K-THEORY SEMINAR. LECTURE 13

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The aim of today's lecture will be to relate our lovely K-theory with another area of mathematics, namely Functional Analysis. A kind of operators in a Hilbert space, called Fredholm operators, turns out to be a classifying space for K-theory. Before getting down into business let us recall some notions of functional analysis that will be treated during the talk:

A **Banach space** is a normed space $(E, ||\cdot||)$ which is complete. A closed subspace of a Banach space is Banach too; and also the quotient by a closed subspace. A linear map $T : E \longrightarrow F$ between Banach spaces is continuous if and only if the image of the unit ball is bounded, and $||T|| := \sup_{||e||\leq 1} ||T(e)|| < \infty$ defines a norm in the space of **operators** or linear and continuous maps from E to F, which we will denote as $\mathscr{L}(E, F)$. If F is Banach, so is $\mathscr{L}(E, F)$, although E is just normed. Therefore, the **dual space** $E^* := \mathscr{L}(E, \mathbb{C})$ is always Banach. If $T : E \longrightarrow F$ is a linear operator, the **adjoint** operator is $T^* : F^* \longrightarrow E^*$, $(T^* f^*)(e) := f^*(T(e))$, and $||T|| = ||T^*||$.

A **Hilbert space** H is a vector space endowed with a complex inner product $\langle\cdot,\cdot\rangle$, such that the norm defined by $||v|| := +\sqrt{\langle v, v\rangle}$ makes H be complete. In these spaces, every closed subspace V is complemented, $H = V \oplus V^{\perp}$. Moreover, the Riesz representation theorem establishes an isometric anti-isomorphism between H and H^*, and the adjoint operator of $T : H \longrightarrow H$ is the unique operator $T^* : H \longrightarrow H$ satisfying the equation

$$\langle T(v), w\rangle = \langle v, T^*(w)\rangle.$$

A **Banach algebra** $\mathcal{A}$ is a Banach space with an associative and distributive product satisfying $\lambda(ab) = (\lambda a)b = a(\lambda b)$, $||ab|| \leq ||a||||b||$ and the unit element has norm 1. If E is Banach, then $\mathscr{L}(E) := \mathscr{L}(E, E)$ is a Banach algebra (with respect the composition).

Finally, recall that an **operator ideal** $\mathscr{C} \subset \mathscr{L} = \mathscr{L}(E)$ is a class of operators which contains the finite-rank operators, $\mathscr{C} + \mathscr{C} \subset \mathscr{C}$ (closed under sums) and $\mathscr{L}\mathscr{C}\mathscr{L} \subset \mathscr{C}$ (closed under composition with any linear operator). For instance, the class $\mathcal{K} \subset \mathscr{L}$ of compact operators (ie, operators such that the image of the unit ball is relatively compact) form an operator ideal, which we will use in the following.

1 Fredholm operators

Definition. Let H be a separable complex Hilbert space. A **Fredholm operator** is a linear operator $T : H \longrightarrow H$ such that $\operatorname{Ker} T$ and $\operatorname{Coker} T$ are finite dimensional. We will denote as $\mathcal{F} = \mathcal{F}(H)$ the set of Fredholm operators.

Since $\operatorname{Ker} T$ is closed, it is Hilbert as well. To see that the $\operatorname{Coker} T$ is Hilbert too, one has to check that the condition about the dimension is enough to guarantee that $\operatorname{Im} T$ is closed.

Example 1.1 Consider

$$\ell_2 := \{x = (x_1, x_2, \dots) : ||x||_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}} < \infty\}.$$

The shift operators

$$\begin{array}{ll} L:\ell_2 \longrightarrow \ell_2 \quad , & L(x_1,x_2,\dots)=(x_2,x_3,\dots) \\ R:\ell_2 \longrightarrow \ell_2 \quad , & R(x_1,x_2,\dots)=(0,x_1,\dots) \end{array}$$

are Fredholm operators, since L is surjective and has a 1-dimensional kernel; and R is injective and has a 1-dimensional cokernel.

Example 1.2 Let H be a separable Hilbert space, let $\{e_0, e_1, e_2, \dots\}$ be an orthonormal basis and let $k \in \mathbb{Z}$. The previous argument generalizes to the **standard operator** of index k,

$$R_k(e_i) := \begin{cases} e_{i-k}, & i \geq k \\ 0, & i < k \end{cases}.$$

–. For $k = 0$, there is no much to say: $R_0 = \mathrm{Id}$.
–. For $k > 0$, we have $R_k(e_0) = \cdots = R_k(e_{k-1}) = 0$ and $R_k(e_{k+p}) = e_p$, $p \geq 0$. In particular, $\operatorname{Ker} R_k =< e_0, \dots, e_{k-1} >$ and $\operatorname{Coker} R_k = 0$ because it is clearly surjective.
–. For $k < 0$, $R_k(e_i) = e_{i-k}$ always, so it is injective and $\operatorname{Coker} R_k =< \bar{e}_0, \dots, \bar{e}_{k-1} >$.

