Lee homology and the Lee spectral sequence

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Handout available at bit.ly/jbecerra

Hopeless outlook

Theorem (Lee 2005, paraphrase)

As a link invariant, Lee homology is terrible: we have

$$Lee^i(K) = \begin{cases} \mathbb{Q}^2, & i = 0 \\ 0, & \text{else} \end{cases}$$

for any knot K, and for a link L the groups $Lee^i(L)$ will be fully determined by the linking matrix of L.

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BUT there is hope!

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   This is called the *Rasmussen s-invariant*, which gives

   - A concordance invariant
   $$s : \mathcal{C}^{sm} \longrightarrow 2\mathbb{Z},$$
   - A lower bound for the slice genus,
   $$|s(K)| \leq 2g_4^{sm}(K),$$
   - Knots that are topologically slice (e.g. $\Delta_K = 1$) but not smoothly slice (e.g. $s(K) \neq 0$) can be used to produce exotic $\mathbb{R}^4$'s.

Bar-Natan's construction of the Khovanov complex of a link diagram D can be split into a two-step functor

$$\{0,1\}^{\times n} \cong \mathbb{P}(\mathrm{cr}(D)) \xrightarrow{\mathrm{res}} \mathsf{Cob}_2 \xrightarrow{\mathrm{TQFT}} \mathsf{grVect}_{\mathbb{Q}}$$

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There is a correspondence

$$2d \text{ TQFTs} \xrightarrow{\simeq} \mathsf{comFrob}_{\mathbb{Q}},$$

and one takes

$$V := \mathbb{Q}[x]/(x^2) = \mathbb{Q}\cdot 1 \oplus \mathbb{Q}\cdot x$$

with 1 primitive and x grouplike. Additionally V is graded with p-degrees defined as $p\text{-}\deg(1) = 1, p\text{-}\deg(x) = -1$.

Quick review of *Kh*

For a link diagram D, let

$n_+ = \#$  , $n_- = \#$  , $n = n_+ + n_-$

and for $\alpha \in \{0,1\}^{\times n}$,

$$|\alpha| = \#\, 1\,'s \qquad , \qquad k_\alpha = \#\, \pi_0(\mathrm{res}(\alpha)) \qquad , \qquad V_\alpha := V^{\otimes k_\alpha}\{|\alpha| + n_+ - 2n_-\}.$$

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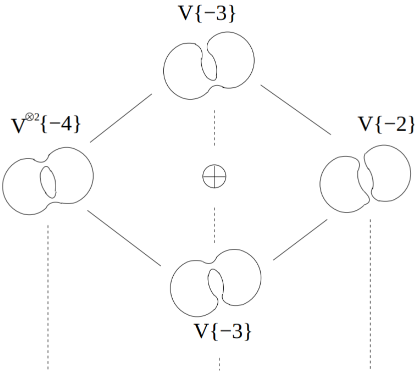
Define

$$C^i_{Kh}(D) := \bigoplus_{\substack{\alpha \\ |\alpha| = i + n_-}} V_\alpha$$

with differentials $d^i_{Kh} : C^i_{Kh}(D) \longrightarrow C^{i+1}_{Kh}(D)$ an alternated sum of tensors of (co)multiplications and id's of V. This is a cochain complex of graded vector spaces.

Quick review of Kh

Example: for $D =$  we have



$$0 \longrightarrow C^{-2}_{Kh}(D) \longrightarrow C^{-1}_{Kh}(D) \longrightarrow C^{0}_{Kh}(D) \longrightarrow 0$$

Theorem (Khovanov, Bar-Natan)

The chain homotopy type of $C^*_{Kh}(D)$ is a link invariant.

Hence the Khovanov homology groups

$$Kh^{i,j}(D) := H^i(C^{*,j}_{Kh}(D))$$

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By construction

$$\chi_q(Kh^*(D)) = \chi_q(C^*_{Kh}(D)) = \widehat{J}(D).$$

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For $c, h, t \in \mathbb{Q}$ with $c \neq 0$, let

$$A_{c,h,t} := \mathbb{Q}[x]/(x^2 - hx - t)$$

(this is 2-dimensional as a vector space) with bialgebra structure given by

$$\begin{aligned} \Delta(1) &= \tfrac{1}{c}(x \otimes 1 + 1 \otimes x - h(1 \otimes 1)) \quad &, \quad \varepsilon(1) &= 0 \\ \Delta(x) &= \tfrac{1}{c}(x \otimes x + t(1 \otimes 1)) \quad &, \quad \varepsilon(x) &= c. \end{aligned}$$

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Note: $A_{1,0,0} = V$.

