LECTURE 8: EXAMPLES OF SULLIVAN MODELS

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Today we will put into practice the machinery developed in the last lectures to compute examples of (minimal) Sullivan models. In the last part of the lecture we will compute a Sullivan model for the pullback of a Serre fibration.

Last weeks there were lots of concepts and results introduced so recalling the ones we will use today will not hurt anyone.

1 Recap of minimal Sullivan models

Let k be a field of char $k \neq 0$, in other words, let k be a field extension of $\mathbb{Q}$.

Definition. Let (B, d) be a cdga with $H^0(B) = k$. A **relative Sullivan algebra** is a cdga of the form $(B \otimes \Lambda V, d)$ where $V = \{V^i : i \geq 1\}$ is a graded vector space together with an increasing sequence $V(0) \subsetneq V(1) \subsetneq \cdots$ of subspaces satisfying $V = \bigcup V(n)$ and such that

$$d : V(n) \longrightarrow B \otimes V(n-1) \qquad , \qquad n \geq 0$$

where $V(-1) := 0$. We say that B is the **base**.

An (absolute) **Sullivan algebra** is a relative Sullivan algebra with $B = k$.

Definition. Let $\varphi : (A, d) \longrightarrow (C, d)$ be a morphism of cdga's, with $H^0(B) = k$. A **Sullivan model** for φ is a quasi-iso

$$m : (B \otimes \Lambda V, d) \xrightarrow{\simeq} (C, d)$$

where $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra with base B and $m_{|B} = \varphi$. [1]

A **Sullivan model** for a cdga (C, d) is a Sullivan model for the morphism $\varphi : k \longrightarrow (C, d)$, that is, a quasi-iso

$$m : (\Lambda V, d) \xrightarrow{\simeq} (C, d)$$

where $(\Lambda V, d)$ is a Sullivan algebra.

If X is a path-connected space, a Sullivan model for X is a Sullivan model for $A_{PL}(X) := A_{PL}(\mathcal{S}(X)) = \mathrm{Hom}_{\mathsf{sSet}}(\mathcal{S}(X)_\bullet, A_{PL}(\Delta^\bullet))$. Here $A_{PL} : \mathsf{Top}^{op} \longrightarrow \mathsf{cdga}$ is the functor of polynomial differential forms.

Definition. A Sullivan algebra $(\Lambda V, d)$ is **minimal** if $\mathrm{Im}\, d \subset \Lambda^{\geq 2} V$.

In general, we will talk about *the* minimal Sullivan algebra of a cdga / space, since

Theorem 1.1 *Every morphism of cdga's* $\varphi : (B, d) \longrightarrow (C, d)$ *with* $H^0(B) = k = H^0(C)$ *and* $\varphi_* : H^1(B) \longrightarrow H^1(C)$ *injective has a unique minimal Sullivan model up to isomorphism.*

Corollary 1.2 *Every cdga* (A, d) *with* $H^0(A) = k$ *has a unique minimal Sullivan model up to isomorphism.*

[1] For cdga's B, ΛV, there is a natural morphism $B \longrightarrow B \otimes \Lambda V$, $b \mapsto b \otimes 1$. Then the restriction $m_{|B}$ means the composite with this morphism.

Corollary 1.3 *Every path-connected space has a unique minimal Sullivan model up to isomorphism.*

Definition. Let $(B \otimes \Lambda V, d)$ be a relative Sullivan algebra and let $\varepsilon : B \longrightarrow k$ be an augmentation. The **Sullivan fibre at** ε is the pushout cdga

$$\begin{array}{ccc} (B,d) & \longrightarrow & k \\ \downarrow & & \downarrow \\ (B \otimes \Lambda V, d) & \longrightarrow & (\Lambda V, \bar{d}) \cong k \otimes_B (B \otimes \Lambda V, d) \end{array}$$

Minimal Sullivan model of a Serre fibration

For the rest of the section we consider the following

Setup: Let X be a path-connected space, let Y be a simply connected space, and let $p : X \longrightarrow Y$ be a Serre fibration. Also, let $y_0 \in Y$ and suppose that the fibre $F := p^{-1}(y_0)$ is path-connected. Lastly, suppose that either X or Y satisfy that all their homology groups with coefficients in k are finite dimensional vector spaces.

