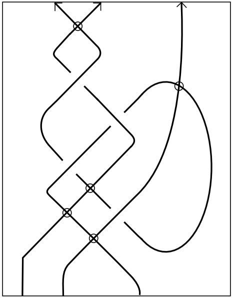

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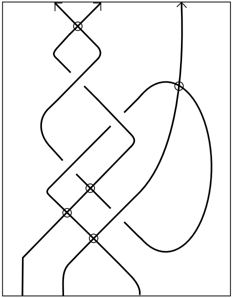

Virtual knot = Gauss diagram on $\uparrow$
Virtual upwards tangle = Gauss diagram on $\amalg_n \uparrow$.

Theorem (B. 2025)

If A is an XC-algebra, then there exists a monoidal category $v\mathcal{E}(A)$ and a functor

$$Z_A : v\mathcal{T}^{\mathrm{up}} \longrightarrow v\mathcal{E}(A)$$

extending the universal invariant of upwards tangles,

$$\begin{array}{ccc} \mathcal{T}^{\mathrm{up}} & \xrightarrow{Z_A} & \mathcal{E}(A) \\ \downarrow & & \downarrow \\ v\mathcal{T}^{\mathrm{up}} & \xrightarrow{Z_A} & v\mathcal{E}(A) \end{array}$$

More naturally, Z_A extends to the category $\mathcal{T}^{\mathrm{XC}}$ of *XC-tangles*, a class of decorated abstract graphs that consists of the exact geometrical counterpart of XC-algebras.

Combining the last two theorems:

Corollary

Let H be a ribbon Hopf algebra and let V be a finite-free H-module. Then the invariant

$$Z_{\mathrm{End}_{\Bbbk}(V)} : v\mathcal{T}^{\mathrm{up}} \longrightarrow v\mathcal{E}(\mathrm{End}_{\Bbbk}(V))$$

extends the Reshetikhin-Turaev invariant $RT_V : \mathcal{T}^{\mathrm{up}} \longrightarrow H\text{-mod}$ to virtual upwards tangles.

¡Gracias por su atención!

Thank you – Dank u wel – Merci