An introduction to knot homology theories

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1 What is categorification?

<u>Category number</u> (Dan Freed): Loose measure of the amount of abstraction involved in a mathematical idea, theorem, construction, ...

$$
\begin{tikzcd}
\vdots \\
\boxed{2} \arrow[d, bend left, "\text{forget functors}"] & \text{(Categories)} \\
\boxed{1} \arrow[u, bend left, "\text{add functors}"] \arrow[d, bend left, "\text{forget maps}"] & \text{(Mathematical structures)} \\
\boxed{0} \arrow[u, bend left, "\text{add maps}"] & \text{(Numbers, polynomials)}
\end{tikzcd}
$$

<u>Categorification</u> consists of taking an object, statement, construction... which happens at a certain category number, and lift it to another such taking place at a higher level, being able to recover the original object, statement, construction, etc. (<u>decategorification</u>)

Example 1 : The category of finite sets is a categorification of the natural numbers : $n \in \mathbb{N}$ is lifted to $S_n :=$ finite set with n elements. The decategorification is simply taking the cardinality, $\# S_n = n$.

Example 2 : The category of finite dimensional chain complexes over a field F (eg $\mathbb{R}, \mathbb{Z}/2, \dots$) is a categorification of the integers : the decategorification sends a chain complex C_* to its Euler characteristic

$$\chi(C_*) := \sum_i (-1)^i \dim_F C_i$$

Actually $\mathbb{Z} \cong K_0\left({}^{\text{fd}\, Ch_F} / {}_{\text{chain htpy}} \right)$, $\quad K_0$ Grothendieck ring

Example 3: Singular homology categorifies the Euler characteristic of finite-dimensional CW-complexes (and hence the (non-)orientable genus of closed surfaces):

$$\chi(X) = \chi\big(H_*(X;F)\big) = \sum_i (-1)^i \dim_F H_i(X;F).$$

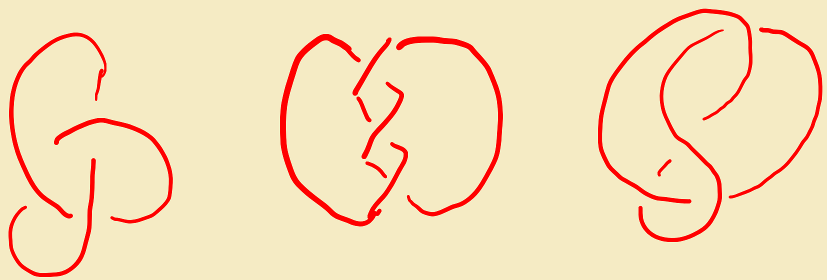
$H(X)$ carries much more topological information than $\chi(X)$:

- $H : \mathrm{Top} \to \mathrm{grVect}_F$ is a functor,
- $H(X)$ only depends on the homotopy type of X, and $H(*) \cong F_{(0)}$
- $H(X \times Y) \cong H(X) \otimes H(Y)$
- Computational tools: Mayer-Vietoris, exact triangles / les, ...

② Knot polynomials

Given a knot $K \subset S^3$ (ie a smooth/PL embedding $S^1 \hookrightarrow S^3$)

a classical problem in knot theory consists of distinguishing knots up to isotopy.



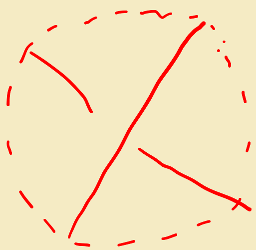
There are two classical knot polynomial invariants, called the

Alexander polynomial $\Delta_K(t)$ and the Jones polynomial $J_K(q)$:

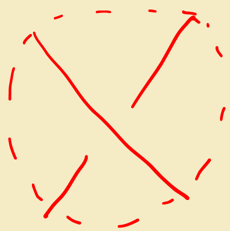
They are characterised by

$\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$	$J_K(q) \in \mathbb{Z}[q, q^{-1}]$
$\Delta_{K_+} - \Delta_{K_-} = (t^{-1/2} - t^{1/2})\,\Delta_{K_0}$	$q^2 J_{K_+} - q^{-2} J_{K_-} = (q - q^{-1})\,J_{K_0}$
$\Delta_{\text{unknot}} = 1$	$J_{\text{unknot}} = q^{-1} + q$