So, in particular, we have a fibration sequence $F \xhookrightarrow{j} X \xrightarrow{p} Y$ and p restricts to $p : F \longrightarrow y_0$. Applying A_{PL} yields the commutative diagram of below. Here ε is viewed as an augmentation.

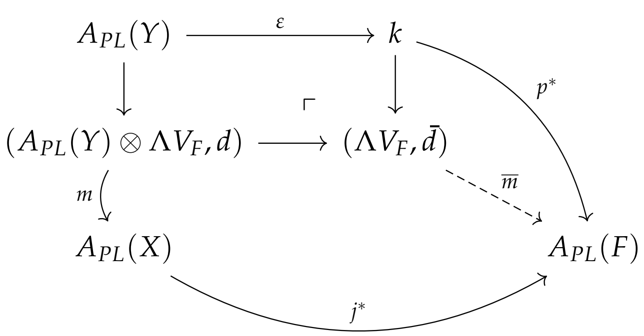
$$\begin{array}{ccc} A_{PL}(F) & \xleftarrow{j^*} & A_{PL}(X) \\ {\scriptstyle p^*}\uparrow & & {\scriptstyle p^*}\uparrow \\ k & \xleftarrow{\varepsilon} & A_{PL}(Y) \end{array}$$

Lemma 1.4 *We have that $p^* : H^1(Y;k) \longrightarrow H^1(X;k)$ is injective, thus there exists a Sullivan model for p*

$$m : (A_{PL}(Y) \otimes \Lambda V_F, d) \xrightarrow{\simeq} (A_{PL}(X), d).$$

Proof. By hypothesis $0 = \pi_1(Y) \cong H_1(Y;\mathbb{Z})$, thus $H_1(Y;k) \cong H_1(Y;Z) \otimes_{\mathbb{Z}} k \cong 0$, thus $H^1(Y;k) \cong H_1(Y;k)^* = 0$, as k is a field. Now apply 1.1. □

Next we consider the Sullivan fibre at ε, that is, the following pushout diagram of cdga's. By the commutativity of the previous diagram and the universal property of the pushout, there is a unique $\overline{m}$ making the diagram commutative.



Theorem 1.5 *Under the previous setup, the Sullivan fibre at ε is a Sullivan model for the fibre of p, that is,*

$$\overline{m} : (\Lambda V_F, \overline{d}) \xrightarrow{\simeq} A_{PL}(F)$$

is a quasi-isomorphism. Moreover, the Sullivan algebra $(A_{PL}(Y) \otimes \Lambda V_F, d)$ can be taken minimal, and in that case $\overline{m}$ is the minimal Sullivan model of F.

This was shown by Yuqing last time. There are two further results that we will use today:

Theorem 1.6 *Let $(\Lambda V_Y, d)$ be a Sullivan model for Y and let $(\Lambda V_F, \overline{d})$ be the minimal Sullivan algebra for F. Then X has a Sullivan model of the form $(\Lambda V_Y \otimes \Lambda V_F, d)$.*

Theorem 1.7 *Let $m_Y : (\Lambda V_Y, d) \longrightarrow A_{PL}(Y)$ be a Sullivan model for Y. Given a relative Sullivan algebra $(\Lambda V_Y \otimes \Lambda W, d)$ and a cdga morphism*

$$n : (\Lambda V_Y \otimes \Lambda W, d) \longrightarrow A_{PL}(X)$$

restricting to $p^* m_Y$ in $(\Lambda V_Y, d)$, then

(i) The map n induces a morphism $\overline{n} : (\Lambda W, d) \longrightarrow A_{PL}(F)$.

(ii) If $\overline{n}$ is a quasi-isomorphism, so is n, thus $(\Lambda V_Y \otimes \Lambda W, d)$ is a Sullivan model for X.

2 Examples of minimal Sullivan models

We will start with some easy examples where fibrations are not needed yet. I will denote by $H^\bullet(X;k)$ the cohomology ring of a topological space X with coefficients in k. This is a cdga with trivial differential and the cup product is the graded commutative product.

