K-THEORY SEMINAR. LECTURE 4

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In this lecture we will introduce the external product and state the Fundamental Product theorem. We will analyse its immediate consequences and we will start its proof, which Luka will continue next week.

Recall that for a compact, Hausdorff topological space X its **(complex) K-theory** is given by the Grothendieck group of isomorphism classes of (complex) vector bundles, $K(X) := K(\mathrm{Vect}^*{}_{\mathbb{C}}(X))$. Its **real K-theory** is the same construction but with real vector bundles, $KO(X) := K(\mathrm{Vect}^*{}_{\mathbb{R}}(X))$, and both are commutative rings (with unit). For convenience we also defined the **reduced K-theory** as

$$\widetilde{K}(X) := K(X)/\mathbb{Z} \qquad , \qquad \widetilde{KO}(X) := K(X)/\mathbb{Z}$$

where $\mathbb{Z}$ is viewed as subring via the canonical injection

$$\mathbb{Z} \hookrightarrow K(X) \qquad , \qquad n \mapsto \underline{n},$$

which also endows $K(X)$ with a natural structure of $\mathbb{Z}$-algebra. We also recall that we have a split short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow K(X) \longrightarrow \widetilde{K}(X) \longrightarrow 0$$

(and similar with KO) and therefore we have splittings (as abelian groups)

$$K(X) \simeq \widetilde{K}(X) \oplus \mathbb{Z} \qquad , \qquad KO(X) \simeq \widetilde{KO}(X) \oplus \mathbb{Z}.$$

Lastly, keep also in mind that the Grothendieck group construction defines a functor

$$K(-) : \mathbf{CH}^{\mathrm{op}} \longrightarrow \mathbf{CRing}$$

from the category of compact, Hausdorff topological spaces whose arrows are homotopy classes of continuous maps, to the category of commutative rings (with unit).

1 The External Product

We will firstly set up some algebraic tools: let R be a commutative ring and let A, B be two R-algebras. In particular, they are R-modules and we can perform its tensor product module $A \otimes_R B$. However, this tensor product has a natural structure of R-algebra: the product is given by

$$\begin{aligned} A\otimes_R B\times A\otimes_R B &\longrightarrow A\otimes_R B\\ (a\otimes b\ ,\ a'\otimes b') &\longmapsto aa'\otimes bb' \end{aligned}$$

and the map $R\longrightarrow A\otimes_R B$, $r\mapsto r\otimes 1=1\otimes r$ is a ring homomorphism, thus $A\otimes_R B$ becomes a R-algebra. In particular,

Proposition 1.1 *Let A,B,C be R-algebras. Then there is an R-algebra isomorphism*

$$\begin{aligned} \mathrm{Hom}_{R-\mathrm{alg}}(A\otimes_R B,C) &\xrightarrow{\ \simeq\ } \mathrm{Hom}_{R-\mathrm{alg}}(A,C)\times \mathrm{Hom}_{R-\mathrm{alg}}(B,C)\\ \phi &\longmapsto (\phi_1,\phi_2) \end{aligned}$$

where $\phi_1(a):=\phi(a\otimes 1)$ and $\phi_2(b):=\phi(1\otimes b)$.

Proof. It is easy (and tedious) to check that both morphisms are R-algebra homomorphisms. The inverse of this map is the one which assigns to a pair (ϕ_1,ϕ_2) the morphism

$$\begin{aligned} A\otimes_R B &\longrightarrow C\\ a\otimes b &\longmapsto \phi_1(a)\phi_2(b). \end{aligned}$$

Clearly both morphisms are inverse of one another. $\square$

Definition. Let X,Y be compact, Hausdorff topological spaces, and consider their product space with projections $p_1:X\times Y\longrightarrow X$, $p_2:X\times Y\longrightarrow Y$. The **external product** is the unique $\mathbb{Z}$-algebra homomorphism

$$\mu: K(X)\otimes K(Y)\longrightarrow K(X\times Y)$$

which corresponds with the pair (p_1^*,p_2^*) given by the previous isomorphism, where

$$p_1^*: K(X\times Y)\longrightarrow K(X) \qquad , \qquad p_2^*: K(X\times Y)\longrightarrow K(Y).$$

Explicitly, $\mu(a\otimes b):=p_1^*(a)p_2^*(b)$ (where the last product refers to the tensor product in $K(X\times Y)$).

Example 1.2 If $E\longrightarrow X$ is a vector bundle over X, and $\underline{1}=Y\times\mathbb{C}$ is the trivial line bundle over Y, then we have that $\mu(E\otimes\underline{1})\simeq p_1^*(E)$, because the pullback of the trivial line bundle over Y through p_2 is precisely the trivial line bundle over $X\times Y$.

