THE WITTEN-RESHETIKHIN-TURAEV 3-MANIFOLD INVARIANT

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Abstract. These notes where prepared for a talk in the School "Between the Waves VI" in Spring 2026. The goal is to explain, in a motivated and natural way, how one can obtain a 3-manifold invariant out of a modular fusion category. We also give a blueprint of the construction of a modular fusion category out of the quantum group of $\mathfrak{sl}_2$ at the root of unity.

Contents

1. Introduction

There is a close relation between the topology of 3-manifolds and knot theory, made explicit by two celebrated theorems: one by Lickorish-Wallace in the early 60s, stating that any closed, connected, oriented 3-manifold can be obtain by surgery on a framed link; and another one by Kirby-Fenn-Rourke in the late 70s stating that two framed links will produce the same 3-manifold if and only if they are related by a sequence of two moves.

The main goal of this notes is to show that, the aforemention connection establishes a deep connection between the topology of 3-manifolds and a categorical structure known as *modular fusion category*.

2. Modular fusion categories

We assume that the reader is familiar with braided pivotal categories, which were explained in previous talks.

2.1. Ribbon categories.

Definition 2.1.
- Let $\mathcal{C}$ be a strict braided monoidal category. We call a *twist* for $\mathcal{C}$ to a family θ of natural isomorphisms $\theta_V : V \longrightarrow V$ satisfying

$$\theta_{V\otimes W} = c_{W,V}c_{V,W}(\theta_V \otimes \theta_W)$$

- A *ribbon category* is a strict pivotal braided monoidal category endowed with a twist such is compatible with the rigid structure in the sense that

$$\theta_{V^*} = \theta_V^*.$$

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Definition 2.2. Let $\mathcal{C}$ be a ribbon category. We write $c_{V,W}$ for the braiding, b_V, d_V for the birth and death maps (aka coevaluation and evaluation) of the left duality and b_V, d'_V for the corresponding maps of the right duality. Recall that in a ribbon category, both are related by

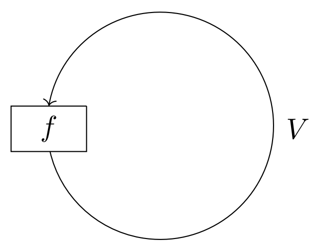
$$b'_V = (\mathrm{id}_{V^*} \otimes \theta_V) c_{V,V^*} b_V \qquad , \qquad d'_V = d_V c_{V,V^*} (\theta_V \otimes \mathrm{id}_{V^*}).$$

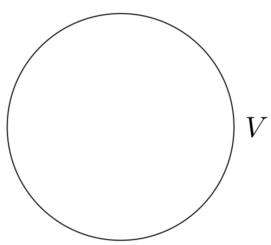
Let $V \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}(V, V)$.

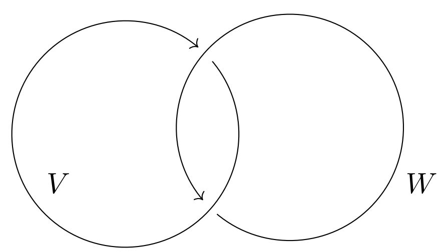
- The *trace* of f, denoted $tr(f)$ is the element of $\mathrm{Hom}(\mathbb{1}, \mathbb{1})$ defined by
$$tr(f) = d'_V (f \otimes \mathrm{Id}_{V^*}) b_V$$
- The *dimension* of V, $dim(V)$, is defined as the trace of the identity,
$$dim(V) = tr(\mathrm{Id}_V) = d'_V b_V.$$

Proposition 2.3. *Let $\mathcal{C}$ be a ribbon category and $V, W \in \mathrm{Ob}(\mathcal{C})$:*

(1) *If $f : V \longrightarrow W$ and $g : W \longrightarrow V$, then $tr(fg) = tr(gf)$.*
(2) *If $f \in \mathrm{Hom}(V, V)$ and $g \in \mathrm{Hom}(W, W)$, then $tr(f \otimes g) = tr(f)tr(g)$.*
(3) *For $k \in Hom(\mathbb{1}, \mathbb{1})$, we have $tr(k) = k$.*
(4) *If $V \simeq W$, then $dim(V) = dim(W)$.*
(5) *$dim(V) = dim(V^*)$.*
(6) *$dim(V \otimes W) = dim(V)dim(W)$.*
(7) *$dim(\mathbb{1}) = \mathrm{Id}_{\mathbb{1}}$.*

$tr(f) =$ 

$tr(\mathrm{Id}_V) =$ 

$tr(c_{V,W} \circ c_{W,V}) =$ 

2.2. **Linear, abelian and semisimple categories.** To make our lives easier we are going to assume that our underlying category $\mathcal{C}$ is $\mathbb{k}$-*linear abelian and semisimple*, where $\mathbb{k}$ is a fixed algebraically closed field of characteristic zero. For those who are not familiar with these concepts we include some backgroup

Definition 2.4. Let $\mathbb{k}$ be a algebraically closed field of characteristic 0.

- A category $\mathcal{C}$ is called *additive* over $\mathbb{k}$ if
  (1) The Hom sets are $\mathbb{k}$-vector spaces such that the composition of morphisms if $\mathbb{k}$-bilinear.
  (2) It admits a zero object $0_{\mathcal{C}} \in \mathrm{Ob}(\mathcal{C})$, ie $\mathrm{Hom}(0_{\mathcal{C}}, V) = \mathrm{Hom}(V, 0_{\mathcal{C}}) = 0$.
  (3) It admits finite direct sums.
- An additive category $\mathcal{C}$ is called *abelian* if every morphism $f : V \longrightarrow W$ admits a kernel and a cokernel and factors as a composition:
$$V \longrightarrow\!\!\!\!\!\rightarrow Im(f) \hookrightarrow W$$

where $Im(f) = ker(coker(f))$.

Definition 2.5.
- An object V in an abelian category is called *simple* if any injection $X \longrightarrow V$ is either zero or an isomorphism.
- An abelian category is called *semisimple* if every object V is isomorphic to a finite direct sum of simple objects

$$V = \bigoplus_{i \in I} N_i V_i$$

where $N_i \in \mathbb{Z}$ and V_i are simple objects .

Remark 2.6.
- If V is a simple object, we have the Schur lemma $\mathrm{Hom}(V) \cong \mathbb{k}$, that is every endomorphism $f : V \longrightarrow V$ of a simple object is a multiple of the identity, $f = \lambda \mathrm{Id}_V$ for some $\lambda \in \mathbb{k}$.
- $\mathrm{Hom}(V_i, V_j) = 0$ for $i \neq j$.

Let $\mathcal{C}$ be a $\mathbb{k}$-linear abelian semisimple ribbon category. We will always assume that the tensor product is bilinear on morphisms and that the unit object is simple, so in particular by the remark above we freely identify

$$\mathrm{End}(\mathbb{1}) \cong \mathbb{k}.$$

Denote by I the set of all isomorphism classes of simple objects of $\mathcal{C}$ and V_i a representative of the class corresponding to i. We chose V_0 to stand for $\mathbb{1}$ and $V_{i^*} = V_i^*$. We also have the *fusion rule*

$$V_i \otimes V_j = \bigoplus_{k \in I} N_{ij}^k V_k$$

where the fusion coefficients N_{ij}^k satisfy:

$$N_{ij}^k = dim \mathrm{Hom}(V_k, V_i \otimes V_j)$$

$$N_{ij}^k = N_{ji}^k = N_{ik^*}^{j^*} = N_{i^* j^*}^{k^*}, \qquad N_{ij}^0 = \delta_{ij^*}$$

We call θ_i and d_i the elements of $\mathbb{k}$ such that

$$\theta_{V_i} = \theta_i \cdot \mathrm{Id}_{V_i}, \qquad d_i := \dim V_i$$

and they satisfy

$$\theta_0 = 1, \quad \theta_{i^*} = \theta_i$$

$$d_0 = 1, \quad d_{i^*} = d_i, \quad d_i d_j = \sum_{k \in I} N_{ij}^k d_k$$

Moreover, $d_i \neq 0$ for all simple objects $V_i \in \mathrm{Ob}(\mathcal{C})$.