Moreover, I will use the following (useful) notation: if V is a graded vector space with basis $e_1, \ldots, e_n$, $|e_i| = r_i$, and differentials $de_1 = e_1 e_2, de_2 = \cdots$, I will write the exterior algebra of V as

$$(\Lambda V, d) = \Lambda(e_1, \ldots e_n; de_1 = e_1 e_2, de_2 = \cdots).$$

Example 2.1 (Spheres) In the course of Algebraic Topology 2 it is shown that $H^\bullet(S^n;k) \cong k[x]/(x^2)$, generated by a class x of degree n. In particular, since the cup product is graded commutative, we might as well write $H^\bullet(S^n;k) \cong \Lambda(x)/(x^2)$. More specifically, for n odd, we have $x \smile x = (-1)^{n^2} x \smile x = -x \smile x$, so $x \smile x = 0$ directly and there is no even need to kill x^2, that is, if n is odd then $H^\bullet(S^n;k) \cong \Lambda(x)$.

Recall from Jaco's talk that there is a quasi-isomorphism $C^\bullet(X;k) \simeq A_{PL}(X)$ inducing an isomorphism $H^\bullet(X;k) \cong H^\bullet(A_{PL}(X))$. Then let $x_n \in A_{PL}(S^n)^n$ be a representative of $x \in H^n(S^n;k)$.

- (Case n odd): There is a natural morphism

  $$m : (\Lambda(e), 0) \xrightarrow{\simeq} A_{PL}(S^n) \quad , \quad m(e) = x_n, \quad |e| = n,$$

  which happens to be a quasi-isomorphism trivially by the observation done above.

- (Case n even): Now $x_n^2 \in A_{PL}(S^n)^{2n}$ represents the 0 class in cohomology, thus it must be a coboundary, ie, there is $x_{2n-1} \in A_{PL}(S^n)^{2n-1}$ such that $dx_{2n-1} = x_n^2$. Now the claim is that the map

  $$m : \Lambda(e, e'; de' = e^2) \xrightarrow{\simeq} A_{PL}(S^n) \quad , \quad m(e) = x_n, m(e') = x_{2n-1}, \quad |e| = n, |e'| = 2n-1$$

  is a quasi-isomorphism. Indeed, the cochain complex looks like

  $$\overset{0}{k} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \overset{n}{\langle x_n \rangle} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \overset{2n-1}{\langle x_{2n-1} \rangle} \xrightarrow{d} \overset{2n}{\langle x_n^2 \rangle} \longrightarrow \cdots$$

  so it has only non vanishing cohomology in degrees 0 and n.

Example 2.2 (Complex projective spaces) Again, we take from Algebraic Topology 2 that $H^\bullet(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$ (with graded commutative product). Then

$$H^\bullet(\mathbb{CP}^n;k) \cong H^\bullet(\mathbb{CP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} k \cong \Lambda(x)/(x^{n+1}),$$

where $|x| = 2$.

Let $x_2 \in A_{PL}(\mathbb{CP}^n)^2$ be a representative of $x \in H^2(\mathbb{CP}^n;k)$. Then $x_2^{n+1} \in A_{PL}(\mathbb{CP}^n)^{2n+2}$ represents the 0 class in cohomology, thus it is a coboundary, ie, there is $x_{2n+1} \in A_{PL}(\mathbb{CP}^n)^{2n+1}$ such that $dx_{2n+1} = x_2^{n+1}$. As before, consider the map

$$m : \Lambda(e, e'; de' = e^{n+1}) \xrightarrow{\simeq} A_{PL}(S^n) \quad , \quad m(e) = x_2, m(e') = x_{2n+1}, \quad |e| = 2, |e'| = 2n+1.$$

A similar computation as before shows that this is a quasi-isomorphism and therefore a minimal Sullivan model for $\mathbb{CP}^n$.

Example 2.3 (Product of spaces) Let X, Y be path-connected topological spaces and suppose that the homology groups with coefficients in k of both X and Y are finite dimensional vector spaces.