2 The Fundamental Product Theorem

Lemma 2.1 *Let E_f,E_g be vector bundles over $\mathbb{S}^k$ coming from clutching functions $f,g:\mathbb{S}^{k-1}\longrightarrow GL_n(\mathbb{C})$. Denote $fg:\mathbb{S}^{k-1}\longrightarrow GL_n(\mathbb{C})$, $(fg)(x):=f(x)g(x)$. Then it holds*

$$E_{fg}\oplus\underline{n}\simeq E_f\oplus E_g.$$

Proof. By exercise sheet 2 we have that the vector bundle $E_f \oplus E_g$ can be constructed with the clutching function

$$f \oplus g : \mathbb{S}^{k-1} \longrightarrow GL_{2n}(\mathbb{C}) \qquad , \qquad (f \oplus g)(x) := \begin{pmatrix} f(x) & 0 \\ 0 & g(x) \end{pmatrix}.$$

For the same reason $E_{fg} \oplus \underline{n}$ comes from the clutching function

$$(fg \oplus \mathrm{Id})(x) := \begin{pmatrix} f(x)g(x) & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Now, since $GL_{2n}(\mathbb{C})$ is path-connected, let σ a path joining the identity matrix with the matrix

$$\left(\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ \hline 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{array}\right)$$

which permutes the basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}\}$ of $\mathbb{C}^{2n}$ to $\{e_{n+1}, \dots, e_{2n}, e_1, \dots, e_n\}$. The homotopy

$$H : GL_{2n}(\mathbb{C}) \times I \longrightarrow GL_{2n}(\mathbb{C}) \qquad , \qquad H(-,t) := (f \oplus \mathrm{Id})\sigma(t)(\mathrm{Id} \oplus g)\sigma(t)$$

satisfies that $H(-,0) = f \oplus g$ and $H(-,1) = fg \oplus \mathrm{Id}$, so both maps are homotopic. We conclude because homotopic clutching functions induce isomorphic vector bundles. □

Now let H be the tautological line bundle over $\mathbb{CP}^1 \simeq \mathbb{S}^2$,

$$H = \{([v], v) \in \mathbb{CP}^1 \times (\mathbb{C}^2 - 0) : v \in \mathbb{C}^2 - 0\}.$$

In Joost's talk we already proved that this vector bundle can be constructed by the clutching function $f : \mathbb{S}^1 \longrightarrow GL_1(\mathbb{C})$, $f(z) := (z)$. Since it is a line bundle, by exercise sheet 2 we have that $H \otimes H = E_f \otimes E_f \simeq E_{ff}$ and applying the lemma

$$(H \otimes H) \oplus \underline{1} \simeq H \oplus H.$$

This means that in $K(\mathbb{S}^2)$, denoting "$\otimes$" as "$\cdot$" and "$\oplus$" as "$+$" we have the equation $H \cdot H + 1 = 2H$, ie, $(H-1)^2 = 0$. Consider the subalgebra $\mathbb{Z}[H] \subset K(\mathbb{S}^2)$, since every polynomial $p(x) = \sum_{i=0}^{k} a_i x^i$ gives rise[1] to a vector bundle $(\underline{a_k} \otimes H^{\otimes k}) \oplus \cdots \oplus (\underline{a_1} \otimes H) \oplus \underline{a_0}$. Since $(H-1)^2 = 0$ in the K-theory of the sphere, we have a well-defined $\mathbb{Z}$-algebra homomorphism

$$\begin{aligned} \frac{\mathbb{Z}[H]}{(H-1)^2} &\xrightarrow{\ i\ } K(\mathbb{S}^2) \\ [p(H)] &\longmapsto p(H). \end{aligned}$$

[1]Observe that for a vector bundle $p : E \longrightarrow X$, the distributive property implies that $\underline{2} \otimes E = (\underline{1} \oplus \underline{1}) \otimes E = (\underline{1} \otimes E) \oplus (\underline{1} \otimes E) = E \oplus E$ so for the vector bundle H of $\mathbb{S}^2$ we have that $(\underline{a_k} \otimes H^{\otimes k}) \oplus \cdots \oplus (\underline{a_1} \otimes H) \oplus \underline{a_0} \simeq (H^{\otimes k} \oplus \overset{a_k}{\cdots} \oplus H^{\otimes k}) \oplus \cdots \oplus (H \oplus \overset{a_1}{\cdots} \oplus H) \oplus \underline{a_0}$.

Theorem 2.2 (Fundamental Product) *Let X be a compact, Hausdorff topological space. Then the composite*

$$\varphi : K(X) \otimes \frac{\mathbb{Z}[H]}{(H-1)^2} \xrightarrow{\mathrm{Id}\otimes i} K(X) \otimes K(\mathbb{S}^2) \xrightarrow{\mu} K(X \times \mathbb{S}^2)$$

is a $\mathbb{Z}$-algebra isomorphism.

This is one of the most important results in K-theory. The proof is long and technical and we will split it in two lectures. Basically it consists in doing several reductions in the sort of functions one handles. Before starting with the proof (which we will divide in several lemmas) it is worth regarding its immediate consequences:

1. $K(\mathbb{S}^2) \simeq \dfrac{\mathbb{Z}[H]}{(H-1)^2}$.

   Proof. Setting $X = *$ the one-point space, since $K(*) \simeq \mathbb{Z}$ (because every vector bundle over the one-point space must be trivial) and $* \times \mathbb{S}^2 \simeq \mathbb{S}^2$ we obtain

   $$\frac{\mathbb{Z}[H]}{(H-1)^2} \simeq \mathbb{Z} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}[H]}{(H-1)^2} \underset{\sim}{\xrightarrow{\varphi}} K(* \times \mathbb{S}^2) = K(\mathbb{S}^2).$$

   □

2. The external product $\mu : K(X) \otimes K(\mathbb{S}^2) \longrightarrow K(X \times \mathbb{S}^2)$ is a $\mathbb{Z}$-algebra isomorphism.