2.3. **Modular fusion categories.**

Definition 2.7. A *modular fusion category* is a $\mathbb{k}$-linear abelian semisimple ribbon category $\mathcal{C}$ satisfying the following properties:

(1) The set I is finite, ie $\mathcal{C}$ has a finite number of isomorphism classes of simple objects.
(2) The matrix $S = (s_{ij})$, where

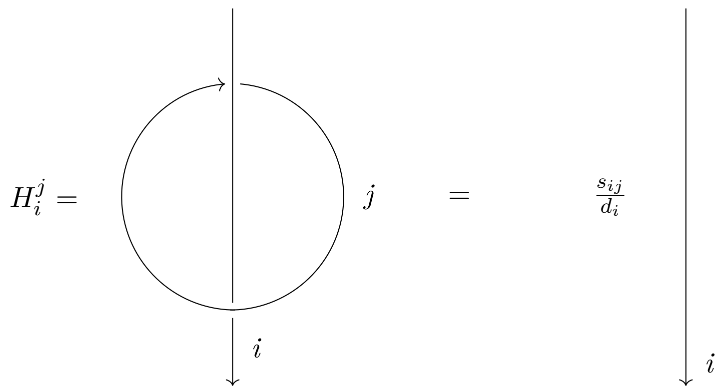
$$s_{ij} := tr(c_{V_i, V_j} \circ c_{V_j, V_i})$$

is invertible, ie $\det S \neq 0$.

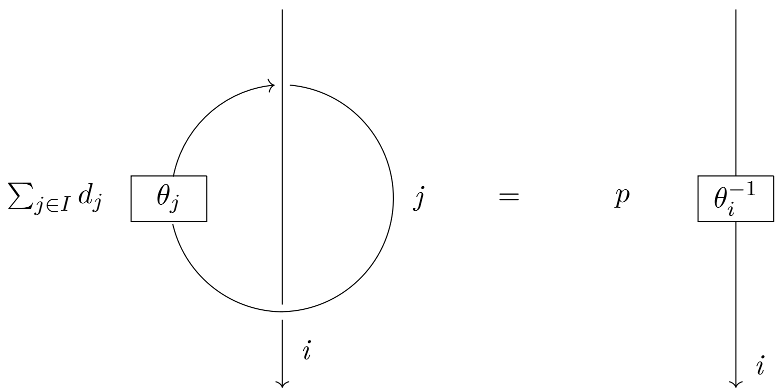
Remark 2.8. The matrix S has the following properties
- S is a symmetric matrix.
- $s_{i0} = s_{0i} = d_i$.
- $s_{ij} = s_{i^* j^*}$

Lemma 2.9. *Let $\mathcal{C}$ be a semisimple ribbon category, then*

(1) Let H_i^j be the morphism shown below, then $H_i^j = \frac{s_{ij}}{d_i}\mathrm{Id}_{V_i}$



(2) Set $p = \sum_{i\in I} \theta_i d_i^2$, then we have



Proof. (1) On one hand, we have

$$tr(H_i^j) = s_{ij}$$

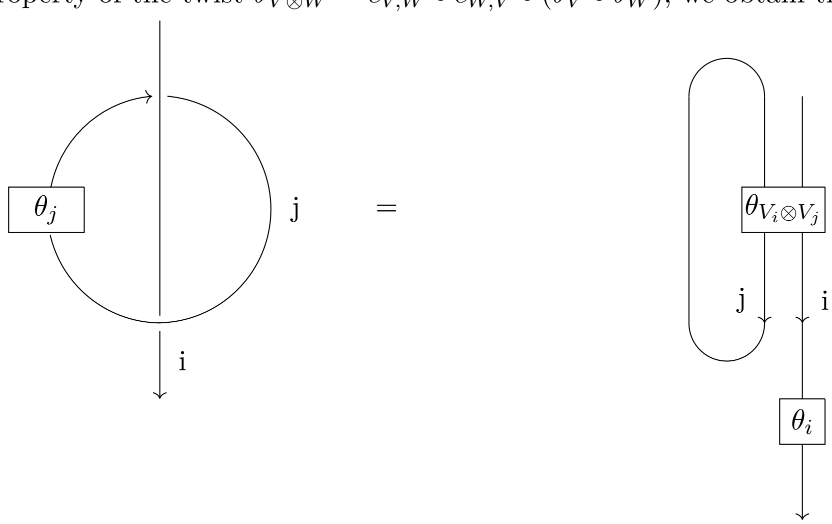
On the other hand, $H_i^j \in \mathrm{End}(V_i)$, and since V_i is a simple object $H_i^j = \alpha \mathrm{Id}_{\mathrm{V_i}}$ where $\alpha \in \mathrm{K}$. It follows that

$$tr(H_i^j) = tr(\alpha \mathrm{Id}_{V_i}) = \alpha tr(\mathrm{Id}_{V_i}) = \alpha . d_i$$

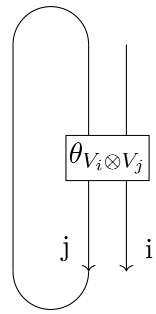
d_i is invertible, so we obtain

$$\alpha = \frac{s_{ij}}{d_i}$$

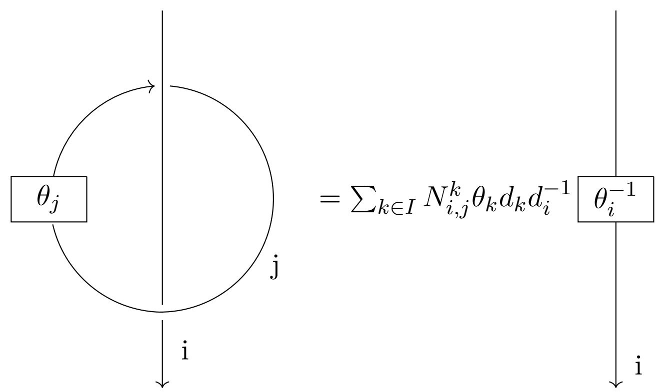
(2) Using the property of the twist $\theta_{V\otimes W} = c_{V,W} \circ c_{W,V} \circ (\theta_V \circ \theta_W)$, we obtain the equality



Let's first analyse the element L_i^j

$L_i^j =$ 

On one hand, since V_i is a simple object, then $L_i^j = \beta Id_{V_i}$ for some $\beta \in K$, which implies that $tr(L_i^j) = \beta d_i$. One the other hand, we know that $V_i \otimes V_j$ decomposes as a finite direct sum of the form $V_i \otimes V_j = \sum_k N_{i,j}^k V_k$, hence $tr_{V_i \otimes V_j}\theta = \sum_k N_{i,j}^k \theta_k d_k$. It follows that $\beta = \sum_k N_{i,j}^k \theta_k d_k d_i^{-1}$. Then for the above equivalence, we obtain



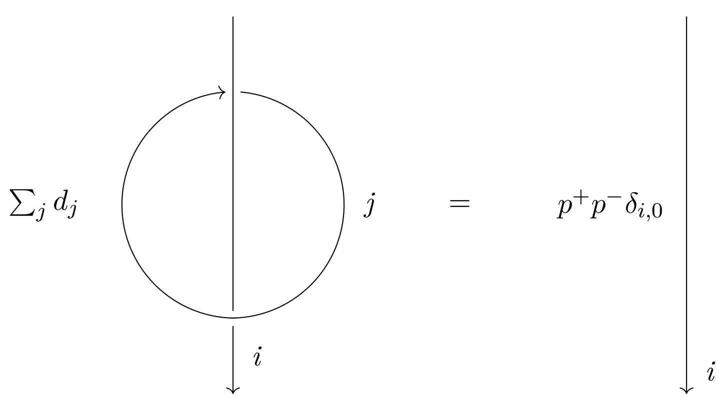
Finally, multiplying by d_j and taking the sum over I, we get

$$\sum_{j \in I} d_j \Big(\sum_{k \in I} N_{i,j}^k \theta_k d_k d_i^{-1} \Big) = \sum_{k \in I} \Big(\sum_{j \in I} N_{i,k^*}^{j^*} d_j \Big) \theta_k d_k d_i^{-1} = \sum_{k \in I} d_i d_k^* \theta_k d_k d_{i^{-1}} = \sum_{k \in I} \theta_k d_k^2$$

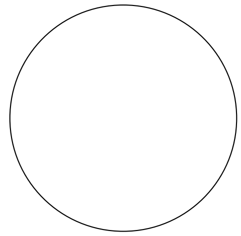
which completes the proof.