Let $m_1 : (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$, $m_2 : (\Lambda W, d) \xrightarrow{\simeq} A_{PL}(Y)$ be the minimal Sullivan models for X and Y and let $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ be the projections. The chain of quasi-isomorphisms

$$(\Lambda V \otimes \Lambda W, d) \xrightarrow[\simeq]{m_1 \otimes m_2} A_{PL}(X) \otimes A_{PL}(Y) \xrightarrow[\simeq]{\pi_1^* \otimes \pi_2^*} A_{PL}(X \times Y)$$

shows that $(\Lambda V \otimes \Lambda W, d)$ is the minimal Sullivan model for $X \times Y$. The second one is also a quasi-isomorphism because the induced map in cohomology

$$H^\bullet(X;k) \otimes H^\bullet(Y;k) \xrightarrow{\cong} H^\bullet(X \times Y;k) \quad , \quad \alpha \otimes \beta \mapsto \pi_1^*\alpha \smile \pi_2^*\beta$$

is an isomorphism by the Künneth theorem (for fields), provided that the homology groups of both spaces are finite dimensional vector spaces.

Example 2.4 (Loop space of spheres) Let X be a space and consider $X^I := F(X, I)$ the space of continuous maps $I \longrightarrow X$, endowed with the compact-open topology. For $x_0 \in X$, let $P_{x_0}X := \{\sigma \in X^I : \sigma(0) = x_0\}$ and $\Omega_{x_0}X := \{\sigma \in X^I : \sigma(0) = \sigma(1) = x_0\}$, with the subspace topology. Then it is a fact that $P_{x_0}X$ is contractible and

$$p : P_{x_0}X \longrightarrow X \quad , \quad \sigma \mapsto \sigma(1)$$

is a Hurewicz fibration with fibre over x_0 $\Omega_{x_0}X$. We usually fix the point and drop it from the notation.

In the case of the spheres $S^n, n \geq 2$, we have a fibration sequence

$$\Omega S^n \hookrightarrow PS^n \xrightarrow{p} S^n.$$

We again distinguish between two cases:

- (Case n odd): By example 2.1, $m' : (\Lambda(e), 0) \xrightarrow{\simeq} A_{PL}(S^n)$ is a minimal Sullivan model for S^n. Now let

  $$m : \Lambda(e, u; du = e) \longrightarrow A_{PL}(PS^n) \quad , \quad m(e) = p^*m'(e), \quad m(u) = t$$

  with $|e| = n, |u| = n-1$ and $t \in A_{PL}(PS^n)^{n-1}$ any cochain such that $dt = p^*m'(e)$ (there exists as cocycles and coboundaries are the same, as PS^n is contractible). By inspection, m is a quasi-isomorphism . Therefore, by theorem 1.5, the minimal Sullivan model for ΩS^n is

  $$\overline{m} : (\Lambda(u), 0) \xrightarrow{\simeq} A_{PL}(\Omega S^n).$$

- (Case n even): In this case the minimal Sullivan model for S^n was

$$m' : (\Lambda(e,e'), de' = e^2) \xrightarrow{\simeq} A_{PL}(S^n) \quad , \quad m'(e) = x_n, m'(e') = x_{2n-1}, \quad |e| = n, |e'| = 2n-1.$$

Now define

$$m : \Lambda(e, e', u, u', du = e, du' = e' - eu) \xrightarrow{\simeq} A_{PL}(PS^n)$$

with

$$|e| = n \quad , \quad |e'| = 2n-1 \quad , \quad |u| = n-1 \quad , \quad |u'| = 2(n-1)$$

and

$$m(e) = p^* m'(e) \quad , \quad m(e') = p^* m'(e') \quad , \quad m(u) = t \quad , \quad m(u') = t'$$

where t is again a cochain such that $d(t) = p^* m'(e)$ and t' is a cochain such that $d(t') = p^* m'(e') - t \cdot p^* m'(e)$. After a painful checking, one sees that this is a quasi-isomorphism, and again by 1.5 we get that

$$\overline{m} : (\Lambda(u, u'), 0) \xrightarrow{\simeq} A_{PL}(\Omega S^n)$$

is the minimal Sullivan model for ΩS^n.