   Proof. Since the tensor product respects isomorphisms, by 1. we have that in the composite φ of the Product theorem the first arrow $\mathrm{Id} \otimes i : K(X) \otimes \frac{\mathbb{Z}[H]}{(H-1)^2} \longrightarrow K(X) \otimes K(\mathbb{S}^2)$ is isomorphism. Now the two out of three property for isomorphisms follows. □

3. $\widetilde{K}(\mathbb{S}^2) \simeq \mathbb{Z}$.

   Proof. As abelian group (ie, $\mathbb{Z}$-module), $\frac{\mathbb{Z}[H]}{(H-1)^2} \simeq 1\mathbb{Z} \oplus H\mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z}$, thus $\widetilde{K}(\mathbb{S}^2) = \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}} \simeq \mathbb{Z}$. [2] □

4. **(Bott periodicity)** $\widetilde{K}(\mathbb{S}^n) \simeq \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$

   Proof. This was Corollary 35 in Thomas' lecture notes for the previous talk, using a split short exact sequence and the key fact that $\mathbb{S}^n \wedge \mathbb{S}^2 = \mathbb{S}^{n+2}$. □

[2] Observe that changing the $\mathbb{Z}$-basis to $\{H-1, 1\}$, one has that $\widetilde{K}(\mathbb{S}^2) \simeq (H-1)\mathbb{Z}$, ie, $\widetilde{K}(\mathbb{S}^2)$ is generated as abelian group by $H-1$. But being $(H-1)^2 = 0$, it says that the product of any two elements must be null in reduced K-theory. This is the same situation as in cohomology where $H-1$ plays the same role as the generator of $H^2(\mathbb{S}^2;\mathbb{Z})$ with respect to the cup product.

3 Generalized clutching functions

In Lecture 2 we constructed vector bundles over k-spheres using the so-called clutching functions, maps $f : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{C})$ which gave rise to a vector bundle over $\mathbb{S}^k$ as the pushout of the following diagram:

$$\begin{array}{ccc} \{a,b\} \times \partial\mathbb{D}^k \times \mathbb{C}^n & \xrightarrow{F} & \mathbb{S}^{k-1} \times \mathbb{C}^n \\ \big\downarrow & & \big\downarrow \\ \{a,b\} \times \mathbb{D}^k \times \mathbb{C}^n & \longrightarrow & E_f \end{array}$$

where $F_{\{a\}\times\partial\mathbb{D}^k\times\mathbb{C}^n} = \mathrm{Id}$ and $F_{\{b\}\times\partial\mathbb{D}^k\times\mathbb{C}^n}(x,v) = (x, f(x)(v))$, or in other words,

$$E_f = \frac{\mathbb{D}^k_+ \times \mathbb{C}^n \coprod \mathbb{D}^k_- \times \mathbb{C}^n}{(x,v) \sim (x, f(x)(v))}$$

for $(x,v) \in \partial\mathbb{D}^n_+ \times \mathbb{C}^n$ and $(x, f(x)(v)) \in \partial\mathbb{D}^n_- \times \mathbb{C}^n$. We want to enlarge this class of clutching functions to construct vector bundles over $X \times \mathbb{S}^2$ out of a vector bundle over a compact Hausdorff space X. This is as follows:

Let $p : E \longrightarrow X$ be a vector bundle over X. The identity $\mathrm{Id} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ can be viewed as the 0-dimensional trivial bundle, $\underline{0} = \mathbb{S}^1 \times 0 \simeq \mathbb{S}^1 \longrightarrow \mathbb{S}^1$. Therefore we have a product bundle[3] $E \times \underline{0}$ over $X \times \mathbb{S}^1$,

$$p \times \mathrm{Id} : E \times \mathbb{S}^1 \longrightarrow X \times \mathbb{S}^1.$$

Any automorphism of vector bundles $f : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$ induces a vector bundle over $X \times \mathbb{S}^2$, arising as the pushout of the diagram

$$\begin{array}{ccc} \{a,b\} \times E \times \partial\mathbb{D}^2 & \xrightarrow{F} & E \times \mathbb{S}^1 \\ \big\downarrow & & \big\downarrow \\ \{a,b\} \times E \times \partial\mathbb{D}^2 & \longrightarrow & [E,f] \end{array}$$

where $F_{\{a\}\times E\times\partial\mathbb{D}^2} = \mathrm{Id}$ and $F_{\{b\}\times E\times\partial\mathbb{D}^2} = f$. Explicitly, this is

$$[E,f] = \frac{E \times \mathbb{D}^2_+ \coprod E \times \mathbb{D}^2_-}{(e,x) \sim f(e,x)}$$

for $(e,x) \in E \times \mathbb{D}^2_+$ and $f(e,x) \in E \times \mathbb{D}^2_-$

Definition. In the terms of above, we say that the automorphism f is a **clutching function** for $[E,f]$.