□

Corollary 2.10. *In a semisimple modular category $\mathcal{C}$, we have:*
1.



2.

$$p^+p^- = \sum_i d_i^2 = \sum_i d_i$$ 

Proof. (1) By taking the trace on the left hand side, then using 4. theorem 2.7, we get

$$\sum_j d_j s_{ij} = \sum_j s_{0j} s_{ij} = (S^2)_{0i} = p^+p^- c_{0i} = p^+p^- \delta_{i,0}$$

(2) We just need to replace i by 0 in 1.

□

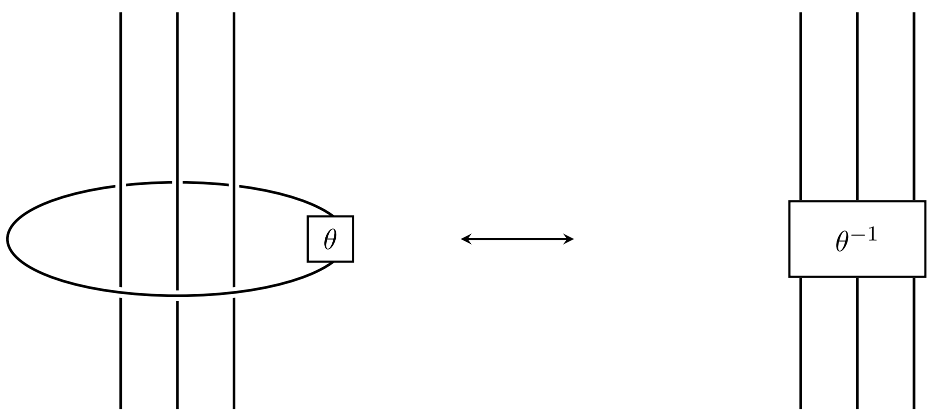
3. The 3-manifold invariant

In this section we will motivate the construction of a 3-manifold invariant in a natural way. The core of the arguments follows [Tur16] but the exposition is somewhat different.

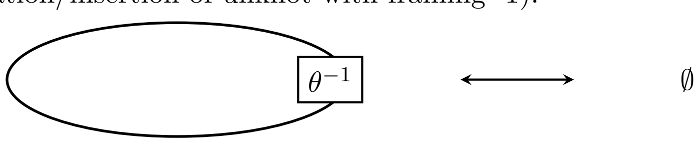
3.1. The Kirby-Fenn-Rourke theorem.

We recall the close relation between 3-manifold and links in S^3 [Kir78, FR79], which was already covered in one of the previous talks.

Let M and M' be two 3-manifolds. Then $M \approxeq M'$ iff their corresponding surgery links L' can be obtained from L by a sequance of Kirby–Fenn–Rourke moves as shown in the following figure.



When the number of inside strands is 0, this move amounts to elimnating an unknot with with a positive twist θ. One can easily see with an induction aurgment on the number of strands that this move is equivelent to the usual kirby moves. In that sense, the move is stable under change of $\theta \longleftarrow \theta^{-1}$. This allows a special kind of move that we will need, special Kirby -1 move which is shown below (elimination/insertion of unknot with framing -1).



3.2. The rank of a modular fusion category and the RT invariant.

Let us fix $\mathcal{C}$ a modular fusion category over an algebraically closed field $\Bbbk$

The *rank* of $\mathcal{C}$ is

$$\mathcal{D} := +\sqrt{\sum_{i \in I} d_i}$$

(such a positive square root exists as $\Bbbk$ is assumed to be algebraically closed). We also put

$$p^{\pm} := \sum_{i \in I} \theta_i^{\pm} \cdot d_i^2.$$

It is also common to denote $\Delta := p^-$.

Let L be a framed, oriented link in S^3. A $\mathcal{C}$*-colouring* of L is a set-theoretical map

$$\lambda : \pi_0(L) \longrightarrow I.$$

We write $\mathrm{Col}(L)$ for the set of $\mathcal{C}$-colourings of L; this is a set of cardinality $|I|^{|\pi_0(L)|}$. A pair (L, λ) is called a $\mathcal{C}$-coloured link.

Recall from one of the previous talks that if $\mathcal{C}$ is a ribbon category, Reshetikhin and Turaev constructed a functor

$$RT : \mathcal{T}_{\mathcal{C}} \longrightarrow \mathcal{C}$$

preserving the ribbon structure. Here $\mathcal{T}_{\mathcal{C}}$ denotes the category of $\mathcal{C}$-coloured tangles. The image of a coloured link (L, λ) under this functor will be denoted by $RT(L, \lambda)$.

3.3. **Towards a 3-manifold invariant.** Let L be a framed link in S^3 and let $S^3(L)$ be the closed, connected, oriented 3-manifold resulting from performing surgery on S^3 along L. We aim to construct a topological invariant for $S^3(L)$ from L, cooking up a quantity that needs to be preserved by the Kirby moves. We also want to use the Reshetikhin-Turaev invariant. For this to make sense first we have to give an arbitrary orientation to L. We also need to choose a $\mathcal{C}$-colouring for L. Which one? There is no a canonical choice... well where is, namely all of them! So we are tempted to consider the quantity

$$\sum_{\lambda \in \mathrm{Col}(L)} RT(L, \lambda). \tag{3.1}$$

In this quantity, we are giving all simple objects the same weight. In the search for a 3-manifold invariant, it might be that not all of the simple objects need to have the same weight and that the above sum need to be weighted.

With this in mind, define a weight on the set of simple objects of $\mathcal{C}$ as a map

$$\delta : I \longrightarrow \Bbbk.$$

We will typically write $\delta_i := \delta(i)$. If $\lambda \in \mathrm{Col}(L)$, the δ-weight of λ is

$$w_\delta(\lambda) := \prod_{\ell \in \pi_0(L)} \delta_{\lambda(\ell)}.$$

Given a weight δ, we define the δ-weighted Reshetikhin-Turaev invariant as

$$wRT(L, \lambda, \delta) := w_\delta(\lambda) RT(L, \lambda).$$

Note that this amounts to "weight" the simple object $\lambda(\ell)$ which labels a given component $\ell \in \pi_0(L)$ with the scalar $\delta_{\lambda(\ell)} \in \Bbbk$.

Therefore, we are going to replace the previous sum (3.1) with the weighted sum

$$\tau_\delta(S^3(L)) := \sum_{\lambda \in \mathrm{Col}(L)} wRT(L, \lambda, \delta). \tag{3.2}$$

Now the main question is: can the quantity $\tau_\delta(M)$ ever be a homeomorphism invariant of M? For any weight? For a special weight?

As it turns out, the Fenn-Rourke move completely determines the weight:

Theorem 3.1. *The quantity $\tau_\delta(S^3(L))$ is preserved under the Fenn-Rourke move if and only if the weights are proportional to the dimensions:*

$$\delta_i = \mathcal{D}^{-2} \Delta d_i$$

for all $i \in I$.

