Lee homology and the Lee spectral sequence

(Rough notes - Use at your own risk!)

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These are notes prepared for a talk in the Matemale school on *Khovanov homology and exotic phenomena*, held on 19–23 May 2025. I will explain Lee's "degenerate" version of Khovanov homology[1], which was first introduced in [Lee05]. Unlike the usual Khovanov homology groups $Kh^{i,j}(L)$, which are bigraded, the Lee homology groups $Lee^i(L)$ will be single-graded. It will turn out that Lee homology is a terrible link invariant: for any knot K we will have

$$Lee^i(K) = \begin{cases} \mathbb{Q}^2, & i = 0 \\ 0, & \text{else} \end{cases}$$

(so it is as bad as the homology groups of the knot complement, it does not distinguish any pair of knots at all!), and for a link $L = \cup_i L_i$ the groups $Lee^i(L)$ will be fully determined by the linking matrix of L. But keep your hopes up: by the end of the talk I will try to convince you why the h*ck this might be useful. Sneak peek (probably already mentioned by Sardor):

1. There is a spectral sequence with E_1-page the Khovanov homology groups converging to Lee homology. Good news: in almost cases known, this sequence collapses at the first page where there are non-trivial differentials[2].

2. In the Lee spectral sequence of a knot K, there is an even integer $s(K)$ such that the two surviving generators have filtration degrees $s(K) \pm 1$. This is the celebrated *Rasmussen s-invariant*. Alexis will tell us that this gives rise to a group homomorphism

   $$s : \mathcal{C}^{\mathrm{sm}} \longrightarrow 2\mathbb{Z}$$

   from the smooth knot concordance group to the even integers, and also that magically this value (produced combinatorially) gives a lower bound for the smooth slice genus of a knot,

   $$|s(K)| \leq 2g_4^{\mathrm{sm}}(K).$$

   Later on Edwin will make use of this inequality to produce exotic structures on $\mathbb{R}^4$ from knots that are topologically slice (e.g. when $\Delta_K = 1$) but not smoothly slice (e.g. when $s(K) \neq 0$).

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1 The two-step Khovanov construction

Bar-Natan's influential construction of Khovanov complex of a link diagram D can be split into a two-step functor

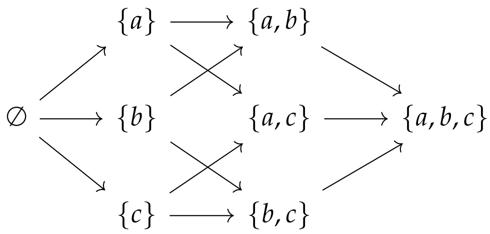
$$\mathbb{P}(\mathsf{cr}(D)) \longrightarrow \mathsf{Cob}_2 \xrightarrow{\mathrm{TQFT}} \mathsf{grVect}_{\mathbb{Q}} \tag{1}$$

that we will now explain.

First recall that if S is a set, its *Boolean lattice* is the poset $\mathbb{P}(S)$ of subsets of S ordered by inclusion, e.g. if $S = \{a, b, c\}$ then

[1] This should have been called Khovanov **co**homology

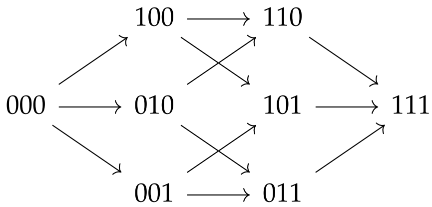
[2] In fact it was believed for some time that this was always the case; the first counterexample was found by Manolescu and Marengon in 2018 [MM20]. They also disproved the so-called Garoufalidis–Khovanov–Bar-Natan's *Knight move conjecture*.



If S is additionally ordered, there is a preferred poset isomorphism

$$\mathbb{P}(S) \xrightarrow{\cong} \{0,1\}^{\#S}$$

so that any subset of S gets identified with the sequence of 0's and 1's of length $\#S$ that has 1 (resp. 0) in the i-th position if the i-th element of $\mathbb{P}(S)$ is (resp. is not) contained in the subset. So if $S = \{a \leq b \leq c\}$ the previous diagram is rewritten as



Also, any poset $(\mathbb{P}, \leq)$ can be viewed as a category with objects the elements of $\mathbb{P}$ and a unique arrow $x \longrightarrow y$ if $x \leq y$.

Now, if D is an (oriented) link diagram, let us write $\mathrm{cr}(D)$ for the set of its crossings, and choose an arbitrary order on it. Replacing every crossing by its 0-resolution or its 1-resolution according to each of the patterns gives rise to a functor

$$\mathbb{P}(\mathrm{cr}(D)) \longrightarrow \mathsf{Cob}_2, \tag{2}$$

using the saddle cobordism between two resolutions that differ in the same position.

The second step is to apply a (rational) 2d TQFT. It is well-known that there is an equivalence of groupoids

$$2d \text{ TQFTs} \xrightarrow{\simeq} \mathsf{comFrob}_{\mathbb{Q}}$$

between 2d TQFTs and (co)commutative[3] Frobenius algebras (if you want a less fancy language, a bijection between isomorphism classes of the two). In the previous talks we have taken a very particular TQFT: the one corresponding to the Frobenius algebra $V := \mathbb{Q}[x]/(x^2)$ with coalgebra structure determined by[4]

$$\begin{array}{lcl} \Delta(1) = x \otimes 1 + 1 \otimes x & , & \varepsilon(1) = 0 \\ \Delta(x) = x \otimes x & , & \varepsilon(x) = 1. \end{array}$$

Additionally, we considered $V = \mathbb{Q}1 \oplus \mathbb{Q}x$ as a graded vector space with $\deg(1) = 1$ and $\deg(x) = -1$ (warning: this is NOT a graded algebra), we will call this the p-degree. Then the image of the combinatorial diagram of 0's and 1's under (1) is a diagram of graded vector spaces.