The following classical result of Functional Analysis relates Fredholm operators with compact operators:

Theorem 1.3 (Fredholm) *Let E be a Banach space. An operator $T : E \longrightarrow E$ is Fredholm if and only if there is $R \in \mathscr{L}(E)$ such that $RT - \mathrm{Id}$ and $TR - \mathrm{Id}$ are compact operators.*

The proof relies on the fact that for any $K : E \longrightarrow E$ compact, $\mathrm{Id} + K$ is Fredholm.

Now denote by $\mathcal{K}$ the class of linear and compact operators, and let $\mathscr{L} = \mathscr{L}(E)$. Since $\mathcal{K}$ is an operator ideal, we can consider the quotient $\mathcal{B} := \mathscr{L}/\mathcal{K}$, which is a Banach algebra (usually called the **Calkin algebra**).

Corollary 1.4 *Denote by $\mathcal{B}^*$ the subset of invertible elements, and $\pi : \mathscr{L} \longrightarrow \mathcal{B}$ the canonical projection. Then we have*

$$\mathcal{F} = \pi^{-1}(\mathcal{B}^*).$$

In particular, $\mathcal{F}$ is open in $\mathscr{L}$.

Proof.

$$\begin{aligned} \pi^{-1}(\mathcal{B}^*) &= \{T \in \mathscr{L} : [T] \text{ is invertible}\} \\ &= \{T \in \mathscr{L} : \exists [S] : [T][S] = [\mathrm{Id}], [S][T] = [\mathrm{Id}]\} \\ &= \{T \in \mathscr{L} : \exists [S] : [TS - \mathrm{Id}] = 0, [ST - \mathrm{Id}] = 0\} \\ &= \{T \in \mathscr{L} : \exists S : TS - \mathrm{Id}, ST - \mathrm{Id} \text{ are compact}\} \\ &= \mathcal{F} \end{aligned}$$

□

From here it also follows that the composite of Fredholm operators is Fredholm: if T_1, T_2 Fredholm then $[T_1], [T_2]$ are invertible, thus $[T_1][T_2] = [T_1 T_2]$ is Fredholm, what happens if and only if $T_1 T_2$ is Fredholm. We will come to a more general fact later.

Definition. Let $T : H \longrightarrow H$ be a Fredholm operator. The **index** of T is the integer

$$\begin{aligned} \operatorname{Ind} T : &= \dim \operatorname{Ker} T - \dim \operatorname{Coker} T \\ &= \dim \operatorname{Ker} T - \dim \operatorname{Ker} T^* \end{aligned}$$

Example 1.5 If H is a finite-dimensional Hilbert space, then $\operatorname{Ind} T = 0$ for all (Fredholm) operator T. Indeed,

$$\begin{aligned}\operatorname{Ind} T &= \dim \operatorname{Ker} T - \dim(H/\operatorname{Im} T) = \dim \operatorname{Ker} T - (\dim H - \dim \operatorname{Im} T) \\ &= \dim \operatorname{Ker} T + \dim \operatorname{Im} T - \dim H = 0\end{aligned}$$

Proposition 1.6 *If two out of three operators T, S, ST are Fredholm, so is the third and it holds*

$$\operatorname{Ind} ST = \operatorname{Ind} T + \operatorname{Ind} S.$$

Proof. For any linear maps T, S, one always has an exact sequence

$$0 \longrightarrow \operatorname{Ker} T \longrightarrow \operatorname{Ker} ST \longrightarrow \operatorname{Ker} S \longrightarrow \operatorname{Coker} T \longrightarrow \operatorname{Coker} ST \longrightarrow \operatorname{Coker} S \longrightarrow 0$$

coming from the snake lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{(\mathrm{Id},T)} & H \oplus H & \xrightarrow{T\pi_1 - \pi_2} & H & \longrightarrow & 0 \\ & & \big\downarrow T & & \big\downarrow ST\oplus\mathrm{Id} & & \big\downarrow S & & \\ 0 & \longrightarrow & H & \xrightarrow{(S,\mathrm{Id})} & H \oplus H & \xrightarrow{\pi_1 - S\pi_2} & H & \longrightarrow & 0 \end{array}$$

If T, S are Fredholm, we already know that so is ST. If ST and either T or S are Fredholm, then in any case $\operatorname{Ker} T$ and $\operatorname{Coker} S$ are finite dimensional too; and in the previous exact sequence 5 out of 6 spaces are finite-dimensional (the only space in question is either $\operatorname{Ker} S$ or $\operatorname{Coker} T$, depending on the operator we assumed at first that was Fredholm, T or S, resp.), so all 6 must be.