Observe that as corollary, we have just shown that for $n \geq 2$,

$$H^\bullet(\Omega S^n; k) \cong \begin{cases} \Lambda(u), \quad |u| = n-1 & n \text{ odd} \\ \Lambda(u, u'), \quad |u| = n-1, |u| = 2(n-1) & n \text{ even.} \end{cases}$$

Example 2.5 (Eilenberg-MacLane spaces) Let A be a finite generated abelian group and let $K(A,n)$ be *the* (up to weak homotopy equivalence) Eilenberg-MacLane space of type (A,n), $n \geq 1$. We will show that

$$m : (\Lambda H^n(K(A,n); k), 0) \xrightarrow{\simeq} A_{PL}(K(A,n))$$

is the minimal Sullivan model of $K(A,n)$. This implies that $H^\bullet(K(A,n); k)$ is the exterior algebra on $H^n(K(A,n); k)$ when n is odd and the polynomial algebra on $H^n(K(A,n); k)$ when n is even (just by degree reasons).

For our purpose, let $V := \mathrm{Hom}_{\mathrm{group}}(A, k)$. Then the first observation is that $H^n(K(A,n); k) \cong V$. Indeed, the Hurewicz theorem implies that $H_n(K(A,n); \mathbb{Z}) \cong A$, thus tensoring with k we get $H_n(K(A,n); k) \cong A \otimes_{\mathbb{Z}} k$, which is a vector space of finite dimension. Since we are working with a field, the dual of homology is cohomology, thus

$$H^n(K(A,n); k) \cong \mathrm{Hom}_{k-\mathrm{vs}}(H_n(K(A,n); k), k) \cong \mathrm{Hom}_{k-\mathrm{vs}}(A \otimes_{\mathbb{Z}} k, k) \cong \mathrm{Hom}_{\mathrm{group}}(A, k) = V.$$

Now we show the statement by induction: for $n = 1$, let $a_1, \ldots, a_r \in A$ represent a basis of $A \otimes_{\mathbb{Z}} k$, and consider the group homomorphism

$$\varphi : \mathbb{Z} \oplus \overset{r}{\cdots} \oplus \mathbb{Z} \longrightarrow A \quad , \quad \varphi(e_i) = a_i.$$

We need the following result from the general theory of Eilenberg-MacLane spaces:

Theorem 2.6 *Let $n \geq 1$ be an integer and let $\varphi : A' \longrightarrow A$ be a group homomorphism between abelian groups. Then there is a unique homotopy class of maps*

$$f : K(A', n) \longrightarrow K(A, n)$$

such that $f_* = \varphi$.

Applying this result to our previous morphism, we get a continuous map $f : K(\mathbb{Z}^r, 2) \longrightarrow K(A, 2)$ such that $f_* = \varphi$. Tensoring with k, we get that

$$f_* \otimes \mathrm{Id} = \varphi \otimes f : \pi_2(K(\mathbb{Z}^2, 2)) \otimes k \cong \mathbb{Z}^r \otimes k \longrightarrow A \otimes k \cong \pi_2(K(A, 2)) \otimes k$$

is an isomorphism of finite-dimensional vector spaces, since it maps basis to basis. At this point we need an extra ingredient:

Theorem 2.7 (Whitehead-Serre) *Let $f : X \longrightarrow Y$ be a map between simply connected spaces. Then the following are equivalent:*

(a) $f_* \otimes \mathrm{Id} : \pi_n(X) \otimes \mathbb{Q} \longrightarrow \pi_n(X) \otimes \mathbb{Q}$ *is an isomorphism for all n.*

(b) $f_* : H_n(X; \mathbb{Q}) \longrightarrow H_n(Y; \mathbb{Q})$ *is an isomorphism for all n.*

(c) $(\Omega f)_* : H_n(\Omega X; \mathbb{Q}) \longrightarrow H_n(\Omega Y; \mathbb{Q})$ *is an isomorphism for all n.*

