K-THEORY SEMINAR. LECTURE 8

JORGE BECERRA GARRIDO

16th April 2018

Proving algebra through topology

Even kindergarten kids (...) know that division is possible in $\mathbb{R}$. By division we understand that there is a multiplication and for every non-zero element $x \in \mathbb{R}$ there is another element $x^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$. Once at school one discovers the plane $\mathbb{R}^2$, the space $\mathbb{R}^3, \ldots$ and wonders if for these spaces one can also define a multiplication with the property that every non-zero element has an inverse. To ease the problem, instead of asking for a field structure (as in $\mathbb{R}$), we will not be so coarse and we will just ask for a *division ring* structure, that is, a field but multiplication might be neither commutative nor associative. Moreover, we also want to multiply by scalars $\lambda \in \mathbb{R}$ componentwise, giving rise to a *division algebra* over $\mathbb{R}$ instead.

Question. For what $n \in \mathbb{N}$ is there a division algebra structure on $\mathbb{R}^n$?

For $\mathbb{R}^2$, the answer seems easy: in the moment that we want to solve the equation $x^2 + 1 = 0$ we come up with a solution i which is not a real number, so we start to consider pairs $x + yi$, ie, pairs (x, y) on $\mathbb{R}^2$ with a multiplication

$$(x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

Observe that the neutral element is still $(1, 0)$. For $z = (x, y)$, setting $\bar{z} := (x, -y)$ one soon discovers that $z\bar{z} = (x^2 + y^2, 0)$ and therefore

$$(x, y)^{-1} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right)$$

whenever $(x, y) \neq (0, 0)$. By a simple computation one even realizes that such operation is associative and commutative, endowing $\mathbb{R}^2$ with a field structure, and working it out a bit more one sees that it is even algebraically closed.

Ok, small victory. What happens with $\mathbb{R}^3$? The Irish mathematician W.R. Hamilton (1805–1865) tried to answer this question. Although he did not succeed, on his way he came up with a division algebra structure on $\mathbb{R}^4$, which is called *quaternions* (denoted with $\mathbb{H}$ after him). Later J.T. Gravess and A. Cayley discovered independently a division algebra structure on $\mathbb{R}^8$, called *octonions* $\mathbb{O}$.

Are there more? Throughout these notes we will try to solve this question translating this algebraic problem to a topological problem. F.G. Frobenius showed in 1877 that $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only finite-dimensional associative division algebras over $\mathbb{R}$ with unit, with a algebraic proof. What happens if we drop the associativity? In 1964 F. Adams and M. Atiyah gave a very short proof of this fact using topological K-theory and the Adams operations. This was one major victory for the K-theory.

1 Division algebras

Let us make rigorous the description we did before:

Warning. In these notes all rings will be considered with (left and right) unit and not necessarily either associative or commutative. In other words, a ring $(R, +, \cdot)$ will be a set with two operations such that $(R, +)$ is an abelian group and $(R, \cdot)$ is a magma with unit, such that the product is distributive with respect to the sum.

Definition. A **division ring** R is a non-zero ring R without zero divisors except 0 and where every non-zero element is invertible. If R is also a k-algebra, we say that R is a **division algebra** (over k).

Observe that if the ring is associative, then the condition of not having zero divisors except 0 follows from the second property. It is clear that an associative, commutative division ring is a field. With more effort, one can show that every finite division ring is a field (Wedderburn's Theorem).

Something surprising about division rings is that geometry makes possible check if the associative or commutative property hold: if we consider $\mathbb{P}^2 := (R^3 - 0)/\sim$, , where R is a division ring, then we have:

$$R \textit{ is commutative} \iff \textit{Pappus' theorem holds in } \mathbb{P}^2,$$

$$R \textit{ is associative} \iff \textit{Desargues' theorem holds in } \mathbb{P}^2.$$

Examples 1.1 1. $\mathbb{R}$ and $\mathbb{C} \simeq \mathbb{R}^2$ are fields, thus division algebras.

2. **(Quaternions $\mathbb{H}$, Hamilton 1843)** Consider $\mathbb{R}^4$ with basis $\{1, i, j, k\}$ and define a product "$\cdot$" determined by the identities

$$i^2 = j^2 = k^2 = -1 \qquad , \qquad ijk = -1$$

and with unit 1 (so in general it is dropped). Every element can be expressed as $u = x + yi + zj + tk$, and if the conjugate of u is $\bar{u} := x - yi - jz - tk$, a simple computation yields $u\bar{u} = x^2 + y^2 + z^2 + t^2$, thus by calling $|u| := +\sqrt{u\bar{u}}$ we obtain that

$$u^{-1} = \frac{\bar{u}}{|u|^2} = \frac{x - yi - zk - tk}{x^2 + y^2 + z^2 + t^2}$$

for all non-zero u, then it is a division algebra. $\mathbb{R}^4$ endowed with this product is called the **quaternions** and it is denoted by $\mathbb{H}$. It is easy to check that the product is associative, but observe that it is not commutative: $ij = k, ji = -k$.

3. **(Octonions $\mathbb{O}$, Graves 1844; Cayley 1845)** Consider $\mathbb{R}^8 \simeq \mathbb{H}^2$ with a multiplication given by

$$(u_1, v_1)(u_2, v_2) := (u_1u_2 - \bar{v}_2v_1, v_2u_1 + v_1\bar{u}_2)$$

(Cayley-Dickson construction). For a more explicit description, consider $\{1, e_1, \ldots, e_7\}$ basis of $\mathbb{R}^8$, and define the product according to the following table:

e_ie_j	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_6	e_5	e_7	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_7	e_3	e_5	$-e_1$	e_5	$-e_4$	$-e_2$	-1

One can do exactly the same trick as before checking that for $\alpha = x_0 + x_1e_1 + \cdots + x_7e_7$ and $\bar{\alpha} = x_0 - x_1e_1 - \cdots - x_7e_7$ it holds $\alpha\bar{\alpha} = x_0^2 + \cdots + x_7^2$ and setting $|\alpha| = \sqrt{\alpha\bar{\alpha}}$

$$\alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|}.$$

Observe that this product is neither commutative ($e_1e_2 = e_4$, while $e_2e_1 = -e_4$) nor associative ($(e_1e_2)e_3 = e_4e_3 = -e_6$, while $e_1(e_2e_3) = e_1e_5 = e_6$).