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Theorem (essentially Bar-Natan, explicit by Turner)

If A is a Frobenius algebra producing a link isotopy invariant via the previous construction, then A is isomorphic to $A_{c,h,t}$ for some $c, h, t \in \mathbb{Q}$ with $c \neq 0$.

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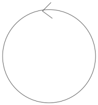
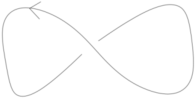
About the proof. Consider the following two diagrams of the unknot:





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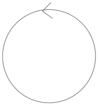

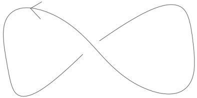

$$0 \longrightarrow \overset{0}{A} \longrightarrow 0 \qquad , \qquad 0 \longrightarrow A \longrightarrow \overset{0}{A \otimes A} \longrightarrow 0.$$

$$\dim A = -\dim A + (\dim A)^2,$$

ie $\dim A = 2$.

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About the proof (cont). So we must have $A = \mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot x$ with $x^2 = hx + t \cdot 1$ for some $h, t \in \mathbb{Q}$, ie as an algebra $A = \mathbb{Q}[x]/(x^2 - hx - t)$.

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Claim: $\varepsilon(1) = 0$. This follows from the fact that the 2-step functor described above must factor through $\mathsf{Add}(\mathbb{Z}\mathsf{Cob}_2)/S, T, 4Tu$ (this defines Bar-Natan's universal geometric theory). S is essentially this condition.

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Remaining of the proof: A 2-dimensional Frobenius algebra with $\varepsilon(1) = 0$ is isomorphic to a $A_{c,h,t}$ (in fact $c = \varepsilon(x)$). See the handout for details. $\square$

Lee homology

Set $W := A_{1,0,1}$. Explicitly,

$$W = \mathbb{Q}[x]/(x^2 - 1)$$

with

$$\begin{aligned} \Delta(1) &= x \otimes 1 + 1 \otimes x \quad , \quad \varepsilon(1) = 0 \\ \Delta(x) &= x \otimes x + 1 \otimes 1 \quad , \quad \varepsilon(x) = 1. \end{aligned}$$

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The cochain complex obtained from this Frobenius algebra W using the construction above will be denoted $C^*_{Lee}(D)$,

$$C^i_{Lee}(D) := \bigoplus_{\substack{\alpha \\ |\alpha| = i + n_-}} W_\alpha,$$

with a new differential d_{Lee}, and its cohomology groups

$$Lee^i(L) := H^i(C^*_{Lee}(D))$$

will be called the *Lee homology* groups.

Lee homology

Surprisingly, the Khovanov and Lee homologies are essentially the only link homology theories that one can produce:

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Theorem (Mackaay, Turner, Vaz 2007)

The (rational) link homology theory produced by the Frobenius algebra $A_{c,h,t}$ is isomorphic to

1. *Khovanov homology, if $h^2 + 4t = 0$,*
2. *Lee homology, if $h^2 + 4t \neq 0$.*

See the handout for a proof.

Lee degeneration

If $L = \cup_i L_i$ ordered, oriented n-component link, denote by L^{un} the underlying unoriented link.
L^{un} admits 2^n possible orientations, and the set of them will be denoted by $\mathrm{Or}(L^{un})$.

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Given $\theta \in \mathrm{Or}(L^{un})$, let $E_\theta \subset \{1, \ldots, n\}$ be the subset of components of L whose original orientation must be reversed to get the orientation θ, and write $\overline{E}_\theta := \{1, \ldots, n\} - E_\theta$.

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Theorem (Lee 2005)

Let $L = \cup_i L_i$ be an oriented n-component link in S^3. There exists a bijection between orientations $\theta \in \mathrm{Or}(L^{un})$ and a set of generators $\mathfrak{s}_\theta$ of the Lee homology of L,

$$Lee^\bullet(L) \cong \bigoplus_{\theta \in \mathrm{Or}(L^{un})} \mathbb{Q} \cdot \mathfrak{s}_\theta,$$

in particular

$$\dim Lee^\bullet(L) = 2^n.$$

Moreover, the (homological) degree of every generator is given by

$$\deg(\mathfrak{s}_\theta) = 2 \sum_{i \in E_\theta,\, j \in \overline{E}_\theta} \ell k(L_i, L_j).$$

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- For a two-component link $L = L_1 \cup L_2$ we have