The claim about the indexes follows from the fact that a chain complex (of vector spaces) which is exact has Euler characteristic 0, thus

$$\begin{aligned}0 = &\dim \operatorname{Ker} T - \dim \operatorname{Ker} ST + \dim \operatorname{Ker} S \\ &- \dim \operatorname{Coker} T + \dim \operatorname{Coker} ST - \dim \operatorname{Coker} S\end{aligned}$$

and rearranging the terms we conclude. □

1.7 (Parametrized families) In general, we will be interested not in a single Fredholm operator, but in a (continuously) parametrized family of operators. Formally this is a continuous map

$$F : X \longrightarrow \mathcal{F}$$

where X is a Hausdorff topological space (thought as the space of parameters) and $\mathcal{F}$ is topologized with the norm topology inhered from $\mathscr{L}$.

Now one wonders how the previous description about the index of a Fredholm operator can be reformulated for a continuous family. Surprisingly the main ideas remain:

Lemma 1.8 *Let $T : H \longrightarrow H$ be a Fredholm operator and let $V \subset H$ be a closed subspace of finite codimension such that $V \cap \operatorname{Ker} T = 0$. Then there is a neighbourhood U of T in $\mathcal{F}$ such that $V \cap \operatorname{Ker} S = 0$ for all $S \in U$ and*

$$\bigcup_{S\in U} H/S(V) := \frac{U \times H}{\{(S, v) \in U \times H : v \in S(V)\}}$$

(topologized as a quotient of $U \times H$) is a trivial vector bundle over U.

Proof. In first place we observe that $H/T(V)$ is finite dimensional, because H/V, $\operatorname{Coker} T$ are too, the obvious map $H/V \longrightarrow T(H)/T(V)$ is surjective and there is a short exact sequence

$$0 \longrightarrow T(H)/T(V) \longrightarrow H/T(V) \longrightarrow \operatorname{Coker} T \longrightarrow 0.$$

Therefore $T(V)$ is closed, and it admits an orthogonal complement $W := T(V)^{\perp}$, which is isomorphic to $H/T(V)$.

Now consider the continuous map

$$\varphi : \mathscr{L}(H) \longrightarrow \mathscr{L}(V \oplus W, H),$$

where $\varphi_S : V \oplus W \longrightarrow H$ is defined by $\varphi_S(v, w) := S(v) + w$. The observation is that $\varphi_T : V \oplus W \longrightarrow H$ is isomorphism. Indeed, obviously it is linear and surjective, but also is injective, because if $T(v) + w = 0$, then $T(v) \in W = T(V)^{\perp}$, so $T(v) = 0$ and then $v = 0$ because $V \cap \operatorname{Ker} T = 0$ (and of course $w = -T(v) = 0$), so it is an isomorphism of vector spaces, and provided that it is continuous (sum of continuous functions) by the Open Mapping theorem it is also an isomorphism of Banach spaces (topological isomorphism). Therefore, since being isomorphism is an open condition, there is a neighbourhood U of T in $\mathcal{F}$ such that φ_S is isomorphism for all $S \in U$.

Now both assertions are easy to check: if $v \in V \cap \operatorname{Ker} S$, then $\varphi_S(v, w) = w = \varphi_S(0, w)$, so $v = 0$. For the second one first note that the the isomorphism φ_S ensures that $S(V)$ is closed and that $H/S(V)$ is finite dimensional. Moreover, $H/S(V) \simeq W$ for all $S \in U$, that is, $\bigcup_{S \in U} H/S(V)$ is isomorphic to the trivial vector bundle $U \times W$. □

Proposition 1.9 *Let X be a compact space, and let $F : X \longrightarrow \mathcal{F}$ be a continuous family of Fredholm operators. Then there exists $V \subset H$ closed and of finite codimension such that*

$$V \cap \operatorname{Ker} F_x = 0 \qquad \forall x \in X$$

and

$$H/F(V) := \bigcup_{x \in X} H/F_x(V) = \frac{X \times H}{\{(x, v) \in X \times H : v \in F_x(V)\}}$$

(topologized as a quotient of $X \times H$) is a vector bundle over X.