For an (oriented) link diagram D, we write n for the number of crossings and n_+ (resp. n_-) for the number of positive (resp. negative) crossings. For each of the resolutions $\alpha \in \{0,1\}^n$, we write $|\alpha|$ for the number of 1's and k_α for the number of circle components in the 1-manifold resulting from applying (2) to α. If for every $\alpha \in \{0,1\}^n$ we set

$$V_\alpha := V^{\otimes k_\alpha}\{|\alpha| + n_+ - 2n_-\},$$

[3] A Frobenius algebra is commutative if and only if it is cocommutative [Koc04, 2.3.29].

[4] For the algebraic topology-minded reader: if M is a oriented closed manifold, then $H^\bullet(M, \mathbb{Q})$ is a Frobinius algebra; and this one is precisely $H^\bullet(\mathbb{CP}^1, \mathbb{Q})$.

(curly brackets denote a shift in the p-degree), then the Khovanov cochain complex is given by

$$C^i_{Kh}(D) := \bigoplus_{\substack{\alpha \\ |\alpha| = i + n_-}} V_\alpha \tag{3}$$

(this is a cochain complex of graded vector spaces), with differentials $d^i_{Kh} : C^i_{Kh}(D) \longrightarrow C^{i+1}_{Kh}(D)$ given by an alternated sum of linear maps combination of multiplications and comultiplications. The unnormalised Jones polynomial can be then recovered taking its graded Eurler characteristic,

$$\chi_q(C^*_{Kh}(D)) = \widehat{J}(D).$$

We have not wasted our time: we get a genuine link invariant.

Theorem 1.1 (proven by Victor)**.** *If D, D' are two link diagrams that differ by one of the Reidemeister moves, then $C^*_{Kh}(D)$ and $C^*_{Kh}(D')$ are chain homotopy equivalent.*

In particular, the chain homotopy type of $C^*_{Kh}(D)$ is a link isotopy invariant.

For a link L with a diagram D we will write $Kh^i(L) := H^i(C^*_{Kh}(D))$, its i-th *Khovanov homology group*. In fact Khovanov homology is finer than the Jones polynomial: there are pairs of knots with the same Jones polynomial but with different Khovanov homology groups.

If $C^{i,j}_{Kh}$ denotes the degree j part of C^i_{Kh}, then i is called the *homological degree* and j the *q-degree*.

Remark 1.2. If $v \in C^{i,j}_{Kh}$ is an homogeneous element, then

$$\begin{cases} i = |\alpha| - n_- \\ j = p(v) + i + n_+ - n_- \end{cases}$$

(here $p(v)$ denotes the p-degree in $V^{\otimes k_\alpha}$).

It turns out that the differential preserves the q-degree, $q(d_{Kh}(v)) = q(v)$ for a homogeneous element v, that is

$$d^i_{Kh} : C^{i,j}_{Kh} \longrightarrow C^{i+1,j}_{Kh}. \tag{4}$$

This is a consequence of the multiplication and comultiplication map satisfying this property. Therefore, the q-degree descends to the cohomology $Kh^i(D) = H^i(C^*_{Kh}(D))$ of the cochain complex $C^*_{Kh}(D)$, and the q-degree j part is denoted by $Kh^{i,j}(D)$, that is $Kh^i(D) = \oplus_j Kh^{i,j}(D)$. Alternatively, $Kh^{i,j}(D) = H^i(C^{*,j}_{Kh}(D))$.

A powerful computational tool of Khovanov homology is the existence of long exact sequences: given a link diagram D, let $\nwarrow\!\!\!\!\nearrow$ be one of its positive crossings. Then its 0-resolution $\rangle\langle$ inherits a canonical orientation. Its 1-resolution $\asymp$ does not, so choose an arbitrary orientation for it. If

$$c := n_-(\asymp) - n_-(\nwarrow\!\!\!\!\nearrow),$$

there is a canonical split short exact sequence of cochain complexes

$$0 \longrightarrow C^{*-c-1,j-3c-2}_{Kh}(\asymp) \longrightarrow C^{*,j}_{Kh}(\nwarrow\!\!\!\!\nearrow) \longrightarrow C^{*,j-1}_{Kh}(\rangle\langle) \longrightarrow 0 \tag{5}$$

which induces induces a long exact sequence

$$\cdots \longrightarrow Kh^{i-c-1,j-3c-2}(\asymp) \longrightarrow Kh^{i,j}(\nwarrow\!\!\!\!\nearrow) \longrightarrow Kh^{i,j-1}(\rangle\langle) \longrightarrow$$
$$\longrightarrow Kh^{i-c,j-3c-2}(\asymp) \longrightarrow Kh^{i+1,j}(\nwarrow\!\!\!\!\nearrow) \longrightarrow \cdots$$

There is a similar one when the crossing is negative, see e.g. [Tur17] for details.