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Theorem (Lee 2005)

$$Lee^{\bullet}(L) \cong \bigoplus_{\theta \in \mathrm{Or}(L^{un})} \mathbb{Q} \cdot \mathfrak{s}_\theta \qquad , \qquad \deg(\mathfrak{s}_\theta) = 2 \sum_{i \in E_\theta,\ j \in \overline{E}_\theta} \ell k(L_i, L_j).$$

About the proof. Given orientation $\theta \in \mathrm{Or}(L^{un})$, let α_θ denote the *oriented resolution* of a diagram D of L, obtained by resolving every or with $)($, equipped with the inherited orientation. $res(\alpha_\theta) = \alpha_\theta$ is a disjoint union of oriented circles on the plane.

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Divide $\pi_0(\alpha_\theta) = \pi_0^A(\alpha_\theta) \amalg \pi_0^B(\alpha_\theta)$ into two groups as follows:

- $\gamma \in \pi_0^A(\alpha_\theta)$ if either
  - it has $\circlearrowright$ orientation and separated from ∞ by an even # of circles, or
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Label $\gamma \in \pi_0^A(\alpha_\theta)$ with $a := x + 1$ and $\gamma \in \pi_0^B(\alpha_\theta)$ with $b := x - 1$. This defines a chain $s_\theta \in C_{Lee}^{|\alpha_\theta| - n_-}(D)$.

Fact of life. The s_θ are cycles, ie $d(s_\theta) = 0$, also $\mathfrak{s}_\theta := [s_\theta] \neq 0$ and they are linearly independent.

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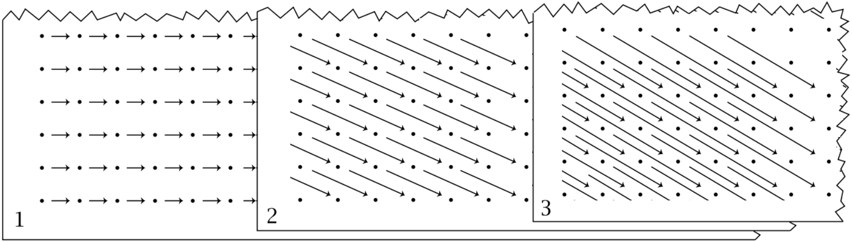
Fact of life. The s_θ are cycles, ie $d(s_\theta) = 0$, also $\mathfrak{s}_\theta := [s_\theta] \neq 0$ and they are linearly independent.
Hence so dim $Lee^{\bullet}(L) \geq 2^n$. For $\leq$ one uses the long exact sequence of Lee homology.
Degree formula: easy computation.

Quick reminder of spectral sequences



A *spectral sequence* is a collection of pairs $(E_r, d_r), r \geq 1$ where $E_r = \bigoplus_{p,q} E_r^{p,q}$ bigraded vector space, $d_r : E_r \longrightarrow E_r$ differential ($d_r^2 = 0$) of bidegree $(r, 1-r)$ and $E_{r+1} = H(E_r, d_r)$. If the E_r are bounded from below & the left, each bullet will reach a stable value $E_\infty^{p,q}$.

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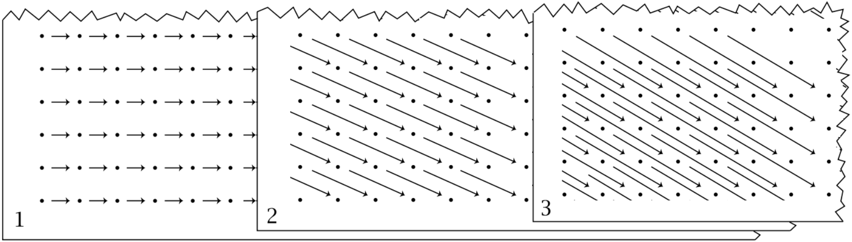

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If $N^\bullet = N^n$ is a graded vector space with each of the N^n equipped with a filtration $F^* N^n$, to say that the spectral sequence converges to $N^\bullet$ is to specify isomorphisms

$$E_\infty^{i,n-i} \xrightarrow{\cong} \mathrm{gr}^i N^n \qquad (:= F^i N^r / F^{i-1} N^n)$$

and we simply write $E_1^{p,q} \Rightarrow N^{p+q}$. Over $\mathbb{Q}$, this implies that $N^n \cong \bigoplus_{p+q=n} E_\infty^{p,q}$.