So how can we face our question? The answer is "by looking at the spheres", and here is where topology comes into play.

2 H-spaces

Definition. An H**-space** (H after H.Hopf) is a pointed topological space (X, e) together with a continuous map

$$\mu : X \times X \longrightarrow X \qquad , \qquad \mu(x,y) \overset{\text{notation}}{=} xy$$

such that e acts as a unit, $xe = x = ex$ for all $x \in X$.

We could have weaken the definition by not letting e be a unit, but letting $X \xrightarrow{\cdot e} X$ and $X \xrightarrow{e\cdot} X$ be homotopic to the identity rel. $\{e\}$, or simply homotopic to the identity. For CW-complexes it can be shown that the three notions are equivalent.

Examples 2.1

1. Every topological group is an $H-$space, thus $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $(\mathbb{R} - 0, \cdot)$, $GL_n(\mathbb{R})$,... are $H-$spaces.

2. The division algebra structures described in 1.1 satisfy that the norm of an element coincides with the euclidean norm on the corresponding $\mathbb{R}^n$, thus the multiplication restricts to a map

$$\mathbb{S}^n \times \mathbb{S}^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{S}^n \subset \mathbb{R}^n$$

because in all cases $|uv| = |u||v|$. Therefore

$$\mathbb{S}^0 \subset \mathbb{R} \quad , \quad \mathbb{S}^1 \subset \mathbb{C} \quad , \quad \mathbb{S}^3 \subset \mathbb{H} \quad , \quad \mathbb{S}^7 \subset \mathbb{O}$$

are H-spaces. The unit elements are the same as in $\mathbb{R}^n$, since they lie in its corresponding sphere. Even more, we see that $\mathbb{S}^0$ and $\mathbb{S}^1$ are in particular abelian topological groups, $\mathbb{S}^3$ is also a topological group (but non-commutative), but $\mathbb{S}^7$ is not, since the operation is not associative.

With the same argument we prove

Lemma 2.2 *If $\mathbb{R}^n$ is a division algebra, then $\mathbb{S}^{n-1}$ is an H-space.*

Proof. We just have to consider the map

$$\begin{aligned}\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{S}^{n-1} \\ (x, y) &\longmapsto \frac{xy}{|xy|},\end{aligned}$$

($|\cdot|$ represents the euclidean norm on $\mathbb{R}^n$) which is continuous and well-defined because every division algebra is a domain. If e is the unit element of $\mathbb{R}^n$, then $e/|e| \in \mathbb{S}^{n-1}$ is the unit of the H-space, since

$$\mu\left(x, \frac{e}{|e|}\right) = \frac{x\frac{e}{|e|}}{\left|x\frac{e}{|e|}\right|} = \frac{xe}{|xe|} = x$$

□

This naive lemma is more powerful than it seems, since translates our original algebraic problem to a topological one. In particular, our machinery on K-theory allows us immediately to discard all $\mathbb{R}^n$ with $n > 1$ odd (and later on will give us the full solution using Adams operations):

Proposition 2.3 *The even spheres $\mathbb{S}^{2k}$ cannot be H-spaces, $k > 0$.*

Proof. Suppose $\mathbb{S}^{2k}$ is an H-space with multiplication $\mu : \mathbb{S}^{2k} \times \mathbb{S}^{2k} \longrightarrow \mathbb{S}^{2k}$ and unit element e. Such a map induces a ring homomorphism in K-theory

$$\mu^* : K(\mathbb{S}^{2k}) = \frac{\mathbb{Z}[\gamma]}{(\gamma^2)} \longrightarrow \frac{\mathbb{Z}[\alpha, \beta]}{(\alpha^2, \beta^2)} = K(\mathbb{S}^{2k} \times \mathbb{S}^{2k})$$

since by Bott periodicity and the product theorem $K(\mathbb{S}^{2k} \times \mathbb{S}^{2k}) \simeq K(\mathbb{S}^{2k}) \otimes K(\mathbb{S}^{2k}) \simeq \mathbb{Z}[\alpha]/(\alpha^2) \otimes \mathbb{Z}[\beta]/(\beta^2) \simeq \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$. Let us see how this map acts: consider the inclusions

$$i_1 : \mathbb{S}^{2k} \hookrightarrow \mathbb{S}^{2k} \times \mathbb{S}^{2k}, x \mapsto (x, e) \quad \text{and} \quad i_2 : \mathbb{S}^{2k} \hookrightarrow \mathbb{S}^{2k} \times \mathbb{S}^{2k}, x \mapsto (e, x).$$

Note that $\mu \circ i_1 = \mathrm{Id} = \mu \circ i_2$, so they also induce the identity in K-theory. But, taking $\{1, \alpha, \beta, \alpha\beta\}$ an additive basis of $K(\mathbb{S}^{2k} \times \mathbb{S}^{2k})$, and observing that $i_1^*(\alpha) = \gamma$, $i_1^*(\beta) = 0$, $i_2^*(\alpha) = 0$, $i_2^*(\beta) = \alpha$, we determine

$$\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$$

for some $m \in \mathbb{Z}$. But this would imply that

$$0 = \mu^*(0) = \mu^*(\gamma^2) = \mu^*(\gamma)^2 = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta,$$

what cannot be. □

3 The Hopf invariant

In general, it is quite hard to give not nullhomotopic maps between spheres (in the end, to compute $\pi_i(\mathbb{S}^m)$). For instance, every map $\mathbb{S}^i \longrightarrow \mathbb{S}^1$ is nullhomotopic[1] for $i \geq 2$.