2 Changing the TQFT

For the usual construction recalled in the previous section, we have taken a very particular Frobenius algebra (=2d TQFT) that indeed gave rise to a link invariant via this construction. There is however a natural question to ask here:

Question 2.1. *For what other Frobenius algebras does this construction produce a link isotopy invariant?*

For $c, h, t \in \mathbb{Q}$ with $c \neq 0$, let

$$A_{c,h,t} := \mathbb{Q}[x]/(x^2 - hx - t) \tag{6}$$

(this is 2-dimensional as a vector space) with bialgebra structure given by

$$\begin{aligned} \Delta(1) &= \tfrac{1}{c}(x \otimes 1 + 1 \otimes x - h(1 \otimes 1)) \quad , \quad \varepsilon(1) = 0 \\ \Delta(x) &= \tfrac{1}{c}(x \otimes x + t(1 \otimes 1)) \quad , \quad \varepsilon(x) = c. \end{aligned}$$

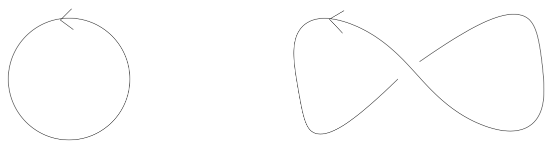
It is routine to check that this indeed defines the structure of a Frobenius algebra on $A_{c,h,t}$.

I roughly learned the following theorem from Turner [Tur17, §4], but this appears somewhat implicit in Bar-Natan's geometric approach to Khovanov homology [BN05].

Theorem 2.2. *If A is a Frobenius algebra producing a link isotopy invariant via the previous construction, then A is isomorphic to $A_{c,h,t}$ for some $c, h, t \in \mathbb{Q}$ with $c \neq 0$.*

Note that $A_{1,0,0}$ is precisely the Frobenius algebra V used in the previous section.

Proof. Let us start showing that $\dim A = 2$. Consider the following two diagrams of the unknot:



These two diagrams produce the following cochain complexes:

$$0 \longrightarrow A \longrightarrow 0 \quad , \quad 0 \longrightarrow A \longrightarrow A \otimes A \longrightarrow 0.$$

In the first complex, the copy of A sits in homological degree 0 whereas in the second sits in homological degree -1 (because the crossing is negative). By hypothesis, these two cochain complexes must be chain homotopy equivalent; in particular they must have the same Euler characteristic:

$$\dim A = -\dim A + (\dim A)^2,$$

ie $\dim A = 2$ (it cannot be $\dim A = 0$ as A must be an algebra, ie there must be a unit).

Since 1 must be one of the generators of A, let us call the other x, so $A = \mathbb{Q}1 \oplus \mathbb{Q}x$ as a vector space. For the multiplicative structure, we must have $1 \cdot 1 = 1$ and $x \cdot 1 = x = 1 \cdot x$ as 1 is the unit; and besides $x^2 = hx + t1$ for some $h, t \in \mathbb{Q}$ by dimension reasons. Ie as an algebra, $A = \mathbb{Q}[x]/(x^2 - hx - t)$.

Next let us see that $\varepsilon(1) = 0$. The need to impose this equality becomes apparent when examining the Reidemeister 2 moves, but it is best explained by Bar-Natan's (universal) geometric theory. Essentially, if one consider the additive closure $\mathsf{Add}(\mathbb{Z}\mathsf{Cob}_2)$, and one mods out the "local relations" $S, T, 4Tu$, then one can defer the application of a TQFT and still get a chain complex whose chain homotopy type is a link invariant[5]. The S relation states that any time a cobordism contains a copy of a sphere (formed by gluing two hemispheres), the entire cobordism is zero. This corresponds exactly to the equation $\varepsilon(1) = \varepsilon \circ \eta = 0$.

The rest of the proof is a standard computation that we include for completeness. Let $c := \varepsilon(x)$. Since A is cocommutative, we can write its action on the basis elements as

$$\begin{aligned} \Delta(1) &= \lambda_1(1 \otimes 1) + \lambda_2(1 \otimes x + x \otimes 1) + \lambda_3(x \otimes x) \\ \Delta(x) &= \mu_1(1 \otimes 1) + \mu_2(1 \otimes x + x \otimes 1) + \mu_3(x \otimes x) \end{aligned}$$

[5] This is [BN05, Theorem 1] stating that the isomorphism class of given object in the onomatopoeia category $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{C}ob^3_{/l}))$ is an invariant of the tangle.

for some coefficients $\lambda_i, \mu_i \in \mathbb{Q}$. The counit axiom $(\varepsilon \otimes \mathrm{id})(\Delta(1)) = 1$ implies that $\lambda_2 = 1/c$ and $\lambda_3 = 0$; whereas from $(\varepsilon \otimes \mathrm{id})(\Delta(x)) = x$ we get $\mu_2 = 0$ and $\mu_3 = 1/c$.

On the other hand, the Frobenius condition $\Delta \circ m = (\mathrm{id} \otimes m) \circ (\Delta \otimes \mathrm{id})$ applied to $1 \otimes x$ yields $\lambda_1 = -h/c$ and $\mu_1 = t/c$, as promised. □

Apart from $A_{1,0,0} = V$, there is another Frobenius algebra we would like to hightlight, and that is $A_{1,0,1} =: W$. Explicitly,

$$W = \mathbb{Q}[x]/(x^2 - 1) \tag{7}$$

with

$$\begin{aligned} \Delta(1) &= x \otimes 1 + 1 \otimes x \quad , \quad \varepsilon(1) = 0 \\ \Delta(x) &= x \otimes x + 1 \otimes 1 \quad , \quad \varepsilon(x) = 1. \end{aligned}$$

The cochain complex obtained from this Frobenius algebra W using the construction above will be denoted $C^*_{Lee}(D)$,

$$C^i_{Lee}(D) := \bigoplus_{\substack{\alpha \\ |\alpha| = i + n_-}} W_\alpha, \tag{8}$$

with a new differential d_{Lee}, and its cohomology groups $Lee^i(L) := H^i(C^*_{Lee}(D))$ will be called the *Lee homology* groups.