Proof. For each $x \in X$ consider $V_x := (\operatorname{Ker} F_x)^{\perp}$, which are closed and finite codimensional. By the lemma, there exist neighbourhoods $\widetilde{U_x}$ of the F_x's in $\mathcal{F}$ satisfying both conditions in the previous lemma. Set $U_x := F^{-1}(\widetilde{U_x})$, and by compactness take $(U_i)_{i=1}^n$ a finite open cover of $(U_x)_{x \in X}$, coming from points $x_1, \ldots, x_n$. Now we claim that $V := \bigcap_{i=1}^n V_{x_i}$ is the desired subspace. Indeed, it is again closed and finite codimensional, and clearly $V \cap \operatorname{Ker} F_x = 0$ for all $x \in X$ provided that we took the intersection of all V_{x_i}'s. For the last assertion regarding the vector bundle one argues as follows: $F_x(V)$ is closed and $H/F_x(V)$ is finite dimensional, by the same argument as in the previous lemma. Moreover, the trivial vector bundles $\bigcup_{S \in \widetilde{U_i}} H/S(V)$ from the lemma provide trivial vector bundles $\bigcup_{x \in U_i} H/F_x(V)$ by the commutativity of the following diagram:

$$\begin{array}{ccc} \bigcup_{x \in U_i} H/F_x(V) & \longrightarrow & \bigcup_{S \in \widetilde{U_i}} H/S(V) \\ \downarrow & & \downarrow \\ U_i & \xrightarrow{\quad F \quad} & \widetilde{U_i} \end{array}$$

(this is a pullback diagram), which stablish the local trivializations of $H/F(V)$, ensuring that it is a vector bundle over X. □

Definition. Let X be a compact space, and $F : X \longrightarrow \mathcal{F}$ continuous. The **index** of F is

$$\operatorname{Ind} F := [H/V] - [H/F(V)] \in K(X)$$

where H/V stands for the trivial bundle $X \times H/V$.

At first sight one might think that this has nothing to do with the notion of index for Fredholm operators. Luckily this is not the case: for $X = *$ the one-point space, then a continuous family $F : * \longrightarrow \mathcal{F}$ is simply an operator $T = F_*$. Using that $K(*) \simeq \mathbb{Z}$, where the isomorphism is given by the dimension of the vector bundle, and taking V to be $(\operatorname{Ker} T)^\perp$, we have

$$\begin{aligned} \operatorname{Ind} F = [H/V] - [H/F(V)] &= \dim H/(\operatorname{Ker} T)^\perp - \dim H/T((\operatorname{Ker} T)^\perp) \\ &= \dim \operatorname{Ker} T - \dim H/\operatorname{Im} T = \dim \operatorname{Ker} T - \dim \operatorname{Coker} T = \operatorname{Ind} T. \end{aligned}$$

recovering the usual notion of the index.

Lemma 1.10 $\operatorname{Ind} F$ *is a well-defined class in K-theory.*

Proof. We need to show that $\operatorname{Ind} F$ is independent of the choice of the subspace V satisfying the conditions of the previous lemma.

Let $W \subset H$ be another subspace with $W \cap \operatorname{Ker} F_x = 0$ for all $x \in X$. In first place suppose that $W \subseteq V$. In such a case the inclusions $W \subset V \subset H$ and $F(W) \subset F(V) \subset H$ induce short exact sequences of vector bundles

$$\begin{gathered} 0 \longrightarrow V/W \longrightarrow H/W \longrightarrow H/V \longrightarrow 0 \\ 0 \longrightarrow V/W \simeq F(V)/F(W) \longrightarrow H/F(W) \longrightarrow H/F(V) \longrightarrow 0 \end{gathered} \tag{1}$$

(the isomorphism in the second sequence comes from the fact that pointwise they do not intersect the kernels). Now applying exercise 1 of Bjarne's notes about the Euler characteristic of an exact sequence of vector bundles we have that

$$[V/W] - [H/W] + [H/V] = 0 = [V/W] - [H/F(W)] + [H/F(V)]$$

so

$$[H/V] - [H/F(V)] = [H/W] - [H/F(W)].$$

For the general case where W is any subspace with the aforementioned property, just observe that $V \cap W$ also satisfies the property, and by the previous step the index computed with $V \cap W \subset V$ is still the same. Arguing with the short exact sequences produced by the inclusions $V \cap W \subset W \subset H$ and $T(V \cap W) \subset T(V) \subset H$ we conclude. □

2 Classifying spaces for K-theory

At this point it is more than reasonable to wonder: why do we care about continuous families of Fredholm operators?

By 1.6, the composite of Fredholm operators is Fredholm, so it defines an associative operation

$$\mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F} \qquad , \qquad (T, S) \mapsto S \circ T$$

endowing $\mathcal{F}$ with a structure of monoid, with unit $\operatorname{const}_{\operatorname{Id}_H}$. Now observe that for any space X, the set $[X, \mathcal{F}]$ of homotopy classes of continuous maps $X \longrightarrow \mathcal{F}$ inherits such a structure, where for two maps $F_1, F_2 : X \longrightarrow \mathcal{F}$, $(F_2 F_1)_x := (F_2)_x (F_1)_x$. The compatibility of the composition with the homotopy relation is straightforward.