Lemma 3.1 *Homotopic clutching functions induce isomorphic vector bundles, that is, if $f, g : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$ are homotopic, then $[E,f] \simeq [E,g]$.*

Proof. Let $H : E \times \mathbb{S}^1 \times I \longrightarrow E \times \mathbb{S}^1$ be a homotopy between f and g, with $H(-,0) = f$ and $H(-,1) = g$. With this homotopy we can construct the following vector bundle over $X \times \mathbb{S}^2 \times I$:

$$F = \frac{E \times \mathbb{D}^2_+ \times I \coprod E \times \mathbb{D}^2_- \times I}{(e,x,t) \sim (H(e,x,t),t)}$$

[3]Recall from Lecture 1 that if $p : E \longrightarrow X$, $p' : E' \longrightarrow Y$ are vector bundles, then $p \times p' : E \times E' \longrightarrow X \times Y$ is a vector bundle over the product of the basis.

for $(e,x,t) \in E \times \partial\mathbb{D}^2_+ \times I$ and $H(e,x,t) \in E \times \partial\mathbb{D}^2_- \times I$. This vector bundle restricts to $[E,f]$ and $[E,g]$ on $X \times \mathbb{S}^2 \times \{0\}$ and $X \times \mathbb{S}^2 \times \{1\}$ respectively. Since $X \times \mathbb{S}^2$ is compact and Hausdorff it is paracompact and applying lemma 33 of Bjarne's notes we are in business. □

Exercise 3.2 Show that $[E,f] \oplus [E',g] \simeq [E \oplus E', f \oplus g]$.

Examples 3.3

1. If $p : E \longrightarrow X$ is a vector bundle and we consider $\mathrm{Id} : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$, then $[E, \mathrm{Id}]$ is the pullback of E via the projection $p_1 : X \times \mathbb{S}^2 \longrightarrow X$, since $[E,\mathrm{Id}] = E \times \mathbb{S}^2$ (immediately from the definition) and on the other hand

   $$p_1^*(E) = (X \times \mathbb{S}^2) \times_X E = \{((x,s),e) \in (X \times \mathbb{S}^2) \times E : x = p(e)\} \simeq E \times \mathbb{S}^2.$$

   This implies that in the K-theory of $X \times \mathbb{S}^2$ $[E,\mathrm{Id}]$ is the external product $\mu(E \otimes \underline{1})$, because the pullback of the trivial line bundle over $\mathbb{S}^2$ via the projection $p_2 : X \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is the trivial line bundle over $X \times \mathbb{S}^2$, ie, the unit of $K(X \times \mathbb{S}^2)$.

2. Recall from lecture 2 that the tautological line bundle over $\mathbb{CP}^1 \simeq \mathbb{S}^2$ can be obtained by a clutching function as

   $$H = \frac{\mathbb{D}^2_\infty \times \mathbb{C} \coprod \mathbb{D}^2_0 \times \mathbb{C}}{(x,\lambda) \sim (x, f(x)(\lambda)}$$

   for $(x,\lambda) \in \partial\mathbb{D}^2_\infty \times \mathbb{C}$ and $(x, f(x)(\lambda)) \in \partial\mathbb{D}^2_0 \times \mathbb{C}$ (where $\mathbb{D}^2_\infty$ and $\mathbb{D}^2_0$ have boundaries identified with $\mathbb{S}^1 \subset \mathbb{C}$), with $f : \mathbb{S}^1 \longrightarrow GL_1(\mathbb{C})$, $f(x)(\lambda) = x\lambda$. If we take $X = *$ the one-point space and $\underline{1} = * \times \mathbb{C} \simeq \mathbb{C}$ the trivial line bundle over $*$, then $[\underline{1}, z\cdot] \simeq H$, where precisely $z\cdot$ is the vector bundle morphism $\mathbb{C} \times \mathbb{S}^1 \longrightarrow \mathbb{C} \times \mathbb{S}^1$, $(\lambda, z) \mapsto (\lambda z, z)$. Since for line bundles the tensor product arises from the product of the clutching functions, then $[\underline{1}, z^n\cdot] \simeq H^{\otimes n} \overset{\text{notation}}{=} H^n$, $n \geq 0$. Because of this, we can set $H^{-1} := [\underline{1}, z^{-1}]$ to have $H \otimes H^{-1} \simeq \underline{1}$ the trivial vector bundle over $\mathbb{S}^2$, and the same argument ensures that the isomorphism $[\underline{1}, z^n\cdot] \simeq H^n$ is true for all $n \in \mathbb{Z}$.

3. For $p : E \longrightarrow X$ a vector bundle, if we consider the automorphism $z\cdot : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$, $(e,z) \longrightarrow (ez^n, z)$ (here $z \in \mathbb{S}^1 \subset \mathbb{C}$ and the product is fiberwise), then we obtain that $[E, z^n\cdot] \simeq \mu(E \otimes H^n)$, $n \in \mathbb{Z}$. To see this, observe that $[E, z^n\cdot]$ can be interpreted as the tensor product of the identity $\mathrm{Id} : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$ and the automorphism $z^n\cdot$ in the trivial line bundle over $X \times \mathbb{S}^1$, giving rise to our first automorphism,

   $$\mathrm{Id} \otimes z^n\cdot : (E \times \mathbb{S}^1) \otimes \underline{1} \simeq E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1 \simeq (E \times \mathbb{S}^1) \otimes \underline{1}.$$

   Therefore we obtain

   $$[E, z^n\cdot] \simeq [E, \mathrm{Id} \otimes z^n\cdot] \simeq \mu(E \otimes [\underline{1}, z^n\cdot]) \simeq \mu(E \otimes H^n).$$

Exercise 3.4 Generalize the previous example to show that $[E, z^n f] \simeq [E,f] \otimes \widehat{H}^n$, $n \in \mathbb{Z}$, where $\widehat{H}^n = p_2^*(H^n)$ is the pullback through $p_2 : X \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2$.