Surprisingly, the Khovanov and Lee homologies are essentially the only link homology theories that one can produce:

Proposition 2.3 ([MTV07]). *The (rational) link homology theory produced by the Frobenius algebra $A_{c,h,t}$ is isomorphic to*

1. *Khovanov homology, if $h^2 + 4t = 0$,*
2. *Lee homology, if $h^2 + 4t \neq 0$.*

Proof. We first claim that the homology theories produced by the Frobenius algebras $A_{c,h,t}$ and $A_{1,h,t}$ are isomorphic for all $c \neq 0$. To see this, note that these two Frobenius algebras have the same multiplicative structure and the coalgebra maps $(\Delta_c, \varepsilon_c)$ and $(\Delta_1, \varepsilon_1)$ of $A_{c,h,t}$ and $A_{1,h,t}$ respectively are related by

$$\Delta_c = \tfrac{1}{c}\Delta_1 \quad , \quad \varepsilon_c = c\varepsilon_1.$$

Now for any link diagram D with n crossings the corresponding n-cubes (the images of $\mathbb{P}(\mathrm{cr}(D))$ under (1)) are isomorphic, as the edges only differ by some scalar multiplication. Therefore the cochain complexes will also be isomorphic. This concludes the claim.

We can therefore focus on $A_{h,t} := A_{1,h,t}$. Let us look at the discriminant $h^2 + 4t$ of $x^2 - hx - t$. If $h^2 + 4t = 0$, then $x^2 - hx - t = (x - h/2)^2$ and obviously there is a Frobenius algebra isomorphism

$$A_{0,0} \longrightarrow A_{h,t} \quad , \quad x \mapsto x - h/2.$$

Now assume $h^2 + 4t \neq 0$, and let $\lambda := 2/\sqrt{h^2 + 4t} \in \overline{\mathbb{Q}}$. Let $A'_{h,t} = A_{h,t} \otimes \overline{\mathbb{Q}}$ as an algebra with coalgebra structure

$$\Delta' = \lambda\Delta \quad , \quad \varepsilon' = \lambda^{-1}\varepsilon.$$

By the argument above, $A'_{h,t}$ and $A_{h,t} \otimes \overline{\mathbb{Q}}$ produce isomorphic homology theories. Now by direct computation one can check that

$$W \otimes \overline{\mathbb{Q}} = A_{0,1} \otimes \overline{\mathbb{Q}} \longrightarrow A'_{h,t} \quad , \quad x \mapsto \lambda(x - h/2)$$

is an isomorphism of Frobenius algebras. We conclude that the homology theories induced by $W \otimes \overline{\mathbb{Q}}$ and by $A_{h,t} \otimes \overline{\mathbb{Q}}$ are isomorphic, and so are the ones induced by W and $A_{h,t}$ as desired. □

We have then obtained the answer to Question 2.1: over the rationals a homology theory constructed as in Section 1 is isomorphic to either Khovanov homology or Lee homology.

3 Lee degeneration, aka "this seems hopeless..."

The goal of this section is to prove that Lee homology is a terrible link invariant, being fully determined by the linking numbers between their components.

Before stating the theorem we need to introduce some notation. If $L = \cup_i L_i$ is an ordered, oriented n-component link in S^3, let us denote by L^{un} the underlying unoriented link. Any unoriented n-component link admits 2^n possible orientations, and the set of them will be denoted by $\mathrm{Or}(L^{un})$.

Given an orientation $\theta \in \mathrm{Or}(L^{un})$, let $E_\theta \subset \{1, \dots, n\}$ be the subset of indices of the components of L whose original orientation must be reversed to get the orientation θ, and write $\overline{E}_\theta := \{1, \dots, n\} - E_\theta$.

Theorem 3.1 ([Lee05]). *Let $L = \cup_i L_i$ be an oriented n-component link in S^3. There exists a bijection between orientations $\theta \in \mathrm{Or}(L^{un})$ and a set of generators $\mathfrak{s}_\theta$ of the Lee homology of L,*

$$Lee^\bullet(L) \cong \bigoplus_{\theta \in \mathrm{Or}(L^{un})} \mathbb{Q} \cdot \mathfrak{s}_\theta,$$

in particular

$$\dim Lee^\bullet(L) = 2^n.$$

Moreover, the (homological) degree of every generator is given by

$$\deg(\mathfrak{s}_\theta) = 2 \sum_{i \in E_\theta,\, j \in \overline{E}_\theta} \ell k(L_i, L_j).$$

Note that in particular $Lee^i(L) \cong 0$ whenever i is odd. It is worth noting some special cases. For any oriented knot K, we have

$$Lee^p(K) = \begin{cases} \mathbb{Q}^2, & p = 0, \\ 0, & \text{else} \end{cases},$$

and for a two-component link $L = L_1 \cup L_2$ we have

$$Lee^p(L) = \begin{cases} \mathbb{Q}^2, & p = 0, 2 \cdot \ell k(L_1, L_2), \\ 0, & \text{else} \end{cases}.$$

In general, according to the theorem, we have that

$$\dim Lee^p(L) = \#\{\theta \in \mathrm{Or}(L^{un}) : p = 2 \sum_{i \in E_\theta,\, j \in \overline{E}_\theta} \ell k(L_i, L_j)\}.$$