The goal of today's lecture is precisely the following theorem:

Theorem 2.1 (Atiyah-Jänich) *Let X be a compact space. There is a natural monoid isomorphism*

$$\mathrm{Ind} : [X, \mathcal{F}] \xrightarrow{\sim} K(X).$$

That is, $\mathcal{F}$ represents K-theory.

Corollary 2.2 $\mathrm{Ind} : \pi_0(\mathcal{F}) \xrightarrow{\sim} \mathbb{Z}$ *is a bijection.*

Proof. Take $X = *$ the one-point space in the previous theorem. □

Almost the rest of the talk will be devoted to prove 2.1. We will divide the proof in several steps.

Lemma 2.3 *If $f : Y \longrightarrow X$ is a continuous map and $F : X \longrightarrow \mathcal{F}$ is a continuous family of Fredholm operators, then*

$$f^*(\mathrm{Ind}\, F) = \mathrm{Ind}(F \circ f).$$

Proof. If $V \subset H$ is a choice for F, then note that V is also a choice for $F \circ f$, since $V \cap \mathrm{Ker}\, F_{f(y)} = 0$ for all $y \in Y$. Therefore

$$\begin{aligned} f^*(\mathrm{Ind}\, F) &= f^*([H/V] - [H/F(V)]) \\ &= [f^*(H/V)] - [f^*(H/F(V))] \\ &= [H/V] - [H/(F \circ f)(V)] = \mathrm{Ind}(F \circ f) \end{aligned}$$

where we used that $(f^*(H/F(V)))_y \simeq H/F_{f(y)}(V) = H/(F \circ f)_y(V)$. □

Proposition 2.4 $\mathrm{Ind} : [X, \mathcal{F}] \longrightarrow K(X)$ *is a well-defined monoid homomorphism, which is natural on X.*

Proof. Firstly we show the independence of the homotopy class. If F_0 is homotopic to F_1 with homotopy $F : X \times I \longrightarrow \mathcal{F}$, consider the inclusions $i_j : X \longrightarrow X \times I$, $i_j(x) = (x, j)$ for $j = 0, 1$. Then we know that $i_0^* = i_1^* : K(X \times I) \longrightarrow K(X)$ because the restrictions of vector bundles to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic, so

$$\mathrm{Ind}\, F_0 = \mathrm{Ind}(F \circ i_0) \overset{2.3}{=} i_0^*(\mathrm{Ind}\, F) = i_1^*(\mathrm{Ind}\, F) \overset{2.3}{=} \mathrm{Ind}(F \circ i_1) = \mathrm{Ind}\, F_1.$$

Now we prove that Ind is a monoid homomorphism. Obviously $[\mathrm{const}_{\mathrm{Id}}]$ maps to 0. Suppose $F, G : X \longrightarrow \mathcal{F}$ are continuous families of Fredholm operators, and let $V, W \subset H$ be the choices of subspaces for F and G, respectively. Write $H = W \oplus W^\perp$ and consider $\pi : H \longrightarrow W$ and $\rho : H \longrightarrow W^\perp$ the orthogonal projections. Observe that $\mathrm{Id} - t\rho : H \longrightarrow H$ is Fredholm for all $t \in I$ (for $t = 1$ is π and else isomorphism), so we can define a homotopy

$$h : X \times I \longrightarrow \mathcal{F} \qquad , \qquad h(x, t) := (\mathrm{Id} - t\rho) \circ F_x$$

from F to $\pi \circ F$. Therefore we can assume that $F_x(H) \subset W$ and in particular that $F_x(V) \subset W$. Concretely, $V \cap \mathrm{Ker}\, G_x F_x \subset V \cap \mathrm{Ker}\, F_x = 0$, what means that V is also a choice for GF. As before the inclusions $F(V) \subset W \subset H$ and $GF(V) \subset G(W) \subset H$ provide short exact sequences

$$0 \longrightarrow W/F(V) \longrightarrow H/F(V) \longrightarrow H/W \longrightarrow 0 \tag{2}$$

$$0 \longrightarrow W/F(V) \simeq G(W)/GF(V) \longrightarrow H/GF(V) \longrightarrow H/G(W) \longrightarrow 0 \tag{3}$$

so using again that the Euler characteristic of an exact sequence of vector bundles is the class 0,

$$\mathrm{Ind}\, GF = [H/V] - [H/GF(V)]$$

$$\begin{aligned}&\overset{(3)}{=} [H/V] - [W/F(V)] - [H/G(W)]\\&\overset{(2)}{=} [H/V] - [W/F(V)] + [H/W] - [H/G(W)]\\&= \operatorname{Ind} F + \operatorname{Ind} G.\end{aligned}$$