We already have all the ingredients we need to develop the proof. Below we have outlined the steps we will go through to finally show that $\varphi : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \longrightarrow K(X \times \mathbb{S}^2)$ is isomorphism. These steps actually will let us work with hypothesis which will get easier and easier:

1. Every vector bundle over $X \times \mathbb{S}^2$ is isomorphic to one of the form $[E, f]$.
2. Every vector bundle $[E, f]$ is isomorphic to $[E, \ell]$ for ℓ a clutching function of the form $\ell(x, z) := \sum_{n=-k}^{k} a_n(x) z^n$.
3. Since such a ℓ as before is $\ell = z^{-k} q$ for a polynomial clutching function q, by exercise 3.4 $[E, \ell] \simeq [E, q] \otimes \widehat{H}^{-k}$.
4. For a degree n polynomial clutching function q, $[E, q] \oplus [\underline{n} \otimes E, \mathrm{Id}] \simeq [\underline{n+1} \otimes E, a(x)z + b(x)]$.
5. For every vector bundle $[E, a(x)z + b(x)]$, there is a splitting $E \simeq E_+ \oplus E_-$ such that $[E, a(x)z + b(x)] \simeq [E_+, \mathrm{Id}] \oplus [E_-, z\cdot]$.
6. Show that φ is surjective (easy) and injective (hard).

4 Proof of the Product Theorem. Part I

Proposition 4.1 *For every vector bundle $E' \longrightarrow X \times \mathbb{S}^2$ there exists a vector bundle E over X and a clutching function $f : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$ such that $E' \simeq [E, f]$.*

Proof. Recall from lecture 2 that we can represent points in $\mathbb{CP}^1$ as ratios $z = z_0/z_1 \in \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$ (the Riemann sphere, or the one-point compactification of $\mathbb{C}$). Denote again $\mathbb{D}_0$ the set of points on the south hemisphere (ie, $|z| \leq 1$) and $\mathbb{D}_\infty$the points in the north hemisphere (ie, $|z^{-1}| \leq 1$), so $\mathbb{S}^2 = \mathbb{D}_0 \cup \mathbb{D}_\infty$. Denote $\alpha = 0$ or ∞, and consider the inclusions

$$i_\alpha : X \times \mathbb{D}_\alpha \hookrightarrow X \times \mathbb{S}^2 \qquad , \qquad i_1 : X \times \{1\} \hookrightarrow X \times \mathbb{S}^2.$$

Set $E_\alpha := i_\alpha^*(E')$ the restriction of E' to the subspace $X \times \mathbb{D}_\alpha$, and let $E := i_1^*(E')$ the restriction to $X \times \{1\}$. Since $\mathbb{D}_\alpha$ is contractible (to $\{1\}$), we have that the retraction $r : X \times \mathbb{D}_\alpha \longrightarrow X \times \{1\}$ is a homotopy equivalence, thus $\mathrm{Id} : X \times \mathbb{D}_\alpha \longrightarrow X \times \mathbb{D}_\alpha$ is homotopic to the composite

$$X \times \mathbb{D}_\alpha \xrightarrow{r} X \times \{1\} \overset{i}{\hookrightarrow} X \times \mathbb{D}_\alpha.$$

Since homotopic maps induce the same pullbacks, $\mathrm{Id}^*(E_\alpha) = E_\alpha$ is isomorphic to $(i \circ r)^*(E_\alpha) = r^* i^*(E_\alpha) = r^*(E) \simeq E \times \mathbb{D}_\alpha$. So we get isomorphisms

$$h_0 : E_0 \xrightarrow{\sim} E \times \mathbb{D}_0 \qquad , \qquad h_\infty : E_\infty \xrightarrow{\sim} E \times \mathbb{D}_\infty.$$

Now setting

$$f := h_0 h_\infty^{-1}{}_{|E \times \mathbb{S}^1} : E \times \mathbb{S}^1 \xrightarrow{\sim} E \times \mathbb{S}^1$$

we obtain the wanted clutching function, and the vector bundle $[E, f]$ is isomorphic to E' by construction. □

Definition. We will call **Laurent polynomial** to a clutching function of the form

$$\ell(x,z) := \sum_{n=-k}^{k} a_n(x) z^n$$

where $a_n : E \longrightarrow E$ are linear maps fiberwise, which we will call **endomorphisms** of E

Note that these maps a_n does *not* need to be isomorphisms of vector bundles. However, the linear combination $\ell(x,z) := \sum_{n=-k}^{k} a_n(x) z^n$ must be because the class of clutching functions we are dealing with is precisely the automorphisms of the vector bundle $E \times \mathbb{S}^1$.

Our next goal (or better, reduction) is to show that every vector bundle arises using a Laurent polynomial as clutching function.