Proof of Theorem 3.1. Let us first define the classes $\mathfrak{s}_\theta$ for a given orientation $\theta \in \mathrm{Or}(L^{un})$. Let α_θ denote the oriented resolution of a diagram D of L, which is obtained by resolving every positive $\overcrossing$ or negative crossing $\undercrossing$ with $\smoothing$, equipped with the inherited orientation. After applying (2), we have that $res(\alpha_\theta) = \alpha_\theta$ is a disjoint union of oriented circles on the plane. We are going to divide $\pi_0(\alpha_\theta)$ into two groups: a circle belongs to Group A (resp. Group B) if it has the counter-clockwise orientation and is separated from infinity by an even (resp. odd) number of circles or if it has the clockwise orientation and is separated from infinity by an odd (resp. even) number of circles. Labelling the components of Group A with $a := x + 1 \in W$ and the components of Group B with $b := x - 1 \in W$ defines a chain $s_\theta \in C^{|\alpha_\theta| - n_-}_{Lee}(D)$. Note:

$$a \cdot b = (x+1)(x-1) = x^2 - 1 = 0$$

and similarly

$$b \cdot a = 0 \quad , \quad a \cdot a = 2a \quad , \quad b \cdot b = 2b,$$

and

$$\Delta(a) = a \otimes a \quad , \quad \Delta(b) = b \otimes b.$$

It is not hard to see that if two circles share a crossing, then they have different labels, cf. [Ras10, Corollary 2.5]. From this and the above relations it follows that this element is a cycle, $d_{Lee}(s_\theta) = 0$. We define $\mathfrak{s}_\theta := [s_\theta]$ to be the corresponding Lee homology class. In fact, these classes are non-zero (i.e. they are not boundaries) and they are linearly independent (we will not show these properties here), so $\dim Lee^\bullet(L) \geq 2^n$. The proof will now consist of showing that the converse inequality also holds.

The first observation is that given a (say positive) crossing in a given link diagram, just as (5) we have a short exact sequence

$$0 \longrightarrow C^{*-c-1}_{Lee}(\asymp) \longrightarrow C^{*}_{Lee}(\times) \longrightarrow C^{*}_{Lee}()() \longrightarrow 0$$

which induces a long exact sequence

$$\cdots \longrightarrow Lee^{i-1}()() \longrightarrow Lee^{i-c-1}(\asymp) \longrightarrow Lee^{i}(\times) \longrightarrow Lee^{i}()() \longrightarrow Lee^{i-c}(\asymp) \longrightarrow \cdots .$$

By exactness,

$$\dim Lee^{\bullet}(\times) \leq \dim Lee^{\bullet}()() + \dim Lee^{\bullet}(\asymp).$$

Let us now prove that $Lee^{\bullet}(L) = 2^{n}$ for any link L. We consider a diagram D of L of minimal number c of crossings (that is, c is the crossing number of L), and we argue using induction on c. It is clear that it holds for the unknot, $\dim Lee^{\bullet}(U) = 2$, and this is the base case.

We have to distinguish several cases depending on the number of components n of L:

- if $n = 1$, pick a (say, positive) crossing $\times$. Now, each of the two resolution $)($ and $\asymp$ of such a crossing has $c - 1$ crossings. Also, one of them, say $)($, is a knot and the other $\asymp$ is a two-component link. Now we claim that the connecting $\delta : Lee^{i}()() \longrightarrow Lee^{i-c}(\asymp)$ is injective. To see this, first note that out of the four possible orientations of the two-component link $\asymp$, two are compatible with the two possible orientations of D whereas other two are compatible with the two possible orientations of the knot $)($. Then, in the long exact sequence, the two generators of $Lee^{i}()()$ map to the two generators of $Lee^{i}(\asymp)$ coming from the two orientations compatible with those of $)($, which yields the claim. Hence by a linear algebra argument and by the induction hypothesis

$$\dim Lee^{\bullet}(\times) \leq \dim Lee^{\bullet}()() + \dim Lee^{\bullet}(\asymp) - 4 = 2 + 4 - 4 = 2.$$

- Now say that $n = 2$. If $D = D_1 \amalg D_2$ is a disjoint union of two knot diagrams (each of them with crossing number lower than c), then

$$Lee^{\bullet}(D) = Lee^{\bullet}(D_1) \cdot Lee^{\bullet}(D_2) = 2 \cdot 2 = 4$$

 and we are done. If D cannot be expressed as the disjoint union of two knot diagrams, then there must be at least a crossing (say positive) $\times$ between the two components. The two resolutions $)($ and $\asymp$ must be knot diagrams, each of them with fewer crossings than c, so

$$4 \leq \dim Lee^{\bullet}(\times) \leq \dim Lee^{\bullet}()() + \dim Lee^{\bullet}(\asymp) = 2 + 2 = 4.$$

The case for general n is mimicked from the former.

Lastly, let us prove the degree formula for each of the generators $\mathfrak{s}_\theta$, this is now easy. Recall that by definition $\deg(\mathfrak{s}_\theta) = |\alpha_\theta| - n_-$, and $|\alpha_\theta|$ is the number of 1-resolutions performed in the oriented resolution α_θ of a diagram D of the link equipped with the orientation θ. Now, negative (resp. positive) crossings are turned into 1- (resp. 0-)resolutions; hence $|\alpha_\theta| = n_-^{\theta}$ the number of negative crossings of D with the orientation θ.

Let us write $D = \cup D_i$ for a oriented diagram of L and $D^{\theta} = \cup D_i^{\theta}$ for the corresponding diagram of D with the orientation θ. Then

$$\begin{aligned}
\deg(\mathfrak{s}_\theta) &= n_-^{\theta} - n_- \\
&= \#\begin{pmatrix}\text{negative crossings}\\ \text{in } D^{\theta}\end{pmatrix} - \#\begin{pmatrix}\text{negative crossings}\\ \text{in } D\end{pmatrix} \\
&= \sum_{i,j \in E_\theta} \left(\#\begin{pmatrix}\text{neg cross betw}\\ D_i^{\theta} \text{ and } D_j^{\theta}\end{pmatrix} - \#\begin{pmatrix}\text{neg cross betw}\\ D_i \text{ and } D_j\end{pmatrix} \right) \\
&\quad + \sum_{i,j \in \overline{E}_\theta} \left(\#\begin{pmatrix}\text{neg cross betw}\\ D_i^{\theta} \text{ and } D_j^{\theta}\end{pmatrix} - \#\begin{pmatrix}\text{neg cross betw}\\ D_i \text{ and } D_j\end{pmatrix} \right) \\
&\quad + \sum_{i \in E_\theta, j \in \overline{E}_\theta,} \left(\#\begin{pmatrix}\text{neg cross betw}\\ D_i^{\theta} \text{ and } D_j^{\theta}\end{pmatrix} - \#\begin{pmatrix}\text{neg cross betw}\\ D_i \text{ and } D_j\end{pmatrix} \right)
\end{aligned}$$