For our purpose, let us take $n \geq 2$ even and a map $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ that we will think as an attaching map for $X := \mathbb{S}^n \cup_f \mathbb{D}^{2n}$, which we can consider as a CW-complex with 1 0-cell, 1 n-cell and 1 $2n$-cell,

$$X_0 = * \subset X_n = \mathbb{S}^n \subset X_{2n} = X.$$

The inclusion of $\mathbb{S}^n$ on X and the collapse of the n-skeleton

$$\mathbb{S}^n \overset{i}{\hookrightarrow} X \overset{\pi}{\longrightarrow} X/\mathbb{S}^n = \mathbb{S}^{2n}$$

induce the long exact sequence of the pair $(X, \mathbb{S}^n)$,

$$\longrightarrow \widetilde{K^1}(\mathbb{S}^{2n}) \xrightarrow{\pi^*} \widetilde{K^1}(X) \xrightarrow{i^*} \widetilde{K^1}(\mathbb{S}^n) \longrightarrow \widetilde{K^0}(\mathbb{S}^{2n}) \xrightarrow{\pi^*} \widetilde{K^0}(X) \xrightarrow{i^*} \widetilde{K^0}(\mathbb{S}^n) \longrightarrow$$

Since $\widetilde{K^1}(\mathbb{S}^{2n}) = \widetilde{K}(\mathbb{S}^{2n+1}) = 0$ and $\widetilde{K^1}(\mathbb{S}^n) = \widetilde{K}(\mathbb{S}^{n+1}) = 0$, by exactness $\widetilde{K^1}(X) = 0$, so it produces a short exact sequence

$$0 \longrightarrow \widetilde{K}(\mathbb{S}^{2n}) = \mathbb{Z} \overset{\pi^*}{\longrightarrow} \widetilde{K}(X) \overset{i^*}{\longrightarrow} \widetilde{K}(\mathbb{S}^n) = \mathbb{Z} \longrightarrow 0.$$

Choose x the generator of $\widetilde{K}(\mathbb{S}^{2n})$ which is the n-fold product of the generator of $\widetilde{K}(\mathbb{S}^2)$; and choose y the generator of $\widetilde{K}(\mathbb{S}^n)$ which is the $n/2$-fold product of the generator of $\widetilde{K}(\mathbb{S}^2)$. Since i^* is surjective, let $a \in \widetilde{K}(X)$ be a lift of y, $i^*a = y$; and let $b := \pi^*x$.

Since $\widetilde{K}(\mathbb{S}^n) = \mathbb{Z}$ is a free $\mathbb{Z}$-module (abelian group), the previous sequence splits,

$$\begin{aligned} \widetilde{K}(\mathbb{S}^{2n}) \oplus \widetilde{K}(\mathbb{S}^n) &\xrightarrow{\simeq} \widetilde{K}(X) \\ (\alpha, \beta) &\longmapsto \pi^*\alpha + s\beta, \end{aligned}$$

where $s : \widetilde{K}(\mathbb{S}^n) \longrightarrow \widetilde{K}(X)$ is a section of i^*. Therefore a and b generate $\widetilde{K}(X)$.

Now, since the ring structure of $\widetilde{K}(\mathbb{S}^n)$ is trivial, in particular $y^2 = 0$, so $a^2 \in \operatorname{Ker} i^* = \operatorname{Im} \pi^*$, and there exists $k \in \mathbb{Z}$ such that $a^2 = \pi^*(kx) = kb$.

Lemma 3.1 *The previous integer k is well-defined.*

[1] It follows from covering theory. With more generality, $\pi_i(X) = 0$, $i \geq 2$, whenever X has a contractible universal covering.

Proof. We need to show that it does not depend on the choice of the lift a of y. If $\bar{a}$ is another lift of y, then $\bar{a} - a \in \operatorname{Ker} i^* = \operatorname{Im} \pi^*$, meaning that there is an integer $m \in \mathbb{Z}$ such that $\bar{a} - a = \pi^*(mx) = mb$, so $\bar{a} = a + mb$. We will show that $\bar{a}^2 = a^2$ and therefore the aforementioned coefficient will be the same for both. We have that

$$\bar{a}^2 = (a + mb)^2 = a^2 + m^2 b^2 + 2mab.$$

Now, $b^2 = 0$ because $b^2 = (\pi^* x)^2 = \pi^* x^2 = 0$. To see that $ab = 0$, we argue as follows: $i^*(ab) = (i^*a)(i^*b) = (i^*a)(i^*\pi^*x) = 0$, so $ab \in \operatorname{Ker} i^* = \operatorname{Im} \pi^*$ and $ab = rb$ for some $r \in \mathbb{Z}$. Multiplying by a we get $rba = aba = a^2 b = kb^2 = 0$, what implies that $ab = 0$, because $ab \in \operatorname{Im} \pi^*$ which is a free $\mathbb{Z}$-submodule and in particular free torsion. $\square$

Definition. The previous integer k is called the **Hopf invariant** of $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$, and it is denoted as $h(f)$.

Example 3.2 Consider $\mathbb{CP}^2$ with the CW-structure given by attaching one cell in dimensions 0, 2 and 4,

$$X_0 = * \hookrightarrow X_2 = \mathbb{CP}^1 = \mathbb{S}^2 \hookrightarrow X_4 = \mathbb{CP}^2.$$

Consider the attaching map for the 4-cell, $\eta : \partial \mathbb{D}^4 = \mathbb{S}^3 \longrightarrow \mathbb{S}^2$ (usually called the **Hopf map**). It is possible to give an explicit description of such a map using homogeneous coordinates: think of $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$, $\mathbb{S}^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$. Then the Hopf map is given by

$$\eta : \mathbb{S}^3 \subset \mathbb{C}^2 \longrightarrow \mathbb{S}^2 = \mathbb{CP}^1 \qquad , \qquad (z_0, z_1) \mapsto [z_0 : z_1].$$

To compute the Hopf invariant of η, we rewrite the previous short exact sequence,

$$0 \longrightarrow \widetilde{K}(\mathbb{S}^4) = \mathbb{Z} \xrightarrow{\pi^*} \widetilde{K}(\mathbb{CP}^2) \xrightarrow{i^*} \widetilde{K}(\mathbb{S}^2) = \mathbb{Z} \longrightarrow 0.$$

Now the generators of $\widetilde{K}(\mathbb{S}^4)$ and $\widetilde{K}(\mathbb{S}^2)$ are $\alpha * \alpha = x$ and α, where $\alpha = H - 1$. By our later discussion in A.2, we see that $\widetilde{K}(\mathbb{CP}^2) = \mathbb{Z}a \oplus \mathbb{Z}a^2$, where $i^* a = \alpha$, and $\pi^*(x) = a^2$. Therefore $h(\eta) = 1$.