Proposition 4.2 *Every vector bundle $[E, f]$ is isomorphic to $[E, \ell]$ for some Laurent polynomial ℓ. Moreover, homotopic Laurent polynomials through clutching functions are homotopic by a Laurent polynomial clutching function homotopy $\mathscr{L}(x,z,t) = \sum_n a_n(x,t) z^n$.*

Before the proof we need a getaway to the analysis world. Suppose that given a continuous function $f : X \times \mathbb{S}^1 \longrightarrow \mathbb{C}$ we want to approximate it by a polynomial expression of the form $\ell(x,z) := \sum_{n \leq |N|} a_n(x) z^n$, where $a_n : X \longrightarrow \mathbb{C}$ is continuous. Firstly rewrite $\ell(x, e^{i\theta}) := \sum_{n \leq |N|} a_n(x) e^{in\theta}$. A first (and as we will find out right) guess one has is to take the coefficients a_n as the Fourier coefficients of f,

$$a_n(x) := \frac{1}{2\pi} \int_0^{2\pi} f(x, e^{i\theta}) e^{-in\theta} \mathrm{d}\theta.$$

Define

$$u(x,r,\theta) := \sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} e^{in\theta} \qquad , \qquad r > 0.$$

For $r < 1$, this series converges absolute and uniformly. Indeed, since X is compact, so is $X \times \mathbb{S}^1$, thus $|f(x, e^{i\theta})|$ is bounded and therefore $|a_n(x)|$ too. Now just observe that $|a_n(x) r^{|n|} e^{-in\theta}| \leq M r^{|n|}$, and by the Weierstrass M-test[4] one concludes comparing the resulting series with the geometric series. We want to show that u converges to f as $r \longrightarrow 1$:

Lemma 4.3 *$u(x,r,\theta) \longrightarrow f(x, e^{i\theta})$ uniformly in x and θ as $r \longrightarrow 1$.*

Proof. In first place note that

$$\begin{aligned} u(x,r,\theta) &= \sum_{n \in \mathbb{Z}} a_n(x) r^{|n|} e^{in\theta} \\ &= \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} \left(\int_0^{2\pi} f(x, e^{it}) e^{int} \mathrm{d}t \right) r^{|n|} e^{in\theta} \end{aligned}$$

[4] *Let (f_n) be a sequence of functions. If for all $n \in \mathbb{N}$ there exists $M_n \geq 0$ such that $|f_n(z)| \leq M_n$ for all z, and $\sum M_n < \infty$, then $\sum f_n$ converges absolute and uniformly.*

$$= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi} \int_0^{2\pi} f(x, e^{it}) e^{in(\theta - t)} r^{|n|} \mathrm{d}t$$
$$= \int_0^{2\pi} \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} f(x, e^{it}) e^{in(\theta - t)} r^{|n|} \mathrm{d}t$$

where in the last equality we interchanged the integral and the summation provided that the series in uniformly convergent (one checks that using again the Weierstrass M-test). Now define

$$P(r,\varphi) := \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\varphi} \qquad , \qquad 0 \le r < 1,\ \varphi \in \mathbb{R},$$

so that $u(x,r,\theta) = \int_0^{2\pi} P(r, \theta - t) f(x, e^{it}) \mathrm{d}t$. We can make $P(x,\varphi)$ explicit as follows: set $a := re^{i\varphi}, b := re^{-i\varphi}$. Then

$$\begin{aligned} P(x,\varphi) &= \frac{1}{2\pi}\left(1 + \sum_{n\in\mathbb{N}} r^n (e^{in\varphi} + e^{-in\varphi})\right) = \frac{1}{2\pi}\left(1 + \sum_{n\in\mathbb{N}} (a^n + b^n)\right) \\ &= \frac{1}{2\pi}\left(1 + \frac{a}{1-a} + \frac{b}{1-b}\right) = \frac{1}{2\pi}\left(\frac{1}{1-a} + \frac{b}{1-b}\right) \\ &= \frac{1}{2\pi}\left(1 + \frac{a}{1-a} + \frac{b}{1-b}\right) = \frac{1}{2\pi}\left(\frac{1-ab}{1+ab-a-b}\right) \\ &= \frac{1}{2\pi}\left(\frac{1-r^2}{1+r^2 - r(e^{i\varphi} + e^{-i\varphi})}\right) = \frac{1}{2\pi}\left(\frac{1-r^2}{1+r^2-2r\cos\varphi}\right). \end{aligned}$$

Now, note also that $\int_0^{2\pi} P(r,\varphi)\mathrm{d}\varphi = 1$: since the series converges uniformly (again by using the Weierstrass M-test), we can permute the integral with the summation and integrate term by term, obtaining

$$\begin{aligned} \int_0^{2\pi} P(r,\varphi)\mathrm{d}\varphi &= \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \frac{1}{2\pi} r^{|n|} e^{in\varphi} \mathrm{d}\varphi \\ &= 1 + \sum_{n=1}^{\infty} r^n \left(\frac{1}{2\pi} \int_0^{2\pi} e^{in\varphi} + \frac{1}{2\pi} \int_0^{2\pi} e^{-in\varphi} \right) = 1, \end{aligned}$$

because all terms in brackets are null since $\{e^{in\varphi} : n \ge 0\}$ forms an orthonormal basis of $L^2[0, 2\pi]$. So we have

$$\begin{aligned} |u(x,r,\theta) - f(x,e^{i\theta})| &= \left| \int_0^{2\pi} P(r,\theta-t) f(x,e^{it})\mathrm{d}t - \int_0^{2\pi} P(r,\theta-t) f(x,e^{i\theta})\mathrm{d}t \right| \\ &\le \int_0^{2\pi} P(r,\theta-t) |f(x,e^{it}) - f(x,e^{i\theta})| \mathrm{d}t, \end{aligned}$$