$$= \sum_{i\in E_\theta, j\in \overline{E}_\theta,} \#\begin{pmatrix}\text{neg cross betw}\\ -D_i \text{ and } D_j\end{pmatrix} - \#\begin{pmatrix}\text{neg cross betw}\\ D_i \text{ and } D_j\end{pmatrix}$$
$$= \sum_{i\in E_\theta, j\in \overline{E}_\theta,} \#\begin{pmatrix}\text{pos cross betw}\\ D_i \text{ and } D_j\end{pmatrix} - \#\begin{pmatrix}\text{neg cross betw}\\ D_i \text{ and } D_j\end{pmatrix}$$
$$= \sum_{i\in E_\theta, j\in \overline{E}_\theta,} 2\cdot \ell k(D_i, D_j)$$

which concludes. □

Each of the generators $\mathfrak{s}_\theta$ constructed in the proof of the previous theorem are called Lee's *canonical generators*. The name is justified by the following theorem, that we state without proof for reference in a forthcoming talk:

Theorem 3.2 ([Ras10])**.** *Let (Σ, L_0, L_1) be a cobordism presented by a movie (M, D_0, D_1), and suppose that every component of Σ has a boundary component in L_0. If θ_0 and θ_1 denote the orientations in D_0 and D_1 induced by the orientation of Σ, then the induced map*

$$\phi_M : Lee^\bullet(D_0) \longrightarrow Lee^\bullet(D_1)$$

satisfies that $\phi_M(\mathfrak{s}_{\theta_0}) = \lambda \cdot \mathfrak{s}_{\theta_1}$ for some $0 \neq \lambda \in \mathbb{Q}$.

4 The Lee spectral sequence, aka "...but it is not!"

Let us pick up the fruits from Sardor's talk and describe the so-called Lee spectral sequence[6].

Here is a friendly reminder about the spectral sequence we will make use of. If C^* is a cochain complex, a (descending) filtration on C^* is a sequence of subcomplexes

$$\cdots \subseteq F^nC^* \subseteq F^{n-1}C^* \subseteq F^{n-2}C^* \subseteq \cdots F^uC^* = C^*$$

(typically $u = 0$ but not necessarily here). Its *associated graded complex* consists of the cochain complexes

$$\mathrm{gr}^n C^* := F^nC^*/F^{n+1}C^*,$$

whose differential is inherited from that of C^*. Note that for every m, the filtration on C^* induces a filtration on the vector space $H^n(C^*)$, namely

$$F^iH^n(C^*) := \mathrm{Im}(H^n(F^iC^*) \longrightarrow H^n(C^*)).$$

In particular we also have the associated graded $\mathrm{gr}^iH^n(C^*) := F^iH^n(C^*)/F^{i+1}H^n(C^*)$, and we can talk of the *filtration degree* of a given element in $H^n(C^*)$.

Filtered complexes give rise to a spectral sequence:

Theorem 4.1 (proven by Sardor)**.** *Let C^* be a filtered cochain complex, and suppose that for every k, the filtration $\{F^iC^k\}_i$ of C^k has finite length. Then there is a spectral sequence*

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}^pC^*) \Rightarrow H^{p+q}(C^*)$$

convering to the cohomology of C^*.

We use the standard bigrading stating that $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$. Recall also that convergence means that we have preferred isomorphisms

$$E_\infty^{i,n-i} \xrightarrow{\cong} \mathrm{gr}^iH^n(C^*),$$

and working over the field $\mathbb{Q}$ this implies that

$$H^n(C^*) \cong \bigoplus_{p+q=n} E_\infty^{p,q}.$$

[6] Aka the Lee-Rasmussen spectral sequence, the Khovanov-to-Lee spectral sequence, or perhaps the Lee-Miller-Thurston spectral sequence, since Lee did not mention at all spectral sequences in the first version of [Lee05], and in a subsequent version she acknowledges that Miller and Thurston pointed out that her computations are really part of a spectral sequence.

Let us see how to apply this to obtain the advertised Lee spectral sequence. Note that, as rational vector spaces,

$$W = V = \mathbb{Q}1 \oplus \mathbb{Q}x,$$

and we had set $\deg(1) = 1$ and $\deg(x) = -1$. We denote as $C^{i,j}_{Lee}$ the q-degree j part. In fact, as vector spaces, $C^{i,j}_{Lee} = C^{i,j}_{Kh}$, so I will denote it simply by $C^{i,j}$.

Key observation 4.2. As a linear map, the Lee differential d_{Lee} restricts to

$$d^i_{Lee} = d_{Kh} + \Phi : C^{i,j} \longrightarrow C^{i+1,j} \oplus C^{i+1,j+4} \tag{9}$$

(compare with (4)), in particular $q(d_{Lee}(v)) \geq q(v)$. That there is such a splitting of d_{Lee} is obvious by the definitions of the multiplication and comultiplications of V and W as Frobenius algebras; and the part about the degrees follows by inspecting the multiplication and comultiplication maps of the Frobenius algebra W that induce the Lee differential.