The following result relates the Hopf invariant with our problem:

Proposition 3.3 *Let $n \geq 2$ be even. If $\mathbb{S}^{n-1}$ has an H-space structure, then there exists a map $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ with Hopf invariant ± 1.*

Proof. Let us see in first place how to construct such a map from an H-space structure $\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$. View $\mathbb{S}^{2n-1}$ and $\mathbb{S}^n$ as

$$\mathbb{S}^{2n-1} = \partial \mathbb{D}^{2n} = \partial(\mathbb{D}^n \times \mathbb{D}^n) = \partial \mathbb{D}^n \times \mathbb{D}^n \cup \mathbb{D}^n \times \partial \mathbb{D}^n$$

and

$$\mathbb{S}^n = \mathbb{S}^{n-1} \cup_{\partial \mathbb{D}^n_+ \amalg \partial \mathbb{D}^n_-} \mathbb{D}^n_+ \amalg \mathbb{D}^n_-,$$

respectively. Now define

$$f : \mathbb{S}^{2n-1} = \partial \mathbb{D}^n \times \mathbb{D}^n \cup \mathbb{D}^n \times \partial \mathbb{D}^n \longrightarrow \mathbb{S}^n$$

as

$$f(x,y) := \begin{cases} |y|\mu\left(x, \frac{y}{|y|}\right) \in \mathbb{D}^n_+, & (x,y) \in \partial\mathbb{D}^n \times \mathbb{D}^n \\ |x|\mu\left(\frac{x}{|x|}, y\right) \in \mathbb{D}^n_-, & (x,y) \in \mathbb{D}^n \times \partial\mathbb{D}^n \end{cases}$$

Note that this map is well-defined and it is continuous, even when $x = 0$ or $y = 0$, and f coincides with the product μ on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.

The goal is to show that $h(f) = \pm 1$. For this, let $e \in \mathbb{S}^{n-1}$ be the unit for the H-space multiplication and as before get $X = \mathbb{S}^n \cup_f \mathbb{D}^{2n}$ and a characteristic map for the $2n$-cell viewed as a map of pairs,

$$\chi : (\mathbb{D}^n \times \mathbb{D}^n, \partial(\mathbb{D}^n \times \mathbb{D}^n)) \longrightarrow (X, \mathbb{S}^n).$$

We consider now the following diagram,where the horizontal maps are the product maps:

$$\begin{array}{ccc}
\widetilde{K}(X) \otimes \widetilde{K}(X) & \xrightarrow{\mu} & \widetilde{K}(X) \\
\uparrow \wr & & \uparrow \pi^* \\
\widetilde{K}(X, \mathbb{D}^n_-) \otimes \widetilde{K}(X, \mathbb{D}^n_+) & \xrightarrow{\mu} & \widetilde{K}(X, \mathbb{S}^n) \\
\varphi \downarrow & & \chi^* \downarrow \wr \\
\widetilde{K}(\{e\} \times \mathbb{D}^n, \{e\} \times \partial\mathbb{D}^n) \otimes \widetilde{K}(\mathbb{D}^n \times \{e\}, \partial\mathbb{D}^n \times \{e\}) & \xrightarrow{\simeq} & \widetilde{K}(\mathbb{D}^n \times \mathbb{D}^n, \partial(\mathbb{D}^n \times \mathbb{D}^n)) \\
\uparrow \wr & & \uparrow \wr \\
\widetilde{K}(\mathbb{S}^n) \otimes \widetilde{K}(\mathbb{S}^n) & \xrightarrow{\simeq} & \widetilde{K}(\mathbb{S}^{2n})
\end{array}$$

The upper left arrow is isomorphism because the disks are contractible; the product in the second line has as target $\widetilde{K}(X, \mathbb{S}^n)$ because $X/\mathbb{D}^n_- \wedge X/\mathbb{D}^n_+ = X/(\mathbb{D}^n_- \cup \mathbb{D}^n_+) = X/\mathbb{S}^n$; χ^* is isomorphism because the quotients are homeomorphic; the lowest horizontal map is isomorphism by Bott periodicity; and the map φ is induced by the composite (maps of pairs)

$$(\mathbb{D}^n \times \{e\}, \partial\mathbb{D}^n \times \{e\}) \hookrightarrow (\mathbb{D}^n \times \mathbb{D}^n, \partial\mathbb{D}^n \times \mathbb{D}^n) \xrightarrow{\chi} (X, \mathbb{D}^n_+).$$

(the other factor is similar). Since χ restricts to a homeomorphism from $\mathbb{D}^n \times \{e\}$ onto $\mathbb{D}^n_-$, the element $a \in \widetilde{K}(X)$ is mapped to a generator of $\widetilde{K}(\mathbb{S}^{2n})$, and therefore the element $a \otimes a$ in the upper left group maps to a generator in the lower left group, which we can take to a generator of $\widetilde{K}(X, \mathbb{S}^n)$, and then it maps to $\pm b$ through π^*. By the commutativity of the diagram, $a \otimes a$ maps to $\pm b$ through the first horizontal map, so we get that $a^2 = \pm b$; that is, $h(f) = \pm 1$. □

The upshot is that the existence of H-space structures on the spheres (and therefore division algebra structures on $\mathbb{R}^n$) are conditioned by the existence of maps $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ with Hopf invariant ± 1. The following theorem finally solves our original question, since it points for what values of n there exists such a map. It was firstly proven by Adams in 1960 using secondary cohomology operations in singular cohomology, but here we will present a much simpler proof making use of K-theory and the Adams operations (primary cohomology operations in K-theory).

Theorem 3.4 (Adams-Atiyah, 1964) *Let $n \geq 2$ be even. If there exists a map $f : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$ with Hopf invariant ± 1, then $n = 2, 4, 8$.*

Proof. Let us write $n = 2m$, $m \in \mathbb{N}$, so $f : \mathbb{S}^{4m-1} \longrightarrow \mathbb{S}^{2m}$, and have Thomas' notes on hand. As before, denote by $i : \mathbb{S}^{2m} \longrightarrow X$ and $\pi : X \longrightarrow \mathbb{S}^{4m}$ the inclusion and collapse of the $2m$-skeleton in $X = \mathbb{S}^{2m} \cup_f \mathbb{D}^{4m}$. We showed that $\psi^k : \widetilde{K}(\mathbb{S}^{2m}) \longrightarrow \widetilde{K}(\mathbb{S}^{2m})$ is the morphism "multiplication by k^m" and by the naturality of the Adams operations we see that the diagram

$$\begin{array}{ccc} \widetilde{K}(X) & \xrightarrow{i^*} & \widetilde{K}(\mathbb{S}^{2m}) \\ {\scriptstyle \psi^k}\downarrow & & \downarrow{\scriptstyle \psi^k = \cdot k^m} \\ \widetilde{K}(X) & \xrightarrow{i^*} & \widetilde{K}(\mathbb{S}^{2m}) \end{array}$$

commutes, what says that $\psi^k(a) = k^m a + r_k b$ for some $r_k \in \mathbb{Z}$. Similarly, π gives rise to

$$\begin{array}{ccc} \widetilde{K}(\mathbb{S}^{4m}) & \xrightarrow{\pi^*} & \widetilde{K}(X) \\ {\scriptstyle \psi^k = \cdot k^{2m}}\downarrow & & \downarrow{\scriptstyle \psi^k} \\ \widetilde{K}(\mathbb{S}^{4m}) & \xrightarrow{\pi^*} & \widetilde{K}(X) \end{array}$$

implying that $\psi^k(b) = k^{2m} b$.