since $P(r,\varphi)$ is decreasing in φ and therefore $P(r,\varphi) \ge P(r,\pi) = \frac{1-r^2}{(1+r)^2} > 0$ for $r < 1$. Let us now show the uniform convergence: given $\varepsilon > 0$, since f is continuous in a compact it is uniformly continuous on $X \times \mathbb{S}^1$, so there exists $\delta > 0$ such that $|f(x,e^{it}) - f(x,e^{i\theta})| \le \varepsilon$ for $|t - \theta| \le \delta$ and all $x \in X$. Set

$$I_\delta := \int_{[\theta-\delta,\theta+\delta]} P(r,\theta-t)|f(x,e^{it}) - f(x,e^{i\theta})|\mathrm{d}t,$$

$$I_\delta^c := \int_{[\theta-\delta,\theta+\delta]^c} P(r,\theta-t)|f(x,e^{it})-f(x,e^{i\theta})|\mathrm{d}t,$$

(where the complement is taken on $[0,2\pi]$). On the one hand,

$$I_\delta \leq \int_{[\theta-\delta,\theta+\delta]} P(r,\theta-t)\varepsilon\mathrm{d}t \leq \varepsilon\int_0^{2\pi} P(r,\theta-t)\mathrm{d}t = \varepsilon,$$

(the second inequality because $P>0$). On the other hand, since P is decreasing in φ, $P(r,\theta-t)$ reaches its maximum on $\{|\theta-t|\geq\delta\}$ at $\theta-t=\delta$, so $\max_{[\theta-\delta,\theta+\delta]^c} P(r,\theta-t) = P(r,\delta)$, and then

$$I_\delta^c \leq P(r,\delta)\int_0^{2\pi} |f(x,e^{it})-f(x,e^{i\theta})|\mathrm{d}t \leq P(r,\delta)M.$$

But observe that for fixed $\varphi\in(0,2\pi)$, $P(r,\varphi)\longrightarrow 0$ as $r\longrightarrow 1$, so we can take r close enough to 1 so that $I_\delta^c\leq\varepsilon$. In conclusion,

$$|u(x,r,\theta)-f(x,e^{it})| \leq I_\delta + I_\delta^c \leq 2\varepsilon$$

for all $x\in X$ and θ, meaning the uniform convergence of u to f. □

Corollary 4.4 *If $f : X\times\mathbb{S}^1 \longrightarrow \mathbb{C}$ is a continuous function, given $\varepsilon>0$, there exists a Laurent polynomial function $\ell(x,z) := \sum_{n\leq|N|} a_n(x)z^n$ such that $|f(x,z)-\ell(x,z)|<\varepsilon$ for all $(x,z)\in X\times\mathbb{S}^1$.*

Proof. By the lemma, such a $u(x,r,\theta)$ converges uniformly to f as $r\longrightarrow 1$. For r close enough to 1, the sum of finitely many terms of u will give us the desired bound. □

Proof of 4.2 . Consider an inner product on E (since the base is paracompact), so we also obtain an inner product on $E\times\mathbb{S}^1$, as both have isomorphic fibers. Now the set $\mathrm{End}(E\times\mathbb{S}^1)$ of endomorphisms of $E\times\mathbb{S}^1$ is a normed vector space, endowed with the norm

$$||\alpha|| := \sup_{|v|=1} |\alpha(v)|.$$

Observe that the subspace $\mathrm{Aut}(E\times\mathbb{S}^1)$ of clutching functions is an open set with the topology induced by this norm, as it is the preimage of $(0,\infty)$ by the continuous map

$$\begin{aligned}\mathrm{End}(E\times\mathbb{S}^1) &\longrightarrow [0,\infty)\\ \alpha &\longmapsto \inf_{(x,z)\in X\times\mathbb{S}^1} |\det\alpha(x,z)|,\end{aligned}$$

where $\alpha(x,z) : E_x \longrightarrow E_x$. With these remarks, all we need to prove then is that the set of Laurent polynomials is dense in $\mathrm{End}(E\times\mathbb{S}^1)$, because then for any $f\in\mathrm{End}(E\times\mathbb{S}^1)$, by taking $B(f,\varepsilon)$ an open ball, there will be a Laurent polynomial $\ell\in B(f,\varepsilon)$, and therefore both are homotopic by

$$H_t := t\ell + (1-t)f,$$

which is contained in $B(f,\varepsilon)$ because balls coming from norms are convex.