Therefore, taking $k = \mathbb{Q}$, we get that

$$(\Omega f)_* : H_n(\Omega K(\mathbb{Z}^r, 2); \mathbb{Q}) \longrightarrow H_n(\Omega K(A, 2); \mathbb{Q})$$

is an isomorphism. But since $\Omega K(A, 2)$ is a $K(A, 1)$, we get after tensoring with k and dualizing that

$$H^\bullet(K(A, 1); k) \xrightarrow{\cong} H^\bullet(K(\mathbb{Z}^r, 1); k)$$

is an isomorphism. But $S^1 \times \overset{r}{\cdots} \times S^1$ is a $K(\mathbb{Z}^r, 1)$, and by Künneth

$$H^\bullet(S^1 \times \overset{r}{\cdots} \times S^1; k) \cong H^\bullet(S^1; k) \otimes \overset{r}{\cdots} \otimes H^\bullet(S^1; k) \cong \Lambda(x_1, \ldots, x_r)$$

with $|x_1| = 1$. The latter is therefore the minimal Sullivan model for a $K(A, 1)$.

For the general case, we first observe that there is a fibration sequence

$$K(A, n-1) \simeq \Omega K(A, n) \hookrightarrow PK(A, n) \longrightarrow K(A, n).$$

By induction, $(\Lambda V^{n-1}, 0)$, where $V^{n-1} = V = \mathrm{Hom}_{\mathrm{group}}(A, k)$, is the minimal Sullivan model for $K(A, n-1)$ (the superscript makes reference to the degree of its elements when viewed as a graded vector space). In particular, its homology groups are finite dimensional.

Let $(\Lambda E, d)$ be the minimal Sullivan model for $K(A, n)$. By theorem 1.7.(ii), we get a quasi-isomorphism

$$(\Lambda E \otimes \Lambda V^{n-1}, d) \xrightarrow{\simeq} A_{PL}(PK(A, n)).$$

We need one more technical result:

Lemma 2.8 *Let (B, d) be a cdga with $H^0(B) = k$ and let $m : (\Lambda V, d) \longrightarrow (B, d)$ be the minimal Sullivan model for (B, d). If $r > 0$ is the least integer such that $H^r(B) \neq 0$, then $V^i = 0$ for all $1 \leq i < r$.*

By Hurewicz, $H^i(K(A, n); k) = 0$ for all $1 \leq i < n$; and by the previous lemma, $E^i = 0$ for all $1 \leq i < n$, and by minimality the differential must be trivial in E^n. On the other hand, the above quasi-isomorphism yields $H^\bullet(\Lambda E \otimes \Lambda V^{n-1}) = k$, which means that $\Lambda E \otimes \Lambda V^{n-1} \cong \Lambda(E \oplus V^{n-1})$ is a contractible Sullivan algebra, that is, the differential induces an isomorphism $d : V^{n-1} \xrightarrow{\cong} E$ and $E = E^n \cong V = \mathrm{Hom}_{\mathrm{group}}(A, k)$ concentrated in degree n.

Example 2.9 (Rational homotopy type of Eilenberg-MacLane spaces) Let $[\sigma_n : S^n \longrightarrow K(\mathbb{Z}, n)]$ be a generator of $\pi_n(K(\mathbb{Z}, n)) \cong \mathbb{Z}$. By the naturality of the Hurewicz homomorphism we get a commutative diagram

$$\begin{array}{ccc} Z = \pi_n(S^n) & \xrightarrow[\cong]{(\sigma_n)_*} & \pi_n(K(\mathbb{Z},n)) = \mathbb{Z} \\ {\scriptstyle h_n}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle h_n} \\ H_n(S^n;\mathbb{Z}) & \xrightarrow{(\sigma_n)_*} & H_n(K(\mathbb{Z},n);\mathbb{Z}) \end{array}$$

so σ_n also induces isomorphism in n-th homology. After tensoring with $\mathbb{Q}$ and dualizing, we get that

$$\sigma_n^* : H^n(K(\mathbb{Z},n);\mathbb{Q}) \longrightarrow H^n(S^n;\mathbb{Q})$$

is an isomorphism as well. Moreover, for $n \geq 2$ consider the map $\Omega\sigma_n : \Omega S^n \longrightarrow \Omega K(\mathbb{Z},n) \simeq K(\mathbb{Z},n-1)$. By the naturality of the long exact sequence of the Serre fibration, we get that

$$(\Omega\sigma_n)_* : \pi_{n-1}(\Omega S^n) \longrightarrow \pi_{n-1}(K(\mathbb{Z},n-1))$$