The spectral sequences of a filtered complex

Suppose C^* is a cochain complex with (descending) filtration

$$\cdots \subseteq F^n C^* \subseteq F^{n-1} C^* \subseteq F^{n-2} C^* \subseteq \cdots F^u C^* = C^*$$

(typically $u = 0$ but not necessarily here). Its *associated graded complex* consists of the cochain complexes $\mathrm{gr}^n C^* := F^n C^* / F^{n+1} C^*$.
For every n, the filtration on C^* induces a filtration on $H^n(C^*)$, namely $F^i H^n(C^*) := \mathrm{Im}(H^n(F^i C^*) \longrightarrow H^n(C^*))$. In particular we also have the associated graded $\mathrm{gr}^i H^n(C^*) := F^i H^n(C^*)/F^{i+1} H^n(C^*)$, and we can talk of the *filtration degree* of a given element in $H^n(C^*)$.

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Theorem (Leray 40s)

Let C^* be a filtered cochain complex, and suppose that for every k, the filtration $\{F^iC^k\}_i$ of C^k has finite length. Then there is a spectral sequence

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}^p C^*) \Rightarrow H^{p+q}(C^*)$$

convering to the cohomology of C^*.

Explicitly, convergence means that

$$H^n(C^*) \cong \bigoplus_{p+q=n} E_\infty^{p,q}.$$

The Lee spectral sequence

As $\mathbb{Q}$-vector spaces,

$$W = V = \mathbb{Q}1 \oplus \mathbb{Q}x, \qquad \deg(1) = 1,\ \deg(x) = -1.$$

We denote as $C^{i,j}_{Lee}$ the q-degree j part. In fact, as vector spaces, $C^{i,j}_{Lee} = C^{i,j}_{Kh} =: C^{i,j}$.

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Key observation. As a linear map, the Lee differential d_{Lee} restricts to

$$d^i_{Lee} = d^i_{Kh} + \Phi : C^{i,j} \longrightarrow C^{i+1,j} \oplus C^{i+1,j+4}$$

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$$d^i_{Lee} = d^i_{Kh} + \Phi : C^{i,j} \longrightarrow C^{i+1,j} \oplus C^{i+1,j+4}$$

This is roughly because

$$\Delta_{Lee}(x) = \underbrace{x \otimes x}_{\Delta_{Kh}(x)} + \underbrace{1 \otimes 1}_{\Phi(x)} \quad , \quad \mu_{Lee}(x \otimes x) = \underbrace{0}_{\mu_{Kh}(x \otimes x)} + \underbrace{1}_{\Phi(x \otimes x)}$$

The Lee spectral sequence

As $\mathbb{Q}$-vector spaces,

$$W = V = \mathbb{Q}1 \oplus \mathbb{Q}x, \qquad \deg(1) = 1,\ \deg(x) = -1.$$

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Upshot. $F^n C^*_{Lee} := \bigoplus_{j \geq n} C^{*,j}_{Lee}$ turns C^*_{Lee} into a filtered complex. Now the associated graded is given by

$$\mathrm{gr}^n C^* = F^n C^* / F^{n+1} C^* = C^{*,n}$$

with differential the q-degree-preserving part of d_{Lee}, that is d_{Kh}. Therefore, the cohomology groups of $\mathrm{gr}^n C^*$ are exactly the Khovanov homology groups. Boom!

Theorem (Lee spectral sequence)

For any link L, there is a spectral sequence with E_1-page given by Khovanov homology which converges to Lee homology,

$$E_1^{p,q} = Kh^{p+q,p}(L) \Rightarrow Lee^{p+q}(L).$$

Furthermore, this spectral sequence has differential $d_r = 0$ in the E_r-page unless $r \in 4\mathbb{Z}$.

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But... why should we care about this?

The Lee spectral sequence in action!

Using isotopy invariance + value of the unknot + behaviour under II + long exact sequence we can compute for the right-handed trefoil $T_{2,3}$

$$Kh^{i,j}(T_{2,3}) = \begin{cases} \mathbb{Q}, & (i,j) = (0,1),(0,3),(2,5),(3,9), \\ 0, & (i,j) = \text{elsewhere except } (2,7),(3,7), \\ ?, & (i,j) = (2,7),(3,7) \end{cases}$$

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Either

- $\longrightarrow$ is the zero map and ? $\cong \mathbb{Q}$
- $\longrightarrow$ is an isomorphism and ? $\cong 0$