Fact of life: Φ is actually a cochain map, squares to zero and anticommutes with d_{Kh}.

This means that setting

$$F^n C^*_{Lee} := \bigoplus_{j \geq n} C^{*,j}_{Lee}$$

turns C^*_{Lee} into a filtered complex, with differential induced by d_{Lee}. Now the associated graded is given by

$$\mathrm{gr}^n C^* = F^n C^* / F^{n+1} C^* = C^{*,n}$$

with differential the q-degree-preserving part of d_{Lee}, that is d_{Kh}. Therefore, the cohomology groups of $\mathrm{gr}^n C^*$ are exactly the Khovanov homology groups. Boom!

It is clear that, for every k, the filtration $\{F^i C^k\}_i$ is finite as we consider at all times finite tensor copies of W in the construction of the Khovanov complex.

The consequence of Theorem 4.1 is now:

Theorem 4.3 (Lee spectral sequence)**.** *For any link L, there is a spectral sequence with E_1-page given by Khovanov homology which converges to Lee homology,*

$$E_1^{p,q} = Kh^{p+q,p}(L) \Rightarrow Lee^{p+q}(L).$$

Furthermore, this spectral sequence has differential $d_r = 0$ in the E_r-page unless $r \in 4\mathbb{Z}$.

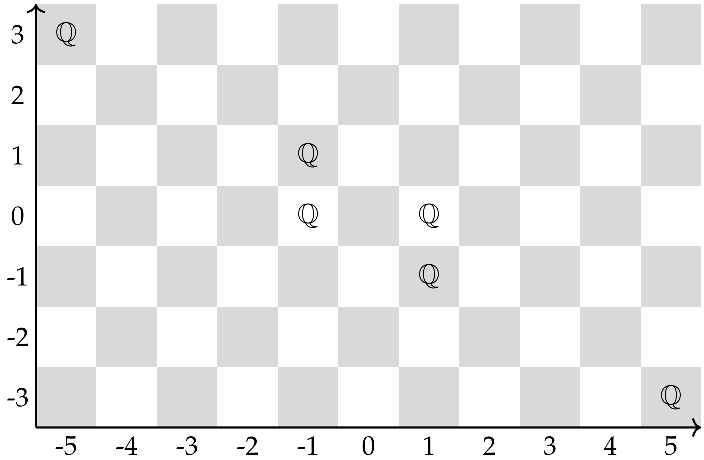
Proof. Only the last claim needs explanation. In the construction of the spectral sequence of a filtered complex C^* one has that the differential $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ is induced by the original differential; more precisely it is defined as a connecting homomorphism (essentially the original differential) followed by a projection. In terms of the Khovanov bidegree, this means that d_r raises the homological degree by 1 and the q-degree by r, and we conclude by the Key observation 4.2. □

Remark 4.4. In fact, one can even show that each of the pages of the spectral sequence is a link invariant itself.

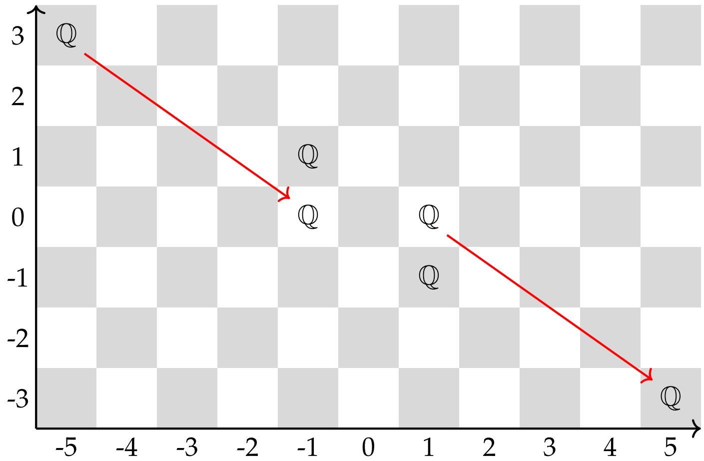
Example 4.5. Let us take 4_1 the figure-of-eight knot. One can compute that

$$Kh^{i,j}(4_1) = \begin{cases} \mathbb{Q}, & (i,j) = (2,5), (1,1), (0,1), (0,-1), (-1,-1), (-2,-5) \\ 0, & \text{else} \end{cases}$$

Therefore the E_1-page looks as follows, where $E_1^{p,q} = Kh^{p+q,p}(4_1)$:



By Theorem 4.3, the first differential that might be non-trivial is d_4, which has bidegree $(4, -3)$ (recall that $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$), so we have



and by degree reasons higher differentials cannot kill other copies. So the two red arrows must be isomorphisms, and the only copies that survive are those in spectral bidegrees $(0, -1)$ and $(0, 1)$. In particular the Rasmussen s-invariant must be 0; recall that the two surviving generators have filtration degrees $s(K) \pm 1$.

Let me explain why on Earth one would be interested in such a spectral sequence if we know the value of $Lee^{\bullet}(L)$ for any link by Theorem 3.1. Typically, one thinks of a spectral sequence as an algebraic gadget which starts with something that we know about and which converges to something that we want to know about. But there are numerous examples where one studies a spectral sequence starting with something we want to know about and converging to something we already know, and by some kind of reverse engineering we infer the values of the early page[7]. This is one such example.