This theorem will follow from three lemmas. In the following we will use locality which allows us to consider parts of colored links instead of the whole link.

Lemma 3.2. *Fix $k \in I$. If $\tau_\delta(S^3(L))$ is preserved under the Fenn-Rourke move, then we have*

$$\frac{\theta_k}{d_k} \sum_{i \in I} \delta_i \theta_i s_{ik} = 1.$$

Proof. Consider kirby move 1 with one strand. By coloring the middle strand with k and using the weight sum defined above for the outer circle with Theorem 2.9.(1). The relation is direct. ☐

Call $x := \sum_{i \in I} \delta_i d_i$.

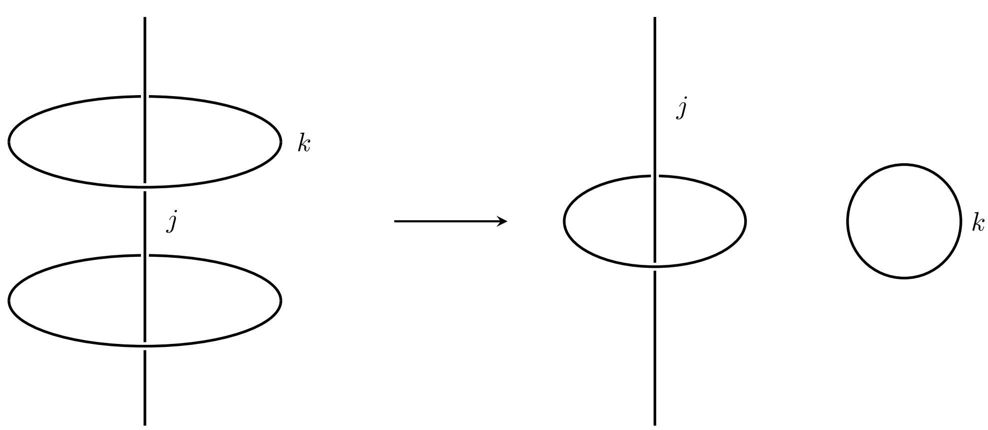
Lemma 3.3. *Suppose that $\tau_\delta(S^3(L))$ is preserved under the Fenn-Rourke move. Fix $j, r \in I$. Then we have*

$$\sum_{i \in I} \delta_i s_{ij} = \begin{cases} x, & j = 0 \\ 0, & else \end{cases} \tag{3.3}$$

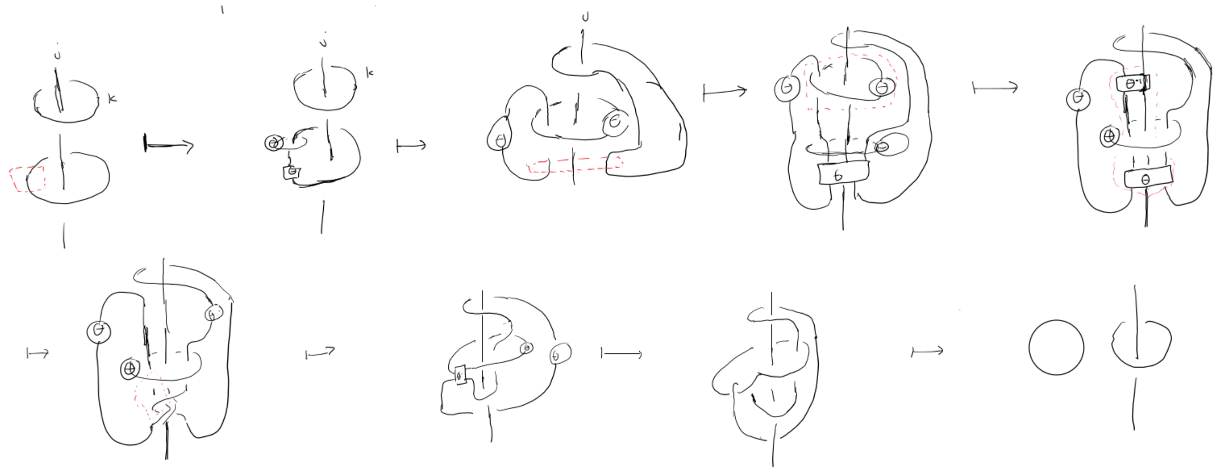
$$\sum_{i \in I} \delta_i d_i^{-1} s_{ji} s_{ir} = \begin{cases} x, & r = j^* \\ 0, & else \end{cases} \tag{3.4}$$

$$\sum_{i \in I} \delta_i \theta_i^{-1} \theta_j^{-1} s_{ij} = x \delta_j \tag{3.5}$$

Proof. For the proof of this lemma we consider a simplified version in the argument found in [Tur16]. The proof follows a simplification of some knot diagrams which are merely a consequence of the choice of the coloring (3.2) and the kibry moves. Let us begin by proving (3.3). This can be done by showing first the whenever we loop colored by (3.2) encricles a strand, the strand acts transparentaly to an other loop that encircle it. Since the category is non degenerate, this forces the strand itself to the identity. The relation that we want to prove is shown in the following figure (an unlabeled component stands for a weighted sum that runs through $i \in I$ labeling such component):



the following is a visual proof of the lemma,



where use used kirby move 1 four times on the colored boxes. In the fifth digram we used the braid relations to similify the digrams. In the above sense, we have reached the following equivalnce by using Theorem 2.9.(1),

$$\sum_i \delta_i \frac{s_{ij}}{d_j}\frac{s_{kj}}{d_j} = d_k \sum_i \delta_i \frac{s_{ij}}{d_j}$$

We denote $\sum_i \delta_i s_{ij} = \omega_j$, and $s'_{kj} = d_k^{-1} d_j^{-1} s_{kj}$. Then, the above relation simplifies to,

$$\omega_j = \omega_j s'_{k,j}$$

Now since the S-matrix is invertible, then s' is also invertible. However s' has unities in the first row and column (notice we divided the s matrix by the dimensions which are exactly these entries since 1 is transparent). Thus, one can easily see that,

$$\delta_{ij} = \sum_k {s'}^{-1}_{ik} {s'}_{kj}$$

when $i = j = 0$, we get,

$$1 = \sum_k {s'}^{-1}_{0k} {s'}_{k0}$$

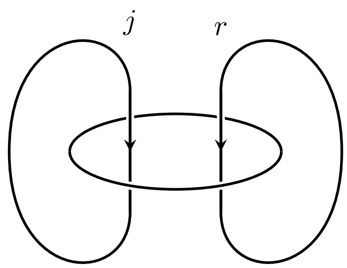
but $s'_{k0} = d_k^{-1} d_0^{-1} s_{k0} = 1$, so we get,

$$1 = \sum_k {s'}^{-1}_{0k}$$

Now plugging this relation inside the w_j equation, one can find the following,

$$w_j = 1.w_j = \sum_k {s'}^{-1}_{ko} w_j = \sum_k {s'}^{-1}_{ko} s'_{kj} w_j = \omega_0 \delta_{0,j}$$

proving (3.3). Now, we move on to prove (3.4). This can be proved by considering the following knot digram.