Specializing for $k = 2$ prime, and using (iv) in Theorem 1 we see that

$$r_2 b \equiv 2^m a + r_2 b = \psi^2(a) \equiv a^2 = h(f)b = \pm b \pmod{2},$$

so r_2 must be odd.

In general for k odd, property (iii) ensures that

$$\psi^k \psi^2(a) = \psi^{k+2}(a) = \psi^2 \psi^k(a).$$

The first term is

$$\psi^k \psi^2(a) = \psi^k(2^m a + r_k b) = k^m 2^m a + (2^m r_k + k^{2m} r_2)b,$$

and the third one

$$\psi^2 \psi^k(a) = \psi^2(k^m a + r_k b) = 2^m k^m a + (k^m r_2 + 2^{2m} r_k)b$$

and since $\widetilde{K}(X) = \mathbb{Z}a \oplus \mathbb{Z}b$ is a free $\mathbb{Z}$-module, the coefficients must coincide. For b this yields

$$2^m(2^m - 1)r_k = (k^m - 1)k^m r_2.$$

This ensures that 2^m divides $(k^m - 1)(k^m r_2)$, but $(k^m r_2)$ is odd, thus it divides $k^m - 1$. Specializing again for $k = 3$ we conclude by the following arithmetic lemma. □

Lemma 3.5 *If 2^m divides $3^m - 1$, then $m = 1, 2, 4$.*

Proof. We start by writing $m = 2^\ell k$, with k odd.

Claim. The highest power of 2 dividing $3^m - 1$ is

$$\begin{cases} 2, & \ell = 0, \\ 2^{\ell+2}, & \ell > 0. \end{cases}.$$

Observe that with the claim we conclude, since it implies that $m \leq \ell + 2$, so $2^\ell \leq 2^\ell k = m \leq \ell + 2$, so it has to be $\ell \leq 2$ and finally $m \leq 4$. Now the four possible cases are checked individually: for $m = 1$, 2 divides 2; for $m = 2$, 4 divides 8; for $m = 3$, 8 does not divide 26; and for $m = 4$, 16 divides 80.

Proof of the claim. We argue by induction on ℓ:

$\ell = 0$ (ie, $m = k$ odd): Since $3 \equiv -1 \pmod 4$, $3^k \equiv (-1)^k = -1 \pmod 4$ and then $3^k - 1 \equiv -2 \equiv 2 \pmod 4$. Therefore the highest power of 2 dividing $3^k - 1$ is 2.

$\ell = 1$ (ie, $m = 2k$, $k = 2p+1$ odd): Write $3^m - 1 = 3^{2k} - 1 = (3^k - 1)(3^k + 1)$. Now note that since $3^2 \equiv 1 \pmod 8$, then $3^{2p} \equiv 1 \pmod 8$, and multiplying by 3 we get $3^k \equiv 3 \pmod 8$ and finally $3^k + 1 \equiv 4 \pmod 8$. We see that, as before, the highest power of 2 diving $3^k - 1$ is 2, and now the highest power dividing $3^k + 1$ is 4; so $8 = 2^{1+2}$ is the highest power of 2 dividing $3^{2k} - 1$.

General case: Suppose that the claim is true for $m = 2^\ell k$, $\ell \geq 1$ and therefore m even. Then for $2^{\ell+1} k = 2m$ we get $3^{2m} - 1 = (3^m - 1)(3^m + 1)$. By the induction hypothesis, the highest power of 2 dividing the first factor is $2^{\ell+2}$, and since m is even, $3 \equiv -1 \pmod 4$ so $3^m \equiv 1 \pmod 4$ and $3^m + 1 \equiv 2 \pmod 4$; what says that the highest power of 2 for the second factor is 2, what ends the proof. $\square$

The latter theorem, together with 3.3, 2.2 and 2.3 conclude that

$$\mathbb{R}^n \text{ is a division algebra} \iff n = 1, 2, 4, 8$$

provided that at least we find one structure on each of these cases (namely, $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$).

4 Parallelizable spheres

The strong result showed in the previous section also allows us to answer a question coming naturally from differential geometry:

Definition. A smooth manifold M is said to be **parallelizable** if there exists a global basis of vector fields, ie, if $\mathfrak{X}(M)$ is a free $\mathcal{C}^\infty(M)$-module of rank $m = \dim M$.

Here $\mathfrak{X}(M)$ denotes the $\mathcal{C}^\infty(M)$-module of vector fields over M. Obviously, this is equivalent to say that the tangent bundle TM is trivializable, that is, diffeomorphic to $M \times \mathbb{R}^m$.

Question. What spheres are parallerizable?

This time we will not defer the solution for so long. We have:

Lemma 4.1 *If $\mathbb{S}^{n-1}$ is parallelizable, then it is an H-space.*

Proof. Let $\{X_1, \dots X_{n-1}\}$ be a global basis of vector fields on $\mathbb{S}^{n-1}$. Then at every point $p \in \mathbb{S}^{n-1}$ we have a basis $\{(X_1)_p, \dots (X_{n-1})_p\}$ of $T_p\mathbb{S}^{n-1}$, and considering the sphere as submanifold of $\mathbb{R}^n$, we see that $\{p, (X_1)_p, \dots (X_{n-1})_p\}$ is basis of $\mathbb{R}^n$, because p is orthogonal to $T_p\mathbb{S}^{n-1}$. Now we can apply the Gram-Schmidt orthonormalization process to that basis to obtain an orthonormal basis of $\mathbb{R}^n$ for every $p \in \mathbb{S}^{n-1}$. Since the process is described in terms of polynomials and non-zero divisions, we obtain an orthonormal basis of vector fields on the sphere (that we will still denote $\{X_1, \dots X_{n-1}\}$).