The assertion about the naturality means that for any continuous map $f : Y \longrightarrow X$ there is a commutative diagram

$$\begin{array}{ccc}[X,\mathcal{F}] & \xrightarrow{\operatorname{Ind}} & K(X)\\ \big\downarrow f^* & & \big\downarrow f^*\\ [Y,\mathcal{F}] & \xrightarrow{\operatorname{Ind}} & K(Y)\end{array}$$

what follows immediately from 2.3. □

Theorem 2.5 *If X is a compact space, then there is an exact sequence of monoids*

$$[X, \mathscr{L}^*] \xrightarrow{\text{incl}_*} [X, \mathcal{F}] \xrightarrow{\operatorname{Ind}} K(X) \longrightarrow 0.$$

Proof. We start by the exactness at $[X, \mathcal{F}]$. On the one hand, for a family of operators $F : X \longrightarrow \mathscr{L}^*$, we have that H is a choice for computing the index, since $\operatorname{Ker} F_x = 0$ for all $x \in X$. Therefore $\operatorname{Ind} F = [H/H] - [H/F(H)] = 0$. On the other hand, for a family $F : X \longrightarrow \mathcal{F}$ such that $\operatorname{Ind} F = 0$, we have that $[H/F(V)] = [H/V]$ for some $V \subset H$ disjoint with the kernels pointwise. The condition is equivalent to say that $H/F(V) \oplus \underline{m} \simeq H/V \oplus \underline{m}$ for some $m \geq 0$. Now for any $W \subset V$ such that $\dim V/W = m$ (in particular $V/W \simeq \mathbb{C}^m$), the split short exact sequences (1) ensure that

$$H/W \overset{(1)}{\simeq} (H/V) \oplus \underline{m} \simeq H/F(V) \oplus \underline{m} \overset{(1)}{\simeq} H/F(W).$$

Note that for every $x \in X$, $F_x(W)$ is still finite codimensional and closed, because the restriction of a Fredholm operator to a closed, finite codimensional subspace is still Fredholm. Because of that, there is a natural isomorphism of vector bundles $H/F(W) \simeq F(W)^\perp$. Consider now the composite

$$H/W \xrightarrow{\sim} H/F(W) \xrightarrow{\sim} F(W)^\perp \hookrightarrow X \times H.$$

It induces a continuous map $\phi : X \longrightarrow \mathscr{L}(H/W, H)$, where ϕ_x takes H/W isomorphically to $F_x(W)^\perp$ for every $x \in X$. But we also have that $F_x : W \longrightarrow F_x(W)$ is isomorphism (being $W \cap \operatorname{Ker} F_x = 0$ for all $x \in X$), so we conclude that the direct sum

$$\phi_x \oplus F_x : H/W \oplus W = H \longrightarrow H = F_x(W) \oplus F_x(W)^\perp$$

is isomorphism as well. This defines a continuous map

$$\phi \oplus F : X \longrightarrow \mathscr{L}^* \subset \mathcal{F}$$

which is homotopic to F via the obvious homotopy $(t\phi) \oplus F$.

It remains to prove the exactness at $K(X)$, that is, to show that Ind is surjective. We already know that every element of $K(X)$ is of the form $[E] - \underline{m}$ for some vector bundle $E \longrightarrow X$. The term $-\underline{m}$ is easy to get: just set $R_{-m} : X \longrightarrow \mathcal{F}$, $(R_{-m})_x := R_{-m}$, where $R_{-m} : H \longrightarrow H$ is the standard operator of index $-m$ defined in 1.2. We saw that it is injective, thus H is a choice for computing the index, so $\operatorname{Ind} R_{-m} = [H/H] - [H/R_{-m}(H)] = -[\operatorname{Coker} R_{-m}] = -\underline{m}$. Now let us find $F : X \longrightarrow \mathcal{F}$ with the property that $\operatorname{Ind} F = [E]$. We can find an orthogonal complement $E^\perp$ to E such that $E \oplus E^\perp \simeq \underline{N}$. For every point write

$$\pi_x : \mathbb{C}^N = E_x \oplus E_x^\perp \longrightarrow E_x$$

for the orthogonal projection. As H is Hilbert, $\mathbb{C}^N \otimes H$ is Hilbert too. Let $\{v_1, \dots, v_N\}$ be an orthonormal basis of $\mathbb{C}^N$ and let $\{e_0, e_1, \dots\}$ be an orthonormal basis of H, so that $\{v_i \otimes e_j\}$ is an orthonormal basis for $\mathbb{C}^N \otimes H$, with respect the inner product $\langle v_i \otimes e_j, v_r \otimes e_s\rangle = \langle v_i, v_r\rangle\langle e_j, e_s\rangle$. Moreover, $\mathbb{C}^N \otimes H$ and H are isomorphic, because they are both separable Hilbert spaces.