is an isomorphism as well. Repeating the argument of the naturality of Hurewicz, we get that

$$(\Omega\sigma_n)^* : H^{n-1}(K(\mathbb{Z},n-1);\mathbb{Q}) \longrightarrow H^{n-1}(\Omega S^n;\mathbb{Q})$$

is also isomorphism. Lastly, observe that the computations done in examples 2.4 and 2.5 imply that

$$\sigma_{2n+1}^* : H^\bullet(K(\mathbb{Z},2n+1);\mathbb{Q}) \longrightarrow H^\bullet(S^{2n+1};\mathbb{Q})$$

and

$$(\Omega\sigma_{2n+1})^* : H^\bullet(K(\mathbb{Z},2n);\mathbb{Q}) \longrightarrow H^\bullet(\Omega S^{2n+1};\mathbb{Q})$$

are isomorphisms, and by the Whitehead-Serre theorem 2.7 we conclude that

$$\sigma_{2n+1} : S^{2n+1} \longrightarrow K(\mathbb{Z},2n+1) \qquad , \qquad \Omega\sigma_{2n+1} : \Omega S^{2n+1} \longrightarrow K(\mathbb{Z},2n)$$

are rational homotopy equivalences.

3 Sullivan model of the pullback of a fibration

For the last part of the talk, we will compute a Sullivan model for the pullback of a Serre fibration. Recall that given a diagram of spaces

$$X \xrightarrow{f} B \xleftarrow{p} E$$

the fibre product or pullback of this diagram is the space

$$E \times_B X := \{(e,x) \in E \times X : p(e) = f(x)\} \subset E \times X$$

endowed with the subspace topology; and it is the the pullback of this diagram in Top.

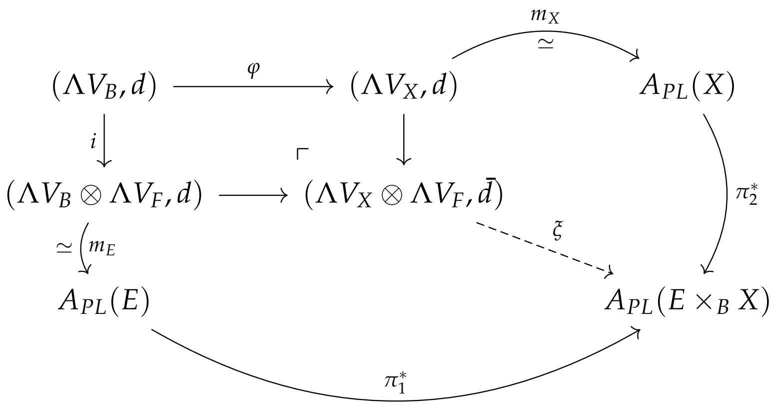
Theorem 3.1 *Let $p : E \longrightarrow B$ be a Serre fibration between a path-connected space E and a simply connected space B with fibre $F := p^{-1}(b_0)$ and $f : X \longrightarrow B$ a continuous map with X simply connected, where $x_0 \in X$ and $b_0 := f(x_0)$. Suppose that B or F have finite dimensional homology groups with coefficients in k, and consider the pullback diagram*

$$\begin{array}{ccc} E \times_B X & \xrightarrow{\pi_1} & E \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

Given a commutative diagram of cdga's

$$
\begin{array}{ccccc}
(\Lambda V_X, d) & \xleftarrow{\varphi} & (\Lambda V_B, d) & \xrightarrow{i} & (\Lambda V_B \otimes \Lambda V_F, d) \\
\simeq \downarrow m_X & & \simeq \downarrow m_B & & \simeq \downarrow m_E \\
A_{PL}(X) & \xleftarrow{f^*} & A_{PL}(B) & \xrightarrow{p^*} & A_{PL}(E)
\end{array}
$$

where $(\Lambda V_X, d)$ is a Sullivan model for X and $(\Lambda V_B, d)$ is a Sullivan model for B, the following pushout diagram of cdga's produces a unique quasi-isomorphism ξ



which is a Sullivan model for $E \times_B X$.