Let $e_1 = (1, 0, \dots, 0) \in \mathbb{S}^{n-1}$ and consider

$$\{e_1, e_2 := (X_1)_{e_1}, \dots e_n := (X_{n-1})_{e_1}\}$$

an orthonormal basis of $\mathbb{R}^n$. For every point $p \in \mathbb{S}^{n-1}$, let ϕ_p be the unique isometry[2] which takes the basis $\{e_1, \dots, e_n\}$ to $\{p, (X_1)_p, \dots (X_{n-1})_p\}$. Now consider the map

$$\begin{aligned} \mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{S}^{n-1} \\ (x, y) &\longmapsto \mu(x, y) := \phi_x(y), \end{aligned}$$

which is continuous because ϕ_x is a linear isomorphism for all $x \in \mathbb{S}^{n-1}$ and the vector fields are continuous. This maps defines the desired H-space structure with neutral element e_1, since $\mu(x, e_1) = \phi_x(e_1) = x$ and $\mu(e_1, x) = \phi_{e_1}(x) = \mathrm{Id}(x) = x$. □

Because of this, we had better only look at $n = 1, 2, 4, 8$. It happens that there are global basis of vector fields for these spheres. For that one can either explicitly give such vector fields or with more elegance show that if $\mathbb{R}^n$ is a division algebra then $\mathbb{S}^{n-1}$ is parallelizable, using a similar argument to 4.1.

Now we know all spheres which have a global basis of vector fields, it is natural to ask: in the rest of spheres, how many linearly independent vector fields can we get at most? The answer is a mixture of linear algebra, which constructs them; and K-theory, which ensures that there are not more (although the proof is beyond the scope of these notes).

Theorem 4.2 (Hurwitz, Radon, Eckmann; Adams 1961) *Let $m = 2^\ell k$, with k odd, and $\ell = 4b + c$, $0 \le c \le 3$, and let $\rho(m) := 8b + 2^c$ be the **Hurwitz-Radon number**.*

On $\mathbb{S}^{m-1}$, there exist $\rho(m) - 1$ linearly independent vector fields (Hurwitz, Radon, Eckmann) and no more (Adams).

ℓ	0	1	2	3	4	5	6	7
2^ℓ	1	2	4	8	16	32	64	128
$\rho(m)$	1	2	4	8	9	10	12	16

We gather all results we proved in these notes in the following

Corollary 4.3 *The following statements occur only for $n = 1, 2, 4, 8$:*

1. *$\mathbb{R}^n$ is a division algebra.*

[2]Observe that in particular $\phi_p \in SO(\mathbb{R}^n)$: the construction gives rise to a continuous map $\mathbb{S}^{n-1} \longrightarrow O(\mathbb{R}^n), p \mapsto \phi_p$, and we conclude because $\mathbb{S}^{n-1}$ is connected and $\phi_{e_1} = \mathrm{Id}$ is in the connected component $SO(\mathbb{R}^n)$.

2. $\mathbb{S}^{n-1}$ *is an* H*-space.*

3. $\mathbb{S}^{n-1}$ *is parallelizable.*

4. $\pi_{2n-1}(\mathbb{S}^n)$ *contains an element with Hopf invariant* ± 1 *(although not for* $n = 1$ *with the definition we gave).*

A The K-theory of $\mathbb{CP}^n$

Let us make explicit the K-theory of the complex projective space. In Lecture 6 Matthijs showed, using the long exact sequence and induction, that the reduced K-theory of $\mathbb{CP}^n$ is given by

$$\widetilde{K^0}(\mathbb{CP}^n) = \mathbb{Z}^n \qquad , \qquad \widetilde{K^1}(\mathbb{CP}^n) = 0,$$

because $\mathbb{CP}^n$ has only cells in even dimensions. To be able to show the ring structure of $K(\mathbb{CP}^n)$ we need to develop higher unreduced K-theory groups.

Let X be a topological space and denote by $X_+ := X \amalg +$, where $+$ is a disjoint basepoint.

Definition. The **higher unreduced K-theory groups** of X are

$$K^n(X) := \widetilde{K^n}(X_+).$$

For a pair of spaces (X, A) define

$$K^n(X, A) := \widetilde{K^n}(X, A).$$

Note that this definition is consistent with our previous theory:

Lemma A.1 $K^0(X) = K(X)$, $K^1(X) = \widetilde{K^1}(X)$, *and in particular the six-term long exact sequence is valid for unreduced groups.*

Proof. For $n = 0$ we have

$$K^0(X) = \widetilde{K^0}(X_+) = \widetilde{K}(X_+) = \mathrm{Ker}(K(X_+) \longrightarrow K(+)) = K(X),$$

and for $n = 1$

$$K^1(X) = \widetilde{K^1}(X_+) = \widetilde{K}(SX_+) = \widetilde{K}(SX \vee \mathbb{S}^1) = \widetilde{K}(SX) \oplus \widetilde{K}(\mathbb{S}^1) = \widetilde{K^1}(X)$$

□

Theorem A.2 $K(\mathbb{CP}^n) = \mathbb{Z}[x]/(x^{n+1})$, *where* $x = L - 1$ *and* L *is the canonical line bundle over* $\mathbb{CP}^n$.

Proof. Using the computations of above and A.1, we see that the long exact sequence of the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ produces a short exact sequence

$$0 \longrightarrow K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \xrightarrow{\pi^*} K(\mathbb{CP}^n) \xrightarrow{i^*} K(\mathbb{CP}^{n-1}) \longrightarrow 0,$$

where as usual $i : \mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ and $\pi : \mathbb{CP}^n \longrightarrow \mathbb{CP}^n/\mathbb{CP}^{n-1} = \mathbb{S}^{2n}$.

Claim. $(L-1)^n$ generates $\mathrm{Ker}\, i^*$ for all n.

With this claim we can prove the theorem by induction: for $n = 1$, $\mathbb{CP}^1 = \mathbb{S}^2$ and the statement was a consequence of the product theorem. For the general case, suppose $K(\mathbb{CP}^{n-1}) = \mathbb{Z}[x]/(x^n)$. Since it is free as $\mathbb{Z}$-module, the previous short exact sequence splits, so it says that

$$\{1, (L-1), \dots, (L-1)^{n-1}, (L-1)^n\}$$

generates $K(\mathbb{CP}^n)$, by the induction hypothesis, the claim and the fact that $\operatorname{Ker} i^* = \operatorname{Im} \pi^* \simeq K(\mathbb{CP}^n, \mathbb{CP}^{n-1})$. But the claim for $n+1$ says that $(L-1)^{n+1} = 0$ in $K(\mathbb{CP}^n)$, so the result follows.