Let $n < N$ be the rank of E, and suppose $\{v_1, \dots, v_n\}$ is a basis of E_x (we are viewing $\mathbb{C}^N = E_x \oplus E_x^{\perp}$). Define

$$\psi : X \longrightarrow \mathcal{F}(\mathbb{C}^N \otimes H) \simeq \mathcal{F}(H) = \mathcal{F} \qquad , \qquad \psi_x := \pi_x \otimes R_1 + (\mathrm{Id}_{\mathbb{C}^N} - \pi_x) \otimes \mathrm{Id}_H.$$

Now the claim is that ψ_x is surjective and its kernel is spanned by $\{v_1 \otimes e_0, \dots, v_n \otimes e_0\}$. Indeed, for $c = \sum_{i=1}^N \lambda_i v_i \in \mathbb{C}^N$ and $u \in H$ we have

$$\begin{aligned}\psi_x(c \otimes u) &= \psi_x(v_1 \otimes \lambda_1 u + \cdots + v_N \otimes \lambda_n u) \\ &= v_1 \otimes \lambda_1 R_1(u) + \cdots + v_n \otimes \lambda_n R_1(u) + v_{n+1} \otimes \lambda_{n+1} u + \cdots + v_N \otimes \lambda_N u.\end{aligned}$$

This shows that ψ_x is surjective, because R_1 is surjective and $\mathrm{Ker}\, R_1 =< e_0 >$. In particular, to get $\lambda v_i \otimes e_j$ one just needs to choose $c = \lambda v_i$ and $u = e_{j+1}$. For the kernel of ψ_x one argues as follows: if $\psi_x(c \otimes u) = 0$, then one of the following three conditions should hold: (i) $u = 0$; (ii) $c = 0$, or (iii) $u = e_0$ and $c \in E_x$, that is, $\lambda_{n+1} = \cdots = \lambda_N = 0$. In particular, $\mathrm{Ker}\, \psi_x =< v_1 \otimes e_0, \dots, v_n \otimes e_0 >\simeq E_x$. The upshot is that we can take $V =< v_1 \otimes e_0, \dots, v_n \otimes e_0 >^{\perp} \subset \mathbb{C}^N \otimes H \simeq H$ and therefore

$$\mathrm{Ind}\, \psi = [\mathbb{C}^N \otimes H/V] - [\mathbb{C}^N \otimes H/\psi(V)] = [E] - 0 = [E],$$

and we conclude that the composite ψR_{-m} is the desired map,

$$\mathrm{Ind}(\psi R_{-m}) = \mathrm{Ind}\, \psi + \mathrm{Ind}\, R_{-m} = [E] - \underline{m}.$$

□

The proof was quite hard but now 2.1 is an immediate consequence of a well-known result of Functional Analysis:

Theorem 2.6 (Kuiper) *$\mathscr{L}^*$ is contractible.*

The only objection we could have is that we obtained an isomorphism of monoids, whereas $K(X)$ has a richer structure (ring structure). If instead of maps to $\mathcal{F}$ we consider maps to $\mathcal{B}^*$ the set of invertible elements of $\mathcal{B} = \mathscr{L}/\mathcal{K})$ we get a group structure on $[X, \mathcal{B}^*]$ (now we have inverses and the unit is the class $[\mathrm{const}_{\mathrm{Id}_H}]$). This happens to be compatible with the previous isomorphism:

Theorem 2.7 *Let X be compact. There is a natural group isomorphism*

$$\mathrm{Ind} : [X, \mathcal{B}^*] \xrightarrow{\sim} K(X).$$

That is, $\mathcal{B}^*$ represents K-theory.