This digram can be evalauted by two different methods, either by using the fusion rule to simiplify the double strand or by using Theorem 2.9.(1) two times. The former produces the following equalities,

$$\begin{aligned}\sum_{i,k}\delta_i N^k_{j*r}\frac{s_{ki}}{d_k} &= \sum_k N^k_{j*r}\omega_k d_k^{-1}\\ &= N^0_{j*r}\omega_0 d_0^{-1}\\ &= \omega_0\delta_{j*r}\\ &= x\delta_{j*r}\end{aligned}$$

Now if we apply Theorem 2.9.(1) twice, once easily obtain,

$$\sum_i d_i\delta_i\frac{s_{ji}}{d_i}\frac{s_{ir}}{d_i} = \sum_i d_i^{-1}\delta_i s_{ji}s_{ir} = x\delta_{j*r}$$

proving (3.4). Now for (3.5), one considers lemmma 3.2 multiply by $\delta_j\theta_r^{-1}\theta_j^{-1}s_{jr}/d_j$ then summs over j.

$$\begin{aligned}\sum_i \delta_i\theta_j\theta_i s_{ij} &= d_j\\ \sum_{i,j}\delta_i\delta_j\theta_i\theta_r^{-1}\frac{s_{jr}}{d_j}s_{ij} &= \sum_j \theta_r^{-1}\theta_j^{-1}\delta_j s_{jr}\\ \sum_i \theta_r^{-1}\delta_i\theta_i\underbrace{\sum_j d_j^{-1}\delta_j s_{jr}s_{ij}}_{x\delta_{ri*}} &= \sum_j \theta_r^{-1}\theta_j^{-1}\delta_j s_{jr}\end{aligned}$$

but since $\theta_i = \theta_{i*}$ one gets,

$$x\delta_r = \sum_j \theta_r^{-1}\theta_j^{-1}\delta_j s_{jr}$$

□

Corollary 3.4. *Suppose that $\tau_\delta(S^3(L))$ is preserved under the Fenn-Rourke move. Then $\delta_i \neq 0$ for all $i \in I$.*

Proof. The first observation is that the weights δ_i cannot be all simultaneously zero, for (3.2) would imply $0 = 1$. Now, (3.3) can be written in matrix form as

$$\begin{pmatrix}\delta_0 & ,\dots, & \delta_{|I|}\end{pmatrix}\cdot S = \begin{pmatrix}x\\0\\ \vdots\\0\end{pmatrix}$$

Since the S-matrix S is invertible,

$$\begin{pmatrix}\delta_0 & ,\ldots, & \delta_{|I|}\end{pmatrix} = \begin{pmatrix} x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot S^{-1},$$

which implies that $x \neq 0$ (as otherwise all the δ_i vanish, contradicting the previous claim.

In order to see that none of the δ_i vanish, we will show that the inverse of the S-matrix is given by

$$S^{-1} = \left(\delta_i x^{-1} d_i^{-1} s_{i,j^*}\right)_{i,j}.$$

Clearly this is enough as the i-th row of S^{-1} is a multiple of δ_i.

Let us prove the last claim. Put $\widetilde{S} = \left(\delta_i x^{-1} d_i^{-1} s_{i,j^*}\right)_{i,j}$. The coefficient (p,q) of $S\widetilde{S}$ is given by

$$\begin{aligned}(S\widetilde{S})_{p,q} &= \sum_{k\in I} s_{p,k}\delta_k x^{-1} d_k^{-1} s_{s,q^*} \\ &= x^{-1}\sum_{k\in I}\delta_k d_k^{-1} s_{p,k} s_{s,q^*} \\ &= \begin{cases} 1, & p = q^{**} = q \\ 0, & else \end{cases}\end{aligned}$$

where in the last equality we have used (3.4).

□

Lemma 3.5. *Suppose that $\tau_\delta(S^3(L))$ is preserved under the Fenn-Rourke move. Then we have*

$$\delta_i = \delta_0 d_i$$

for all $i \in I$.

Proof. Consider the following jth surgery componeent of in S^3 of the following a hopf link with -1 framming on both components, For the folllwing, we will consider the standard kirby 2 move which is equivelent to the Fenn Rourke theorem stated above. Then, by inverse sliding the link colored by $\tau_d elta$ over j, one finds the following equaility,

$$\sum_i \delta_i \theta_i^{-1}\theta_j^{-1} s_{ij} = d_j \sum_i \delta_i \theta_i^{-1} d_i$$

Now, using (3.5), the left hand side is just $x\delta_j$, while the right hand side is $d_j x\delta_0$ since $\sum_i \delta_i\theta_i^{-1}\theta_0^{-1}s_{i0} = \sum_i \delta_i d_i \theta_i^{-1}$. Thus, we reach the following conculsion,

$$x\delta_0 d_j = x\delta_j$$

From which one can conclude that,

$$\delta_j = \delta_0 d_j$$

making it proportional to the dimension of the simple objects. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Recall that we put

$$\mathcal{D}^2 = \sum_i d_i^2 \qquad , \qquad \Delta = \sum_i \theta_i^{-1} d_i^2.$$

We will start by showing that these two elements are non-zero. Indeed (3.5) for $j = 0$ gives

$$\sum_i \delta_i \theta_i^{-1} d_i = x\delta_0$$

as $s_{i,0} = d_i$ and $\theta_0 = 1$. According to Theorem 3.5, the previous equation can be written as

$$\sum_i \delta_0 d_i \theta_i^{-1} d_i = x\delta_0$$

and since $\delta_0 \neq 0$ by Theorem 3.4, we get

$$\Delta = \theta_i^{-1} d_i^2 = x, \tag{3.6}$$

which also means that $\Delta \neq 0$. In particular,

$$\Delta = x = \sum_i \delta_i d_i = \delta_0 \sum_i d_i^2 = \delta_0 \mathcal{D}^2,$$

so $\mathcal{D} \neq 0$, and most importantly

$$\delta_0 = \mathcal{D}^{-2}\Delta.$$

This equality together with Theorem 3.5 concludes the direct direction of Theorem 3.1.

The converse follows from the following argument: when we only have a single component traversing the unknotted component, it follows directly from Theorem 2.9.(2) that the quantity τ is preserved because $p = \mathcal{D}^2\Delta^{-1}$ by Theorem 2.10. Regarding the general move, we can consider the local morphism H_j^i as in Theorem 2.9 but with the loop colored by our weighted sum with a $+1$ framing. Upon forming the link, this part decomposes as sum of the fusion coefficients. Then without loss of generality, one can write,

$$\sum_{i_1,i_2,\cdots,i_n} N_{i_1,i_2}^{j_1} N_{j_1,i_3}^{j_2} \cdots N_{j_{n-2},i_n}^{j_{n+1}} \theta \circ H_{n+1}^{\Omega+1} = \sum_{i_1,i_2,\cdots,i_n} N_{i_1,i_2}^{j_1} N_{j_1,i_3}^{j_2} \cdots N_{j_{n-2},i_n}^{j_{n+1}} id_{j_{n+1}} = id_{i_1} \otimes id_{i_2} \otimes \cdots id_{i_n}$$

where in the second equaility we just used the fact that it is true for one strand. Please note again that the above relation is meant to be understood upon forming the whole link. ☐

In view of Theorem 3.1, let us put

$$wRT(L,\lambda) := wRT(L,\lambda,\delta) \qquad , \qquad \tau(M) := \tau_\delta(M)$$

for δ the weight of simple objects given by $\delta_i = \mathcal{D}^{-2}\Delta d_i$.

Recall that we picked an arbitrary orientation for the link L in order to be able to compute the Reshetikhin-Turaev invariant and the quantity $\tau(M)$. At this moment we can already verify that with the unique choice of weight δ that preserves the Fenn-Rourke move, the choice of orientation for L does not matter.

Lemma 3.6. *The quantity*

$$\tau(S^3(L)) = \sum_{\lambda \in \mathrm{Col}(L)} wRT(L,\lambda)$$

is independent of the choice of orientation for L.

Proof. One just needs to consider the change induced whenver the components of L change orientation. Let L_n be an arbitrary component of L. Consider the change in orientation of this single component L_n. The coloring on all other compoents concide and for L_n, one gets $\delta(L_n)^*$. Howver, since the evaulation inside the categoy does not depend on the orientation furthermore $d_i = d_{i^*}$, its clear that the claim follows. ☐

So, if we want τ to be a 3-manifold invariant, all we have to check is whether it is also preserved under the negative stabilisation or not. BAD NEWS: it is not, but it changes in a very controlled way.