Proof of the claim. Note that the complex projective space can be viewed as the orbit space $\mathbb{S}^{2n+1}/\mathbb{S}^1$, where we consider $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$, and $\mathbb{S}^1$ acts under multiplication. Now set

$$\mathbb{S}^{2n-1} = \partial \mathbb{D}^{2n+2} = \partial(\mathbb{D}^2_0 \times \cdots \times \mathbb{D}^2_n) = \bigcup_i (\mathbb{D}^2_0 \times \cdots \times \partial \mathbb{D}^2_i \times \cdots \times \mathbb{D}^2_n) = \bigcup_i \partial_i \mathbb{D}^{2n+2}.$$

Call C_i to the orbit space of the factor $\partial_i \mathbb{D}^{2n+2}$, and observe that is homeomorphic to $\mathbb{D}^2_0 \times \cdots \times \widehat{\mathbb{D}^2_i} \times \cdots \times \mathbb{D}^2_n$, since the action identifies all points in $\partial \mathbb{D}^2_i$. Thus $\mathbb{CP}^n = \cup_i C_i$ where each C_i is homeomorphic to $\mathbb{D}^{2n}$. We again decompose C_0 as $C_0 = \cup_i \partial_i C_0$, with $\partial_i C_0 = \mathbb{D}^2_1 \times \cdots \times \partial \mathbb{D}^2_i \times \cdots \times \mathbb{D}^2_n$. Now the inclusions of pairs

$$(\mathbb{D}^2_i, \partial \mathbb{D}^2_i) \hookrightarrow (C_0, \partial_i C_0) \hookrightarrow (\mathbb{CP}^n, C_i)$$

induce the following commutative diagram:

$$\begin{array}{ccccc}
K(\mathbb{D}^2_1, \partial \mathbb{D}^2_1) \otimes \cdots \otimes K(\mathbb{D}^2_n, \partial \mathbb{D}^2_n) & \overset{\simeq}{\searrow} & & & \\
\uparrow \wr & & & & \\
K(C_0, \partial_1 C_0) \otimes \cdots \otimes K(C_0, \partial_n C_0) & \xrightarrow{\simeq} & K(C_0, \partial C_0) & & \\
\uparrow & & \downarrow \wr & & \\
K(\mathbb{CP}^n, C_1) \otimes \cdots \otimes K(\mathbb{CP}^n, C_n) & \longrightarrow & K(\mathbb{CP}^n, C_1 \cup \cdots \cup C_n) & \xrightarrow{\simeq} & K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \\
\downarrow & & \downarrow & \overset{\pi^*}{\swarrow} & \\
K(\mathbb{CP}^n) \otimes \cdots \otimes K(\mathbb{CP}^n) & \longrightarrow & K(\mathbb{CP}^n) & &
\end{array}$$

Here all maps from the first column to the second one are n-fold products. The upper diagonal map is isomorphism[3], the upper map in the second column is too because the inclusion $C_0 \longrightarrow \mathbb{CP}^n$ induces a homeomorphism $C_0/\partial C_0 = \mathbb{CP}^n/(C_0 \cup \cdots \cup C_n)$. Since $\mathbb{CP}^n$ sits in the last n coordinates of $\mathbb{C}^{n+1}$, it is disjoint from C_0, inducing a homotopy equivalence $\mathbb{CP}^n/\mathbb{CP}^{n-1} \longrightarrow \mathbb{CP}^n/(C_1 \cup \cdots \cup C_n)$, and therefore the isomorphism in K-theory.

Now: take $x_i \in K(\mathbb{CP}^n, C_i)$ a lift of $L-1 \in K(\mathbb{CP}^n)$. Such lift exists because the pair $(\mathbb{CP}^n, C_i)$ induces a short exact sequence

$$0 \longrightarrow K(\mathbb{CP}^n, *) \longrightarrow K(\mathbb{CP}^n) \xrightarrow{\dim} K(*) = \mathbb{Z} \longrightarrow 0,$$

[3] Here we use that there is a relative product $K^i(X, A) \otimes K^j(Y, B) \longrightarrow K^{i+j}(X \times Y, X \times B \cup A \times Y$ (similar to the situation in cohomology with respect the cup product), defined as the external product $\widetilde{K}(\Sigma^i(X/A) \otimes \widetilde{K}(\Sigma^j(Y/B) \longrightarrow \widetilde{K}(\Sigma^{i+j}(X/A \wedge Y/B)$ using the identification $X/A \wedge Y/B = (X \times Y)/(X \times B \cup A \times Y)$

since C_i is contractible. As $L-1$ is in the kernel of the map dim, it comes from an element x_i in $K(\mathbb{CP}^n, C_i)$. Now we just have to check that it maps to a generator of $K(\mathbb{D}^2_i, \partial\mathbb{D}^2_i)$, because in such a case by the commutativity of the diagram the product $x_1 \cdots x_n$ generates $K(\mathbb{CP}^n, C_1 \cup \cdots \cup C_n)$, and therefore $(L-1)^n$ generates the image of the lower right map, which is $\operatorname{Ker} i^*$ by exactness, and we are done.