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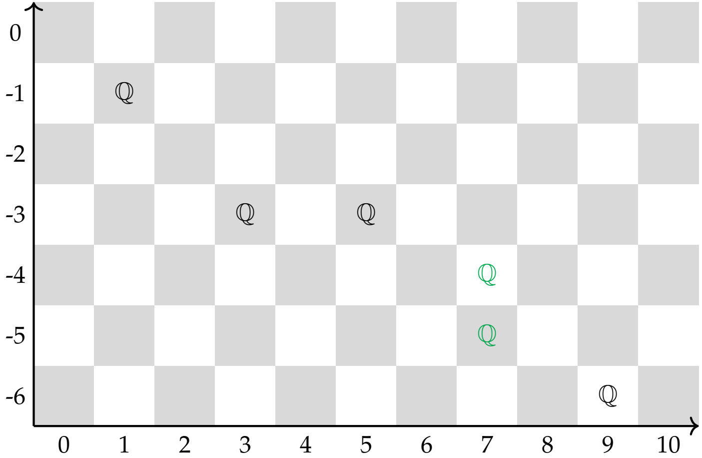
If (?) $\cong \mathbb{Q}$,



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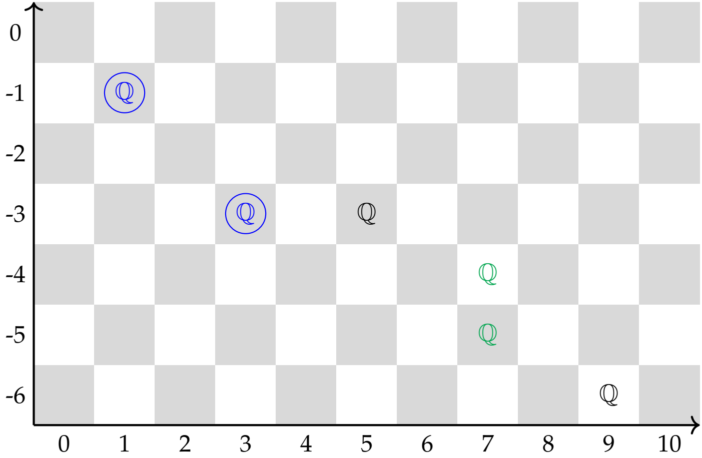

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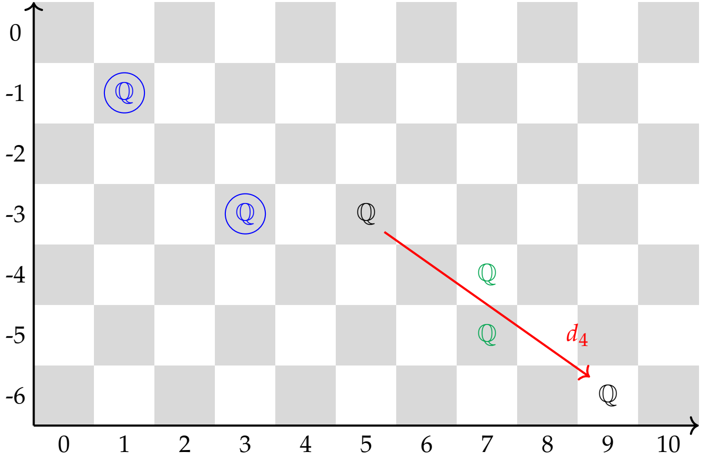

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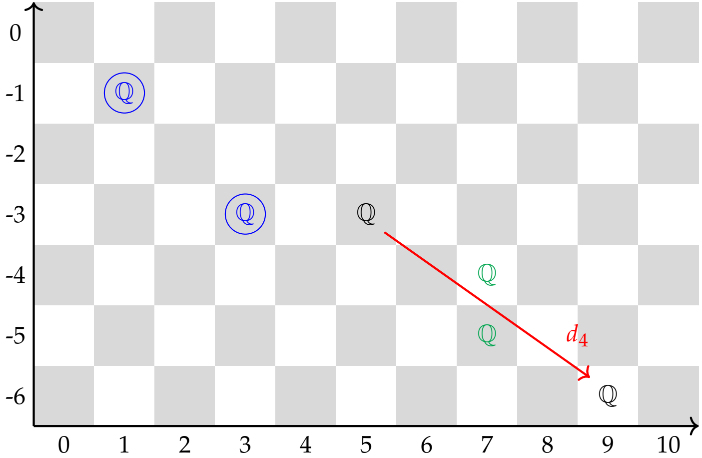

(recall $E_1^{p,q} = Kh^{p+q,p}$). Hence $?$ $\cong 0$, and $s(T_{2,3}) = 2$.