Lemma 3.7. *Let L be a framed link and let L_{NS} be the link obtained from L by adding an additional unknotted component U_- with framing -1 (that is, a negative stabilisation). Then*

$$\tau(S^3(L_{NS})) = \mathcal{D}^{-2}\Delta^2\tau(S^3(L)).$$

Proof. Recall that the Reshetikhin-Turaev invariant of coloured links is multiplicative under disjoint unions,

$$RT((L,\lambda) \amalg (L',\lambda')) = RT(L,\lambda)RT(L',\lambda').$$

It is immediate to realise that this implies the multiplicativity of the weighted Reshetikhin-Turaev invariant,

$$wRT((L,\lambda) \amalg (L',\lambda')) = wRT(L,\lambda)wRT(L',\lambda').$$

Note that for a disjoint union of two links L, L', there is a canonical bijection

$$\mathrm{Col}(L \amalg L') = \mathrm{Col}(L) \times \mathrm{Col}(L').$$

With this in mind, we compute

$$\begin{aligned}
\tau(S^3(L_{NS})) &= \sum_{\lambda\in\mathrm{Col}(L_{NS})} wRT(L_{NS},\lambda) \\
&= \sum_{(\lambda_1,\lambda_2)\in\mathrm{Col}(L)\times\mathrm{Col}(U_-)} wRT(L_{NS},(\lambda_1,\lambda_2)) \\
&= \sum_{(\lambda_1,\lambda_2)\in\mathrm{Col}(L)\times\mathrm{Col}(U_-)} wRT(L,\lambda_1)wRT(U_-,\lambda_2) \\
&= \left(\sum_{\lambda_2\in\mathrm{Col}(U_-)} wRT(U_-,\lambda_2)\right)\left(\sum_{\lambda\in\mathrm{Col}(L)} wRT(L,\lambda_1)\right) \\
&= \left(\sum_{i\in I}\delta_i d_i\theta_i^{-1}\right)\tau(S^3(L)) \\
&= \left(\delta_0\sum_{i\in I} d_i^2\theta_i^{-1}\right)\tau(S^3(L)) \\
&= \delta_0\Delta\tau(S^3(L)) \\
&= \mathcal{D}^{-2}\Delta^2\tau(S^3(L))
\end{aligned}$$

as claimed. □

So: the quantity $\tau(S^3(L))$ is preserved under the Fenn-Rourke move but changes (in a controlled way) under negative stabilisation. Now the game is similar to that of finding a (unframed) knot invariant (aka the Jones polynomial) from the Kauffman bracket: we need a multiplicative factor that balances out the factor $\mathcal{D}^{-2}\Delta^2$ from the previous lemma.

Given a framed, oriented link $L = L_1\cup\cdots\cup L_m$ with m components, recall that its linking matrix $A = A_L = (a_{ij})$ is the $m\times m$ matrix whose coefficients are given by

$$a_{ij} = \begin{cases} lk(L_i, L, j), & i\neq j \\ fr(L_i), & i=j \end{cases}.$$

This is a symmetric matrix with integer coefficients. In particular, it is diagonalisable. Let $\sigma_\pm(L)$ denote the number of positive/negative eigenvalues of A, and set $\sigma(L) := \sigma_+(L) - \sigma_-(L)$ for the signature of the matrix A.

Remark 3.8. By classical 4-manifold topology, the linking matrix A of L is a matrix for the intersection form of the 4-manifold W_L obtained from D^4 by attaching m 2-handles along L. The boundary of this 4-manifold is precisely $S^3(L)$. In particular, the signature of the 4-manifold W_L is exactly $\sigma(L)$.

Let us contemplate $\sigma_-(L)$.

Lemma 3.9. *Write L_{FR} and L_{NS} for the links resulting from applying the Fenn-Rourke move and the negative stabilisation to a given framed oriented link L, respectively. Then we have*

(1) $\sigma_-(L_{FR}) = \sigma_-(L)$,
(2) $\sigma_-(L_{NS}) = \sigma_-(L) + 1$

Proof. (2) is immediate so let us check (1). Let us label the m components of L so that the last component is the unknotted component with framing $+1$ and the first k components are those which traverse the unknotted component. A bit of thinking says that for some symmetric matrix A of size $m-1$ and for

$$v = (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0)^T \in \mathbb{R}^n.$$

we have

$$A_L = \begin{pmatrix} A & v \\ v^T & 1 \end{pmatrix}.$$

and $A_{L_{FR}} = B = (b_{ij})$ by

$$b_{ii} = a_{ii},$$

and for $i \neq j$,

$$b_{ij} = \begin{cases} a_{ij} - 1, & \text{if } 1 \leq i, j \leq k, \\ a_{ij}, & \text{otherwise.} \end{cases}$$

We claim that A_L and $A_{L_{FR}}$ have the same number of negative eigenvalues: consider the invertible matrix

$$P = \begin{pmatrix} I_n & -v \\ 0 & 1 \end{pmatrix}.$$

A direct computation gives

$$P^T A_L P = \begin{pmatrix} A_{L_{FR}} & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore A_L is congruent to $A_{L_{FR}} \oplus (1)$. □

In other words: $\sigma_-(L)$ has the same behavior than $\tau(S^3(L))$! Preserved by the Fenn-Rourke move and it changes (in a controlled way) by the negative stabilisation.

Looking at Theorem 3.7 and Theorem 3.9, it is clear how to modify $\tau(S^3(L))$ in order to get a well-defined 3-manifold invariant.

Theorem 3.10. *Let L be a framed link in S^3. Then*

$$WRT(S^3(L)) := (\mathcal{D}^{-2}\Delta^2)^{-\sigma_-(L)}\tau(S^3(L))$$

is a well-defined homeomorphism invariant of $S^3(L)$.

Proof. It follows from Theorem 3.1 and Theorem 3.9.(1) that $WRT(S^3(L))$ is preserved under the Fenn-Rourke move. Now for the negative stabilisation we compute

$$\begin{aligned} WRT(S^3(L_{NS})) &= (\mathcal{D}^{-2}\Delta^2)^{-\sigma_-(L_{NS})}\tau(S^3(L_{NS})) \\ &= (\mathcal{D}^{-2}\Delta^2)^{-\sigma_-(L)-1}\mathcal{D}^{-2}\Delta^2\tau(S^3(L)) \\ &= (\mathcal{D}^{-2}\Delta^2)^{-\sigma_-(L)}\tau(S^3(L)) \end{aligned}$$

$$= WRT(S^3(L)),$$

where we have used Theorem 3.7 and Theorem 3.9.(2) in the second equality. On the other hand, this amount is independent on the choice of orientation of L chosen to compute the right-hand side according to Theorem 3.6. ☐

For a modular fusion category $\mathcal{C}$ and a closed, connected 3-manifold M obtained from surgery along a framed link L, the resulting invariant

$$\boxed{WRT_{\mathcal{C}}(M) := (\mathcal{D}^{-2}\Delta^2)^{-\sigma_-(L)} \sum_{\lambda\in \mathrm{Col}(L)} wRT_{\mathcal{C}}(L,\lambda)} \tag{3.7}$$

is called the *Witten-Reshetikhin-Turaev invariant* of M.

Examples 3.11. (1) The 3-manifold $S^2 \times S^1$ is obtained from surgery along the unknot U_0 with framing zero. Since $\sigma_-(U_0) = 0$, we have

$$WRT(S^2 \times S^1) = \sum_{i\in I} \delta_i d_i = \Delta$$

(2) The 3-manifold S^3 is obtained from surgery along the empty link. So

$$WRT(S^3) = 1.$$

It is common to choose a normalisation WRT^{norm} of WRT that matches Witten's ill-defined 3-manifold invariant using path integrals. More concretely, we seek a normalisation such that

$$WRT^{\mathrm{norm}}(S^2 \times S^1) = 1 \qquad , \qquad WRT^{\mathrm{norm}}(S^3) = \mathcal{D}^{-1}. \tag{3.8}$$

Let $b_1(M) := \dim H_1(M;\mathbb{R})$. Set

$$WRT^{\mathrm{norm}}_{\mathcal{C}}(M) := \Delta^{-b_1(M)}\mathcal{D}^{b_1(M)-1} WRT_{\mathcal{C}}(M). \tag{3.9}$$

This is clearly a 3-manifold invariant (it is a product of 3-manifold invariants) and besides it is easy to see that the equalities (3.8) indeed hold.