Take then $f \in \mathrm{End}(E\times\mathbb{S}^1)$ and let us show that there is a Laurent polynomial ℓ with $||f-\ell|| \leq \varepsilon$, for a given $\varepsilon > 0$. Consider $\{U_i\}$ a cover of X formed by trivializing open sets, with trivializations $h_i : p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^{n_i}$. We can suppose that these h_i are isometries fiberwise (we already saw this, by applying Gram-Schmidt). Let $\{\phi_i\}$ be a partition of unity subordinated to the cover. For $x \in \mathrm{supp}\ \phi_i$, the trivializations h_i allow us to view $f(x,z)$ as a matrix, since they give isomorphisms $E_{(x,z)} \simeq \mathbb{C}^{n_i}$ fiberwise. If we denote $f_{kl} : \mathrm{supp}\ \phi_i \times \mathbb{S}^1 \longrightarrow \mathbb{C}$ the entries of such a matrix, by 4.4 we find Laurent polynomials $\ell^i_{kl} : \mathrm{supp}\ \phi_i \times \mathbb{S}^1 \longrightarrow \mathbb{C}$ with $||f_{kl} - \ell^i_{kl}||_\infty < \varepsilon$, and therefore we can form matrices $\ell^i(x,z)$, for $x \in \mathrm{supp}\ \phi_i$. Therefore, ℓ^i approximates f with the norm, because the entries are uniformly approximated.

Now by setting $\ell := \sum_i \phi_i \ell^i$ we find the desired Laurent polynomial on $X \times \mathbb{S}^1$ which approximates f, since the family $\{\mathrm{supp}\ \phi_i\}$ is locally finite.

For the last part of the proposition, one uses a similar argument approximating a homotopy H from ℓ_0 to ℓ_1 (understood as an automorphism of $E \times \mathbb{S}^1 \times I$) by a Laurent polynomial homotopy $\mathscr{L}'(x,z,t)$, and combining this with linear homotopies from $\mathscr{L}'(-,-,0)$ to ℓ_0 and from $\mathscr{L}'(-,-,1)$ to ℓ_1 we obtain our desired homotopy $\mathscr{L}$ from ℓ_0 to ℓ_1. $\square$

By this last proposition, we are reduced to Laurent polynomials; or more precisely, just to polynomials, since for a Laurent polynomial $\ell(x,z) := \sum_{n\leq|k|} a_n(x)z^n$ we just need to take $q := z^k\ell$ to obtain a polynomial; and then we have that

$$[E,\ell] \simeq [E,q] \otimes \widehat{H}^{-k}$$

by applying 3.4 to $\ell = z^{-k}q$.

Proposition 4.5 *If q is a degree n polynomial clutching function, there is a splitting $[E,q] \oplus [\underline{n} \otimes E, \mathrm{Id}] \simeq [\underline{n+1} \otimes E, a(x)z + b(x)]$ for a linear clutching function $a(x)z+b(x)$.*

Proof. The key observation for this proof is that given a set of endomorphisms $\{f_{ij} : i,j = 1,\dots,n+1\}$ of E, we can construct an endomorphism

$$\underline{n+1} \otimes E \simeq E \oplus \overset{n+1}{\cdots} \oplus E \longrightarrow E \oplus \overset{n+1}{\cdots} \oplus E \simeq \underline{n+1} \otimes E$$

by letting f_{ij} be a linear map from the i-th summand in the source to the j-th summand in the target.

Write $q(x,z) = a_n(x)z^n + \cdots + a_1(x)z + a_0$, and define the matrices

$$A := \begin{pmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{pmatrix}$$

and

$$B := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & q \end{pmatrix}.$$

By the previous remark, these two matrices define endomorphisms of $\underline{n+1} \otimes E$. Let us define a sequence of polynomials $q_r(z) = q_r(z,x)$ inductively as follows:

$$\begin{aligned} q_0(z) &:= q(z) \\ q_1(z) &:= \frac{q_0(z) - q_0(0)}{z} \\ &\vdots \\ q_n(z) &:= \frac{q_{n-1}(z) - q_{n-1}(0)}{z} \end{aligned}$$

(note that we can divide by z at every step because $q_r(z) - q_r(0)$ is always a polynomial of degree greater or equal than 1). It is a straightforward (and tedious) computation that we can write A as the product

$$A = (I + N_1)B(I + N_2)$$

where I denotes the $(n+1) \times (n+1)$ identity matrix and N_1 and N_2 are the following nilpotent matrices:

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ q_1 & 0 & 0 & \dots & 0 \\ q_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_n & 0 & \dots & 0 & 0 \end{pmatrix} \quad , \quad N_2 = \begin{pmatrix} 0 & -z & 0 & \dots & 0 \\ 0 & 0 & -z & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Recall from linear algebra that being N_1, N_2 nilpotent, $I + tN_i$ is invertible (for $0 \le t \le 1$). Now, again by exercise sheet 2 we know that the matrix B defines a clutching function for the vector bundle

$$[E, q] \oplus [\underline{n} \otimes E, \mathrm{Id}],$$

meaning that B is invertible fiberwise (after fixing basis fiberwise), and therefore so is A. In conclusion, we see that A defines an automorphism of $\underline{n+1} \otimes E$ for each $x \in \mathbb{S}^1$, thus a clutching function $L^n q : (\underline{n+1} \otimes E) \times \mathbb{S}^1 \longrightarrow (\underline{n+1} \otimes E) \times \mathbb{S}^1$, which takes the form $L^n q = a(x)z + b(x)$ by the form of A. From the previous relation $A = (I+N_1)B(I+N_2)$ it follows that we can define a homotopy between A and B,

$$H_t = (I + tN_1)B(I + tN_2),$$

showing that the induced vector bundles are isomorphic,

$$[E, q] \oplus [\underline{n} \otimes E, \mathrm{Id}] \simeq [\underline{n+1} \otimes E, a(x)z + b(x)].$$

□

To be continued... next week.

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