A Hopf approach to the universal tangle invariant

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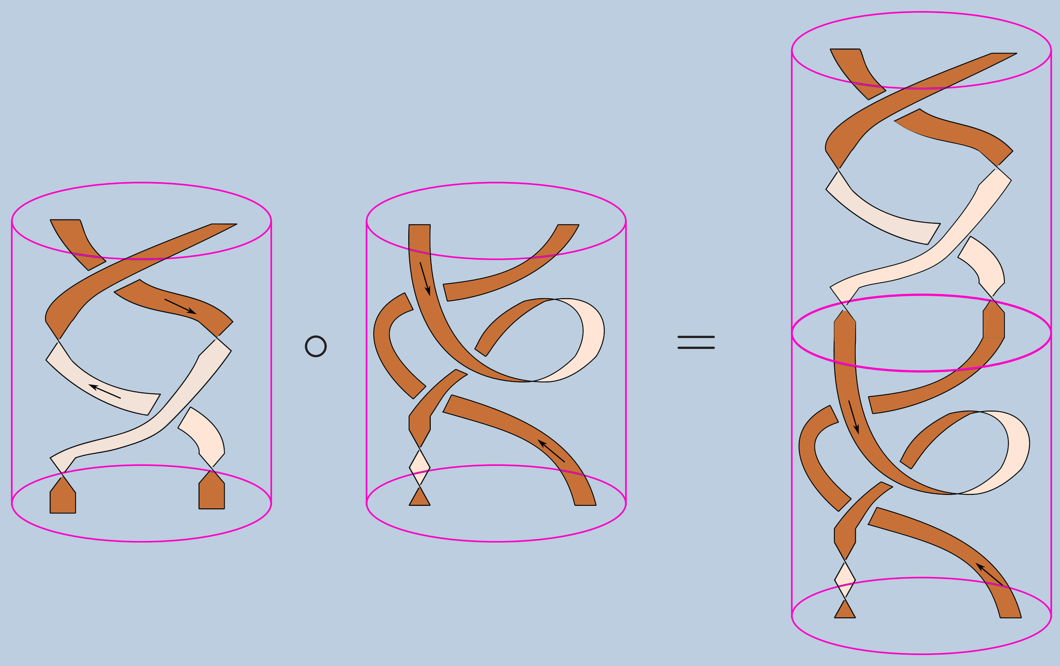
Motivation

Classically, (oriented, framed) tangles are viewed in a vertical position as a (smooth) embedding