We want to wrap up this section by giving a more neat formula for the normalised invariant. Write $d : I \longrightarrow \Bbbk$ for the weight on the set of simple objects given by the dimension, $d(i) := d_i$.

Proposition 3.12. *Let M be a closed, connected 3-manifold M obtained from surgery along a framed link L with m components, and let $\sigma = \sigma(L)$. Then we have*

$$\boxed{WRT^{norm}_{\mathcal{C}}(M) = \Delta^{\sigma}\mathcal{D}^{-m-\sigma-1} \sum_{\lambda\in \mathrm{Col}(L)} wRT_{\mathcal{C}}(L,\lambda,d).}$$

Before proving this formula we need a small non-obvious preliminary lemma that connects the signature of W_L with the first homology group of ∂W_L.

Lemma 3.13. *Let M be a closed, connected 3-manifold M obtained from surgery along a framed, oriented link L. Then we have*

$$\sigma_-(L) = \frac{\#L - b_1(M) - \sigma(L)}{2}.$$

Proof. The key point is that the linking matrix A of L is precisely a presentation matrix for $H_1(M;\mathbb{R})$ [GS99, Corollary 5.3.12]. That is, $H_1(M;\mathbb{R}) \cong \mathrm{coker}(A)$ where we view $A : \mathbb{R}^m \longrightarrow \mathbb{R}^m$, here $m = \#L$. In particular, we can decompose $\mathbb{R}^m$ as

$$\mathbb{R}^m \cong V_+ \oplus V_- \oplus \ker A \tag{3.10}$$

where $V_\pm$ denotes the positive/negative definite subspace, that is the subspace spanned by eigenvectors with positive/negative eigenvalue (we insist that A is diagonalisable as it is a symmetric matrix). Since

$$b_1(M) = \dim H_1(M;\mathbb{R}) = \dim \operatorname{coker}(A) = \dim \ker(A),$$

we obtain directly from (3.10) that

$$m = \sigma_+(L) + \sigma_-(L) + b_1(M).$$

Replacing $\sigma_+(L)$ by $\sigma(L) + \sigma_-(L)$ we obtain the above formula. □

Proof of Theorem 3.12. Let us put $b = b_1(M)$, $\sigma_\pm = \sigma_\pm(L)$, $\sigma = \sigma(L)$. We simply compute

$$\begin{aligned}
WRT^{\mathrm{norm}}(M) &= \Delta^{-b}\mathcal{D}^{b-1}WRT(M)\\
&= \Delta^{-b}\mathcal{D}^{b-1}(\mathcal{D}^{-2}\Delta^2)^{-\sigma_-(L)} \sum_{\lambda\in\mathrm{Col}(L)} wRT(L,\lambda,\delta)\\
&= \Delta^{-b}\mathcal{D}^{b-1}(\mathcal{D}^{-1}\Delta)^{-m+b+\sigma} \sum_{\lambda\in\mathrm{Col}(L)} \left(\prod \delta_{\lambda(i)}\right) RT(L,\lambda)\\
&= \Delta^{-m+\sigma}\mathcal{D}^{m-1-\sigma}(\mathcal{D}^{-2}\Delta)^{m} \sum_{\lambda\in\mathrm{Col}(L)} \left(\prod d_{\lambda(i)}\right) RT(L,\lambda)\\
&= \Delta^{\sigma}\mathcal{D}^{-m-1-\sigma} \sum_{\lambda\in\mathrm{Col}(L)} wRT(L,\lambda,d)
\end{aligned}$$

as claimed. □

4. Example: a modular fusion category out of the quantum group of $\mathfrak{sl}_2$

4.1. **The small quantum group $u_q(\mathfrak{sl}_2)$.** Let us take $r \geq 3$ an odd integer and let $q = e^{2\pi i/r}$, a primitive r-th root of unity. The quantum group $u_q(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra generated by K, E, F with relations

$$E^r = 0 = F^r \quad , \quad K^r = 1$$

$$KE = q^2EK \quad , \quad KF = q^{-2}FK \quad , \quad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}$$

Note: the relation $K^r = 1$ means that K is invertible with $K^{-1} = K^{r-1}$, and $r \geq 3$ implies that $q - q^{-1}$ is non-zero.

Given a finite dimensional algebra A, there is a rough dictionary that translates between properties/structures on A and properties/structures on the category A-mod of finite-dimensional A-modules:

A	A-mod
semisimple	semisimple
bialgebra	monoidal (with $\otimes_{\Bbbk}$)
antipode	rigid
quasi-triangular	braided
factorisable	factorisable
pivotal	pivotal
ribbon	ribbon
factorisable ribbon	modular tensor category
semisimple fact. ribbon	modular fusion category

In the previous talks we have seen that $u_q(\mathfrak{sl}_2)$ is a Hopf algebra, with structure maps

$$\Delta(K) = K\otimes K \qquad \varepsilon(K) = 1 \qquad S(K) = K^{-1}$$

$$\Delta(E) = 1\otimes E + E\otimes K \qquad \varepsilon(E) = 0 \qquad S(E) = -EK^{-1}$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F \qquad \varepsilon(F) = 0 \qquad S(F) = -KF$$

and it is in fact a quasi-triangular Hopf algebra with universal R-matrix

$$R = \frac{1}{r} \sum_{a,b,c=0}^{r-1} \frac{(q-q^{-1})^a}{[a]!} q^{a(a-1)/2-2bc} K^b E^a \otimes K^c F^a.$$

Furthermore, it is a ribbon Hopf algebra, with pivot element (ie a group like element that implements S^2 by conjugation) given by $\omega := K$. It is possible to give a formula for the ribbon element, but it is much uglier, see [BDR24, Lemma B.2].

By a highly non-trivial result of Lybashenko, $u_q(\mathfrak{sl}_2)$ is factorisable when r is odd, which we assumed from the start. That is, $u_q(\mathfrak{sl}_2)$-mod is a (non-semisimple!) modular tensor category.

However, $u_q(\mathfrak{sl}_2)$ is **NOT** a semisimple algebra. Indeed a classical theorem of Larson and Radford states that over a field of characteristic zero, a Hopf algebra is semisimple if and only if its antipode squares to the identity. However we have

$$S^2(F) = S(-KF) = q^2 F.$$

It is true that $u_q(\mathfrak{sl}_2)$-mod has finitely many simple objects, but there are indecomposable representations that are not irreducible. Therefore, in order to construct a modular fusion category out of $u_q(\mathfrak{sl}_2)$, we cannot simply take $u_q(\mathfrak{sl}_2)$-mod.

4.2. **Naive fix: semisimplification.** There is a construction in tensor categories called "semisimplification", which as spoiled by the name takes certain category into a semisimple one.

The story goes as follows: let $\mathcal{C}$ be a $\Bbbk$-linear abelian rigid monoidal category equipped with a spherical pivotal structure, that is left and right traces of morphisms coincide, so I will just denote them by Tr. Say that a morphism $f : X \longrightarrow Y$ is *negligible* if $Tr(f \circ f) = 0$ for all $g : Y \longrightarrow X$. It is easy to see that if we set $\mathcal{N}(X,Y)$ to be the set of negligible morphisms from X to Y, this makes $\mathcal{N}$ a tensor ideal.