Proof. In first place, given the morphisms φ and i, we perform the pushout

$$(\Lambda V_X, d) \otimes_{(\Lambda V_B, d)} (\Lambda V_B \otimes \Lambda V_F, d) \cong (\Lambda V_X \otimes \Lambda V_F, \bar{d}).$$

The outer diagram of above commutes since

$$\pi_2^* m_X \varphi = \pi_2^* f^* m_B = \pi_1^* p^* m_B = \pi_1^* m_E i.$$

By the universal property of the pushout, there is a cdga map

$$\xi : (\Lambda V_X \otimes \Lambda V_F, \bar{d}) \longrightarrow A_{PL}(E \times_B X)$$

such that the above diagram commutes. Now the key observation is that π_1 restricts to a homeomorphism on fibres,

$$\pi_1 : \pi_2^{-1}(x_0) =: F' \xrightarrow{\cong} F = p^{-1}(b_0)$$

as $F' = \{(e, x_0) : p(e) = b_0\}$, so they have isomorphic Sullivan models. Applying theorem 1.7 we conclude that ξ is a quasi-isomorphism, as desired. □

Remark 3.2 The above theorem also holds under weaker hypothesis, where the starting diagram is not necessarily a pullback of a fibration but a commutative square with Serre fibrations as vertical maps. For further references see [1, Prop. 15.8]

Example 3.3 Next week, Kevin will use this theorem to compute a Sullivan model for the free loop space $X^{S^1} = F(S^1, X)$ of a simply connected space X.

References

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[2] HESS, K. Rational homotopy theory: a brief introduction, interactions between homotopy theory and algebra, 175-202. *Contemp. Math 436* (2007).

[3] SHI, Y. Relative sullivan algebra and models of fibrations.

4 Problems

Please send by e-mail to `j.becerragarrido@uu.nl`, or return 23rd April.

1. Compute the minimal Sullivan model of $S^1 \times S^{12} \times S^{123} \times S^{1234}$.

2. Let $\mathbb{H}$ be the division algebra of the quaternions. Consider $S^7 \subset \mathbb{H}^2$ and $S^4 \cong \mathbb{HP}^1$. The latter space is built in a similar fashion as $\mathbb{RP}^1$ or $\mathbb{CP}^1$. Let $p : S^7 \subset \mathbb{H}^2 \longrightarrow S^4 \cong \mathbb{HP}^1$, $p(u_1, u_2) := [u_1 : u_2]$. One can show that this map is a fibre bundle with fibre S^3, and it is one of the so-called *Hopf fibrations*.

   Compute a Sullivan model for the Hopf fibration $S^3 \hookrightarrow S^7 \longrightarrow S^4$.

3. Compute the cohomology ring $H^\bullet(\mathbb{RP}^\infty; k)$.

   (*Hint:* $\mathbb{RP}^\infty$ is an Eilenberg-MacLane space).

4. Check that

$$\Lambda(v_1, v_2, v_3; dv_1 = v_2v_3, dv_2 = v_3v_1, dv_3 = v_1v_2) \qquad , \qquad |v_1| = 1,$$

   is not a Sullivan algebra.

5. *(Bonus)* Consider $S^3 \subset \mathbb{C}^2$ and $S^2 \cong \mathbb{CP}^1$. Under this homeomorphism, the point at infinity corresponds to the north pole. Let $p : S^3 \subset \mathbb{C}^2 \longrightarrow S^2 \cong \mathbb{CP}^1$, $p(z_1, z_2) := [z_1 : z_2]$.

   (a) Show that the fibre at the point at infinite is S^1.

   This is other of the so-called Hopf fibrations, so we have a fibration sequence $S^1 \hookrightarrow S^3 \longrightarrow S^2$. Now let $f : S^1 \longrightarrow S^1$, $f(z) := z^n$ let $\Sigma f : \Sigma S^1 \cong S^2 \longrightarrow S^2 \cong \Sigma S^1$. One can show (e.g. Mayer- Vietoris) that this is also a map of degree n. Set

$$S^2 \times_{S^2} S^3 := \{(x, (z_1, z_2)) \in S^2 \times S^3 : (\Sigma f)(x) = [z_1 : z_2]\}.$$

   (b) Compute a Sullivan model for this space.