$$T : [(\amalg I) \amalg (\amalg S^1)] \times I \hookrightarrow D^2 \times I$$

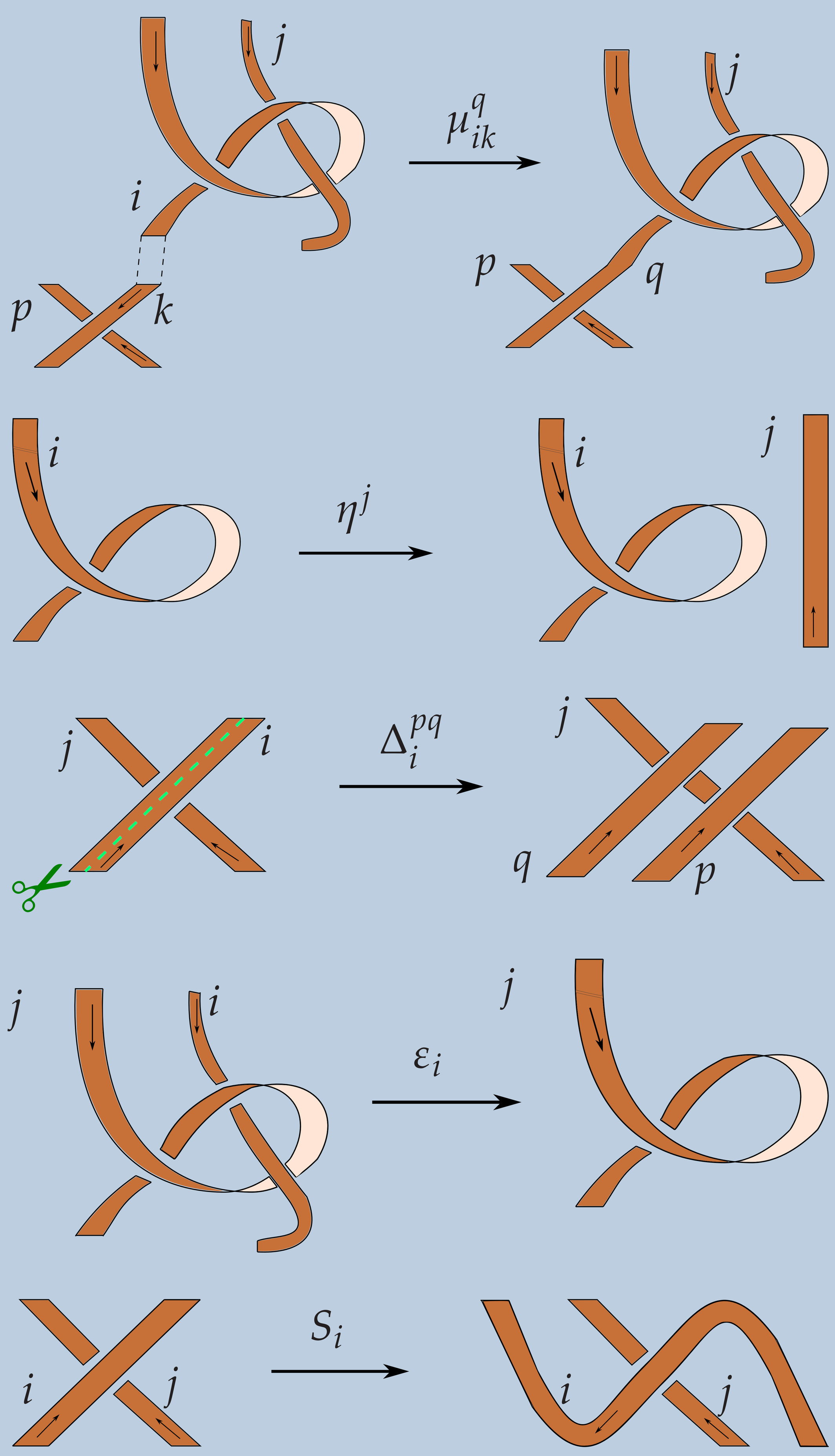
such that the image of $\partial(\amalg I) \times I$ lies in $\{0\} \times [-1,1] \times \{0,1\}$ in a orientation-preserving way and the cores of the bands are also oriented. In this way, tangles form a (ribbon) category Tangle where the composite is given by stacking:



From the representation-theoretic point of view, this realisation of tangles is powerful as it gives rise to a functorial invariant

$$F_{\mathcal{C}} : \mathsf{Tangle}_{\mathcal{C}} \to \mathcal{C}$$

where $\mathcal{C}$ is a ribbon category, typically $\mathcal{C} = \mathsf{Mod}_A$ for a ribbon Hopf algebra A (eg certain quantum groups). However, this vertical representation turns out not to be very efficient if one wishes to work directly with the ribbon Hopf algebra A, which happens to be very desirable as it gives rise to an invariant which encodes F_{Mod_A} and that is much faster to compute than the representation-theoretic one. A novel approach, computable in polynomial time, have been recently developed in [1].
Furthermore, framed tangles allow natural operations which happen to be very hard to keep track if one sticks to the previous representation:

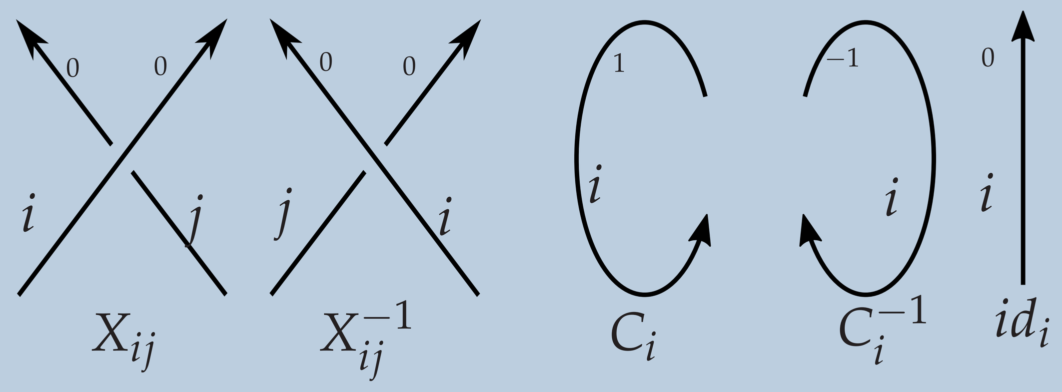


To describe tangles and the invariant we pass to a more general notion of great flexibility and simplicity.

Rotational virtual tangles

Definition 1 (van der Veen). Let J be a finite set. A *rv-tangle labelled by* J is a finite, oriented graph with only four-valent and univalent vertices. Each edge carries an element of J and integer called the *rotation number*. Vertices are cyclically ordered and pairs of opposite edges are labelled with the same element of J and are marked as the overpass or underpass. Edges labelled with the same element of J form connected oriented paths with distinct endpoints, called *strands*. The set of rv-tangles labelled by J is denoted by $\mathcal{T}_J$.

It follows that every rv-tangle can be obtained by merging copies of the following building blocks:



There is an injective realisation map

$$\left(\begin{array}{c} \text{Framed oriented} \\ \text{tangles in } D^2 \times I \\ \text{labelled with } J \end{array} \right) \hookrightarrow \frac{\mathcal{T}_J}{\text{Reidemeister moves}}$$

for a suitable notion of Reidemeister moves for rv-tangles.

The universal invariant

Let A be a ribbon Hopf algebra in Mod_k (or a symmetric monoidal category) with universal R-matrix $R_{ij} \in A^{\otimes\{i,j\}}$ and distinguished grouplike element $\kappa = uv^{-1} \in A$, where u is the Drinfeld' element and v is the ribbon element. Define Z_A on building blocks by

$$Z_A(X^{\pm 1}_{ij}) := R^{\pm 1}_{ij} \quad , \quad Z_A(C^{\pm 1}_i) := \kappa^{\pm 1}_i \quad , \quad Z_A(id_i) := 1_i$$

extended naturally by merging. Note that $Z_A(T) \in A^{\otimes J}$ for $T \in \mathcal{T}_J$.

The Hopf point of view

Let $\mathcal{H}$ be the PROP for Hopf algebra objects. PROPs can arise quite naturally, see [2]. Any Hopf algebra A defines then a lax monoidal functor $A : \mathcal{H} \to \mathsf{Set}$. Likewise, the operations described before give rise to another lax monoidal functor $\mathcal{T} : \mathcal{H} \to \mathsf{Set}$.

Theorem 2. *The universal invariant is a monoidal natural transformation*

$$Z_A : \mathcal{T} \Longrightarrow A.$$

Let $\mathsf{hSet} := \mathsf{Set}^{\mathcal{H}}$ the category of *Hopf sets*. This turns out to be related to simplicial sets [3]:

Theorem 3. *There is an embedding of categories* $\Delta \hookrightarrow \mathcal{H}$ *which induces an adjunction*

$$\mathsf{sSet} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{hSet}$$

Via this adjunction, classical constructions for simplicial sets, eg the geometric realisation, can be lifted to Hopf sets.

References

[1] Dror Bar-Natan and Roland van der Veen: *Perturbed Gaussian generating functions for universal knot invariants*, arXiv: 2109.02057 (2021)
[2] Jorge Becerra: *A combinatorial PROP for bialgebras*, arXiv:2106.13107 (2021)
[3] Jorge Becerra: *On the naturality of the universal invariant*, in preparation.