The semisimplification of $\mathcal{C}$ is the quotient category $\mathcal{C}^{ss} := \mathcal{C}/\mathcal{N}$, and one can easily see that this is again a $\Bbbk$-linear abelian rigid monoidal category equipped with a spherical pivotal structure. Furthermore $\mathcal{C}^{ss}$ has the property that if an object $X \in \mathcal{C}$ is negligible (meaning: Id_X is a negligible morphism), then X becomes the zero object in $\mathcal{C}^{ss}$ (indeed if the identity becomes the zero map this forces X to be zero). However the main change is that $\mathcal{C}^{ss}$ becomes semisiple, with simple objects being in bijection with indecomposable objects of $\mathcal{C}$ of non-zero dimension [EO22]. This is not that good news, because a finite dimensional algebra can have infinitely many indecomposable representations. This is the case for e.g. $\mathbb{C}[x,y]/(x^2,y^2)$, for which Jordan blocks will produce representations of any size; and more in particular for the small quantum group $u_q(\mathfrak{sl}_2)$. So this approach simply won't work, at least not directly.

In order to get a modular fusion category, the first thing that you have to replace is $u_q(\mathfrak{sl}_2)$ itself. We are still keeping the underlying Lie algebra $\mathfrak{sl}_2$ and a root of unity (now we will rather put $q = e^{\pi i/r}$, ie q^2 is a r-th root of unity), but we need a different version of this quantum group.

4.3. **Lusztig's divided powers quantum group.** Let v be an indeterminate and consider the quantum enveloping algebra $U_v(\mathfrak{sl}_2)$ over $\mathbb{Q}(v)$ generated by $E, F, K^{\pm 1}$ with relations

$$KK^{-1} = K^{-1}K = 1 \quad , \quad KE = v^2 EK \quad , \quad KF = v^{-2} FK \quad , \quad [E,F] = \frac{K - K^{-1}}{v - v^{-1}}$$

and with the usual Hopf algebra structure (same formulas as for the small quantum group). For $n \geq 1$, consider the divided powers

$$E^{(n)} = \frac{E^n}{[n]_v!}, \qquad F^{(n)} = \frac{F^n}{[n]_v!}.$$

where as usual

$$[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}, \qquad [n]_v! = [1]_v[2]_v \cdots [n]_v.$$

The *Lusztig integral form* is the $\mathbb{Z}[v, v^{-1}]$-subalgebra

$$U_v^L(\mathfrak{sl}_2) \subset U_v(\mathfrak{sl}_2)$$

generated by

$$K^{\pm 1}, \qquad E^{(n)}, \qquad F^{(n)} \quad (n \geq 0).$$

This is in fact a Hopf algebra over $\mathbb{Z}[v, v^{-1}]$.

Let now $q = e^{\pi i/r}$ with $r \geq 3$ odd. We can specialize $v \mapsto q$ by extension of scalars: the *Lusztig quantum group at q* is

$$U_q^L(\mathfrak{sl}_2) := U_v^L(\mathfrak{sl}_2) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}.$$

The goal is to obtain a modular fusion category out of $U_q^L(\mathfrak{sl}_2)$. The idea is to reduce to a certain class of representations that behave "well".

A $U_q^L(\mathfrak{sl}_2)$-module V is called a *weight module* if it is equipped with a weight decomposition, ie a decomposition

$$V = \bigoplus_{m \in \mathbb{Z}} V^m$$

such that

$$K\big|_{V^m} = q^m \cdot \mathrm{id}_{V^m}, \qquad E^{(n)}(V^m) \subset V^{m+2n}, \qquad F^{(n)}(V^m) \subset V^{m-2n}.$$

We write $U_q^L(\mathfrak{sl}_2)\text{-}\mathsf{mod}^{\text{weight}}$ for the category of finite-dimensional weight modules of $U_q^L(\mathfrak{sl}_2)$. This happens to be a ribbon category; but it is not yet the category we are looking for; for instance one can check that it is not semisimple.

4.4. **Weyl modules.** Now, recall e.g. [Kas95, Theorem VI.3.5] that for every natural number m (aka for every dominant weight), the standard $U_v(\mathfrak{sl}_2)$ (for generic v) has a unique simple (type 1) module W_v^m generated by a highest weight vector of weight m; such module is of dimension $m+1$. More precisely, for $p = 0, \ldots, m$ we have

$$\begin{aligned} Kw_p &= q^{m-2p}\, w_p \\ Ew_p &= [m-p+1]\, w_{p-1} \\ Fw_{p-1} &= [p]\, w_p \end{aligned}$$

where (w_i) is a basis and w_0 is the highest weight vector.

Since $U_v^L(\mathfrak{sl}_2) \subset U_v(\mathfrak{sl}_2)$, we can consider the $U_v^L(\mathfrak{sl}_2)$-submodule of W_v^m generated by w_0,

$$W_L^m := U_q^L(\mathfrak{sl}_2) \cdot w_0,$$

and then specialise it at the root of unity by extension of scalars,

$$W_q^m := W_L^m \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{C}.$$

This $U_q^L(\mathfrak{sl}_2)$-module is called the *Weyl module with dominant weight m*.

It turns out that these Weyl modules are simple for some concrete values of m:

Theorem 4.1 ([APW91])**.** *If m belongs to the principal alcove*

$$C := \{0, 1, 2, \ldots, r-2\},$$

then the Weyl module W_q^m is simple and $\dim(W_q^m) \neq 0$.

Weyl modules with their dominant weights in the principal alcove behave *almost* like a fusion category: by [AP95], for $m, n \in C$ we have

$$W_q^n \otimes W_q^m \cong \left(\bigoplus_{k \in C} N_{n,m}^k W_q^k \right) \oplus Z$$

for some module Z of (quantum) dimension zero.

The goal will be to extract a modular fusion category from $U_q^L(\mathfrak{sl}_2)\text{-}\mathsf{mod}^{\mathrm{weight}}$ that will have as simple objects the Weyl modules W_q^m with $m \in C$.

4.5. **Tilting modules.** A *Weyl filtration* for a $U_q^L(\mathfrak{sl}_2)$-module V is a sequence

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = V,$$

where the successive quotients V_{i+1}/V_i are Weyl modules W_q^m for some $m \geq 0$. If both V and its dual V^* have Weyl filtrations, V is called a *tilting module*.

For instance, one can prove that for $m \in C$ the Weyl modules W_q^m are tilting (this does not work outside the principal alcove in general).

Let us write $U_q^L(\mathfrak{sl}_2)\text{-}\mathsf{mod}^{\mathrm{tilt}}$ for the full subcategory of $U_q^L(\mathfrak{sl}_2)\text{-}\mathsf{mod}^{\mathrm{weight}}$ consisting of tilting modules. This category is quite well-behaved:

Proposition 4.2 ([AP95])**.** *The category $U_q^L(\mathfrak{sl}_2)\text{-}\mathsf{mod}^{\mathrm{tilt}}$ is closed under duals, tensor products, direct sums and direct summands.*

Moreover, there is a bijection between non-negative integers m (dominant weights) and indecomposable tilting modules T^m. If $m \in C$, then $T^m = W_q^m$ (which we already said it has non-zero dimension); and if $m \notin C$, then $\dim T^m = 0$.

Furthermore, $U_q^L(\mathfrak{sl}_2)\text{-}\mathsf{mod}^{\mathrm{tilt}}$ is a ribbon category.

At this point, we can satisfactorily use the semisimplification with which we had our naive, failed fix at first. By the above proposition, we will get finitely many simple objects this time!!!

Theorem 4.3 ([BK01])**.** *The semisimplification $(U_q^L(\mathfrak{sl}_2)\text{-}\mathsf{mod}^{\mathrm{tilt}})^{ss}$ is a modular fusion category with set of simple objects the Weyl modules W_q^m for $m \in C$. Furthermore*

$$\mathcal{D}^2 = \frac{r}{2\sin(\pi/r)^2}.$$

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