The proof will rely on the following general result of Functional Analysis:

Lemma 2.8 *Let $T : E \longrightarrow F$ an operator between Banach spaces, and let $U \subset F$ be open. If* $\mathrm{Im}\, T$ *is dense in F, then for any compact space X there is a bijection*

$$[X, T^{-1}(U)] \xrightarrow{\sim} [X, U].$$

Proof of 2.7. Apply the lemma to $\pi : \mathscr{L} \longrightarrow \mathcal{B} = \mathscr{L}/\mathcal{K}$, recalling that $\mathcal{B}^*$ is open and $\mathcal{F} = \pi^{-1}(\mathcal{B}^*)$. □

Remark 2.9 For reduced K-theory of pointed spaces, 2.1 can be expressed as

$$\widetilde{K}(X) = [(X, x_0), (\mathcal{F}, \mathrm{Id}_H)]_*$$

where the lower star refers to basepoint preserving homotopy classes of basepoint preserving maps. Now the loop-suspension adjunction ensures that

$$\widetilde{K}^n(X) = \widetilde{K}(\Sigma^n X) \simeq [\Sigma^n X, \mathcal{F}]_* \simeq [X, \Omega^n \mathcal{F}]_*.$$

The periodicity of the reduced K-theory $\widetilde{K}^0 = \widetilde{K}^2$ suggests that $\mathcal{F}$ and $\Omega^2\mathcal{F}$ might be homotopy equivalent, but one cannot conclude directly since X is required to be compact and $\mathcal{F}$ is not. In fact the result is true but to show the equivalence one requires more machinery, namely to make sure that $\mathcal{F}$ has the homotopy type of a CW-complex, which is true because $\mathcal{F}$ is an open subset of a locally convex space (see [3]).

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PROBLEMS

1. If $T : H \longrightarrow H$ is Fredholm then $\operatorname{Im} T$ is closed.(*Hint:* Use the Open Mapping theorem.)

2. Let $T : H \longrightarrow H$ be a Fredholm operator.

   (a) Show that $\operatorname{Coker} T \simeq \operatorname{Ker} T^*$, for $T : H \longrightarrow H$ Fredholm. (*Hint:* Show in first place that $\operatorname{Ker} T^* = (\operatorname{Im} T)^{\perp}$).

   (b) Consider an orthonormal basis $\{e_i : i \geq 0\}$ of H and set $H_n := \operatorname{span}\{e_i : i \geq n\} =< e_0, \ldots, e_{n-1} >^{\perp}$, and write $P_n : H \longrightarrow H_n \subset H$ for the orthogonal projection. Show that for $n \in \mathbb{N}$ large enough, $\operatorname{Ind} T = \dim \operatorname{Ker} T_n - n$, where $T_n = P_n \circ T$. (*Hint:* Choose an orthonormal basis such that $H_n \subset \operatorname{Im} T$.)

3. If $T : H \longrightarrow H$ is Fredholm, then $T^* : H \longrightarrow H$ is Fredholm too and $\operatorname{Ind} T^* = -\operatorname{Ind} T$. Show also that the same formula holds for a parametrized family of Fredholm operators: given a continuous map $F : X \longrightarrow \mathcal{F}$, and writing $F^* : X \longrightarrow \mathcal{F}$, $(F^*)_x := (F_x)^*$, show that $\operatorname{Ind} F^* = -\operatorname{Ind} F$.

4. Consider the family $F : X = \{0, 1, 2, 3, 4, 5\} \subseteq \mathbb{Z} \longrightarrow \mathcal{F}(\ell_2)$ defined by $F_k := R_k$. Compute $\operatorname{Ind} F \in K(X)$.

5. Let $T_1 : H_1 \longrightarrow H_1$, $T_2 : H_2 \longrightarrow H_2$ be Fredholm operators. Then $T_1 \oplus T_2 : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2$ is Fredholm and $\operatorname{Ind}(T_1 \oplus T_2) = \operatorname{Ind} T_1 + \operatorname{Ind} T_2$.

6. If $T : H \longrightarrow H$ is Fredholm and $S : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ is an isomorphism, then $T \otimes S : H \otimes \mathbb{C}^N \longrightarrow H \otimes \mathbb{C}^N$ is Fredholm and $\operatorname{Ind}(T \otimes S) = N \cdot \operatorname{Ind} T$. (*Hint:* $\operatorname{Ker}(T \otimes S) = \operatorname{Ker} T \otimes \mathbb{C}^N$, $\operatorname{Im}(T \otimes S) = \operatorname{Im} T \otimes \mathbb{C}^N$ and $\operatorname{Coker}(T \otimes S) \simeq \operatorname{Coker} T \otimes \mathbb{C}^N$).

7. Let $F : X \longrightarrow \mathcal{F}$ be a parametrized family of Fredholm operators. Show that the function $\dim \operatorname{Ker} F : X \longrightarrow \mathbb{Z}$, $x \mapsto \dim \operatorname{Ker} F_x$ is semi-continuous, that is, for any point x_0 there is a neighbourhood U of x_0 such that $\dim \operatorname{Ker} F_{x_0} \geq \dim \operatorname{Ker} F_x$ for all $x \in U$.

Hand-in exercises: 1 – 4.