To see that x_i maps to a generator of $K(\mathbb{D}^2_i, \partial\mathbb{D}^2_i)$, for $i = 1, \ldots, n$ consider the commutative diagram

$$\begin{array}{ccc} (\mathbb{D}^2_i, \partial\mathbb{D}^2_i) & \longrightarrow & (\mathbb{CP}^n, C_i) \\ \downarrow & \nearrow & \\ (\mathbb{CP}^1, \bar{C}_i) & & \end{array}$$

where $\mathbb{CP}^1$ is obtained from $\mathbb{D}^2_0 \times \mathbb{D}^2_1$ similarly, and $\bar{C}_1 = (\mathbb{D}^2_0 \times \partial\mathbb{D}^2_1)/\mathbb{S}^1$ (quotient by the action). Such a diagram induces the following one in K-theory:

$$\begin{array}{ccc} K(\mathbb{D}^2_i, \partial\mathbb{D}^2_i) & \longleftarrow & K(\mathbb{CP}^n, C_i) \\ \uparrow \wr & & \\ K(\mathbb{CP}^1, \bar{C}_i) & & \uparrow \wr \\ \uparrow \wr & & \\ \mathbb{Z} = K(\mathbb{CP}^1, *) & \longleftarrow & K(\mathbb{CP}^n, *) \\ \downarrow & & \downarrow \\ K(\mathbb{CP}^1) & \longleftarrow & K(\mathbb{CP}^n) \end{array}$$

If $x_i \in K(\mathbb{CP}^n, *) \simeq K(\mathbb{CP}^n, C_i)$ is a lift of $L-1 \in K(\mathbb{CP}^n)$, by the commutativity of the lower square we see that x_i maps to a generator of $K(\mathbb{CP}^1, *)$, because the lower horizontal map sends $L-1$ to a generator. Since the whole diagram commutes the upper horizontal arrow maps $x_i \in K(\mathbb{CP}^n, C_i)$ to a generator of $K(\mathbb{D}^2_i, \partial\mathbb{D}^2_i)$, as we wanted. □

References

[1] Banerjee, I. *The Hopf invariant one problem.* http://math.uchicago.edu/~may/REU2016/REUPapers/Banerjee.pdf.

[2] Blom, T. *Lecture notes for the K-Theory seminar. Lecture 3.*

[3] Blom, T. *Lecture notes for the K-Theory seminar. Lecture 7.*

[4] Hatcher, A. *Algebraic Topology.* 2001. https://www.math.cornell.edu/~hatcher/AT/AT.pdf.

[5] Hatcher, A. *Vector Bundles and K-Theory.* 2017. https://www.math.cornell.edu/~hatcher/VBKT/VB.pdf.

[6] Lip, M. *K-theory seminar. Lecture 6.*

[7] Miller, H. *Vector Fields on Spheres, etc.* https://chromotopy.org/latex/misc/haynes-notes/haynes-notes.pdf.

[8] Navarro González, J. A. *Notes for a Degree in Mathematics.* 2018. http://matematicas.unex.es/~navarro/degree.pdf.

[9] Quick, G. *Lecture notes for "Advanced Algebraic Topology".* https://folk.ntnu.no/gereonq/Math231br.html.

[10] Urhammer, E. *Real Division Algebras.* http://www.math.ku.dk/~moller/undervisning/aktuel/rap2/emil2.pdf.

PROBLEMS

1. Show that every real division algebra R of odd dimension has dimension 1 using only linear algebra. (*Hint:* Every endomorphism of a odd dimensional real vector space has a real eigenvalue).

2. Show that $\mathbb{R}^n$, with $n \geq 3$ odd, is not a division algebra over $\mathbb{R}$ by considering how the determinant of the linear map "multiplication by a", $0 \neq a \in \mathbb{R}^n$, varies as a moves along a path in $\mathbb{R}^n - 0$ joining two antipodal points. (*Hint:* Use the mapping degree.)

3. Show that $\mathbb{R}$ and $\mathbb{C}$ are the only finite-dimensional division algebras over $\mathbb{R}$ which are commutative and have identity as follows: for $n \geq 2$, suppose that $\mathbb{R}^n$ is a division algebra. Then use the map $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$, $f(x) = x \cdot x / |x \cdot x|$ to produce a homeomorphism between $\mathbb{RP}^{n-1}$ and $\mathbb{S}^{n-1}$ (which only occurs for $n = 2$). Then conclude that every 2-dimensional commutative division algebra with identity is isomorphic to $\mathbb{C}$. (*Hint:* Use Brower's Invariance of Domain Theorem for topological manifolds).

4. Show that if R is a associative division ring, then its center $Z(R) = \{a \in R : ar = ra \ \forall r \in R\}$ is a field. Compute $Z(\mathbb{H})$.

5. Let A be a ring. A A-module M is *simple* when it only has trivial submodules.

   Show that if M, N are A-modules, every non-zero A-module homomorphism $f : M \longrightarrow N$ is isomorphism, and $\text{End}_A(M)$ is a division ring.

6. Show that if the Pappus' theorem holds in $\mathbb{P}^2(R)$, where R is a division algebra, then R is commutative.

7. Show that the Hopf invariant does not depend on the homotopy class of the map and defines a group homomorphism $h : \pi_{2n-1}(\mathbb{S}^n) \longrightarrow \mathbb{Z}$. (*Hint:* For the second part, given $f, g \in \pi_{2n-1}(\mathbb{S}^n)$ get X_{f+g} and $X_{f \vee g}$, where the latter space arises by attaching two $2n$-cells via f and g through the map $f \vee g : \mathbb{S}^{2n-1} \vee \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^{2n-1}$, and relate the two short exact sequences obtained from X_{f+g} and $X_{f \vee g}$ with the morphism induced in K-theory by $q : X_{f+g} \longrightarrow X_{f \vee g}$ which collapses the equatorial disk of the $2n$-cell of X_{f+g}).

8. If (X, e) is an H-space, show that $\pi_1(X, e)$ is abelian.

9. Let X be a CW-complex and $e \in X$ a 0-cell. Show that it is equivalent:

   (a) (X, e) is an H-space.

   (b) There exists a map $\mu : X \times X \longrightarrow X$ such that the maps $\mu(-, e) : X \longrightarrow X$ and $\mu(e, -) : X \longrightarrow X$ are homotopic to the identity rel. $\{e\}$.

   (c) There exists a map $\mu : X \times X \longrightarrow X$ such that the maps $\mu(-, e) : X \longrightarrow X$ and $\mu(e, -) : X \longrightarrow X$ are homotopic to the identity.

   (*Hint:* Use the HEP)

10. If (X, e) is an H-space and CW-complex with multiplication $\mu : X \times X \longrightarrow X$ as in (b) before, show that by setting $(f + g)(x) = \mu(f(x), g(x))$ we obtain the same the group operation on $\pi_n(X, e)$.

11. Give explicit families of vector fields which form basis of $\mathfrak{X}(\mathbb{S}^1)$ and $\mathfrak{X}(\mathbb{S}^3)$. (*Hint:* Use the multiplicative structure of $\mathbb{C}$ and $\mathbb{H}$.)

Hand-in exercises: 1, 4, 7, 8, 11.