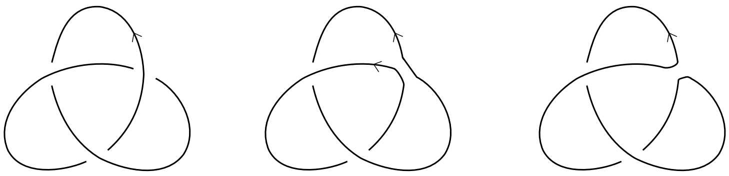
If we think for a moment of Khovanov homology as a link homology theory (in the sense of Eilenberg-Steenrod, cf. [Tur16, §1]), then one should not use the construction sketched in Section 1 to compute the values of Khovanov homology. Instead, one should use the characterising properties of Khovanov homology, namely

- isotopy invariance,
- the value at the unknot (normalisation),
- behaviour under disjoint unions,
- a computational tool, namely the (family of) long exact sequence described at the end of Section 1.

[7] For the algebraic topology-minded reader: this is just another day at the office. Via this reverse engineering it is how one applies the Serre spectral sequence to compute the cohomology of $\mathbb{CP}^n$ or ΩS^n using the fibre sequences $S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n$ and $\Omega S^n \longrightarrow PS^n \longrightarrow S^n$, respectively.

This is standard in algebraic topology: to compute the (co)homology of e.g. spheres, we do not use a combinatorial model –say, simplicial complexes–, and then carry out the cumbersome computation of simplicial (co)homology. Instead, we use the Mayer-Vietoris sequence together with the homotopy invariance of (co)homology and the values at the one-point space.

Let us attempt to compute the Khovanov homology of the right-handed trefoil $T_{2,3}$ using this approach, by means of using the long exact sequences from Section 1 applied to one of its positive crossings. Its 0-resolution)(is the Hopf link H with two positive crossings, whereas its 1-resolution $\asymp$ is the diagram of the unknot U with two negative crossings. We have chosen an arbitrary orientation for it, so that $c = n_-(\asymp) - n_-(\times) = 2$ in this case.



(10)

By the normalisation axiom, $Kh(U) = \mathbb{Q}_{(0,1)} \oplus \mathbb{Q}_{(0,-1)}$, and using e.g. the long exact sequence one can prove that

$$Kh^{i,j}(H) = \begin{cases} \mathbb{Q}, & (i,j) = (0,0),(0,2),(2,4),(2,6), \\ 0, & \text{elsewhere.} \end{cases}$$

The long exact sequence of the given crossing looks as follows:

$$\cdots \longrightarrow Kh^{i-3,j-8}(U) \longrightarrow Kh^{i,j}(T_{2,3}) \longrightarrow Kh^{i,j-1}(H) \longrightarrow Kh^{i-2,j-8}(U) \longrightarrow Kh^{i+1,j}(T_{2,3}) \longrightarrow \cdots$$

It is easy to see that

$$Kh^{i,j}(T_{2,3}) \xrightarrow{\cong} Kh^{i,j-1}(H) \quad , \quad \text{if } (i \neq 3 \text{ and } j \neq 7,9) \text{ or } (i \neq 2 \text{ and } j \neq 7,9),$$

which determines the values of $Kh^{i,j}(T_{2,3})$ for all but 4 pairs of indices. From the exact sequence it also follows that

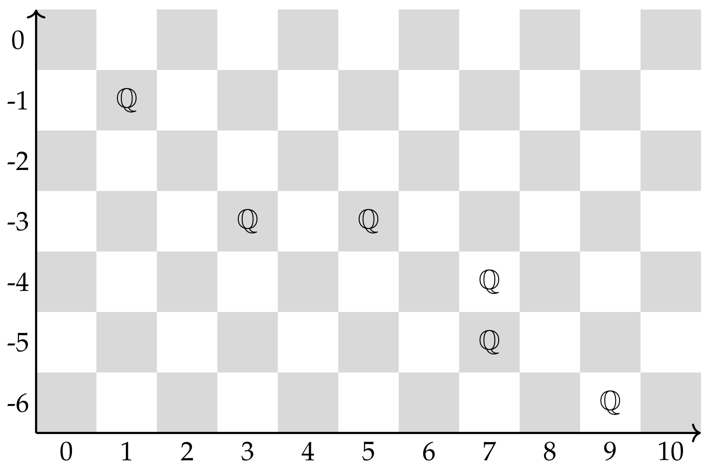
$$\mathbb{Q} \cong Kh^{0,1}(U) \xrightarrow{\cong} Kh^{3,9}(T_{2,3}) \quad , \quad 0 \cong Kh^{1,1}(U) \xrightarrow{\cong} Kh^{2,9}(T_{2,3}).$$

However in order to determine $Kh^{2,7}(T_{2,3})$ and $Kh^{3,7}(T_{2,3})$ we run intro trouble: reading off the long exact sequence we get an exact sequence

$$0 \longrightarrow Kh^{2,7}(T_{2,3}) \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow Kh^{3,7}(T_{2,3}) \longrightarrow 0,$$

and just from this it is not possible to decide whether the middle map is an isomorphism (so $Kh^{2,7}(T_{2,3}) \cong 0 \cong Kh^{3,7}(T_{2,3})$) or the zero map (so $Kh^{2,7}(T_{2,3}) \cong \mathbb{Q} \cong Kh^{3,7}(T_{2,3})$).

We can resolve the dichotomy by inspecting the Lee spectral sequence in one of the two possible scenarios. If the former was true, $Kh^{2,7}(T_{2,3}) \cong \mathbb{Q} \cong Kh^{3,7}(T_{2,3})$, the E_1-page of the Lee spectral sequence would look like



(recall that $E_1^{p,q} = Kh^{p+q,p}$). We know that in the E_∞ page only two copies of $\mathbb{Q}$ must survive. But by degree reasons there are no differentials $d_4, d_8, \ldots$ that could kill the generators in bidegree $(7, -5)$ and $(7, -4)$, so we conclude that $Kh^{2,7}(T_{2,3}) \cong 0 \cong Kh^{3,7}(T_{2,3})$.

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