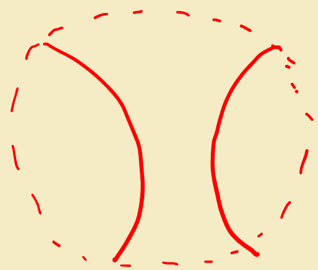
where



K_+



K_-

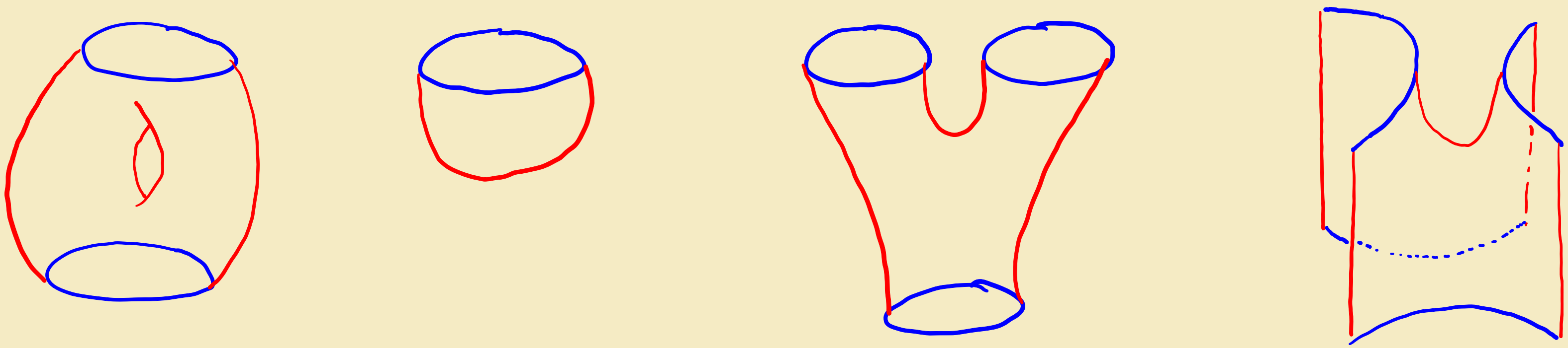


K_0

Just like with the Euler characteristic of CW-complexes, one would like to lift Δ_K and J_K to "homology-like theories", with similar properties to the ones singular homology satisfies.

Let Knots be the category

Knots : $\begin{cases} \underline{\text{objects}} : \text{isotopy classes of oriented knots in } S^3 \\ \underline{\text{arrows } K \to K'} : \text{orientation-preserving homeomorphism classes of bordisms from } K \text{ to } K', \text{ ie, compact oriented surfaces } \Sigma \subseteq S^3 \times I \text{ such that } \partial\Sigma = -K \amalg K'. \end{cases}$



③ Khovanov homology (Khovanov, 99)

Theorem: There exists a functor

$$Kh : \text{Knots} \longrightarrow \text{bigrVect}_{\mathbb{Z}/2}$$

satisfying

1) If $\Sigma : K \to K'$ is an isotopy, then $Kh(\Sigma) : Kh(K) \xrightarrow{\cong} Kh(K')$ is iso
2) $Kh(\text{unknot}) \cong \mathbb{Z}/2_{(0,1)} \oplus \mathbb{Z}/2_{(0,-1)}$ $\quad \left(Kh(\emptyset) = \mathbb{Z}/2_{(0,0)} \right)$
3) $Kh(K \amalg K'^{*}) \cong Kh(K) \otimes Kh(K)$

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*: Not a knot, but a link in S^3 (two components). Kh is more generally defined for links (even tangles).

4) If K is a knot, denote by K_0 and K_∞ knots identical to K except around one crossing of the form $\times$ where they have been modified as $)($ and $\asymp$, resp. Then there is an exact triangle

$$\begin{tikzcd}
Kh(K_0) \arrow[rr] & & Kh(K) \arrow[dl] \\
& Kh(K_\infty) \arrow[ul] &
\end{tikzcd}$$

5) The Jones polynomial is the graded Euler characteristic of Kh:

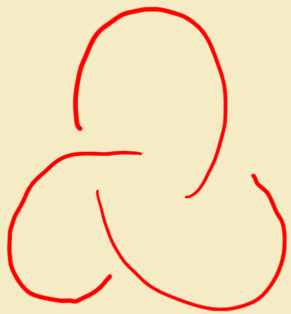
$$\boxed{J_K(q) = \chi_{gr}\big(Kh(K)\big) = \sum_{i,j} (-1)^i q^j \dim_{\mathbb{Z}/2} Kh^{i,j}(K)}$$

<u>Rough construction.</u> Combinatorial: given an n-crossing knot (diagram) K there is a composite of functors

$$\underline{2}^n \xrightarrow{\text{resolutions}} \mathrm{Cob}_2 \xrightarrow{\text{TQFT}} \mathrm{grVect}_{\mathbb{Z}/2}$$

which gives rise to a chain complex $C^{*,*}(K) \in \mathrm{Ch}(\mathrm{grVect}_{\mathbb{Z}/2})$, whose homology is $Kh^{*,*}(K)$.

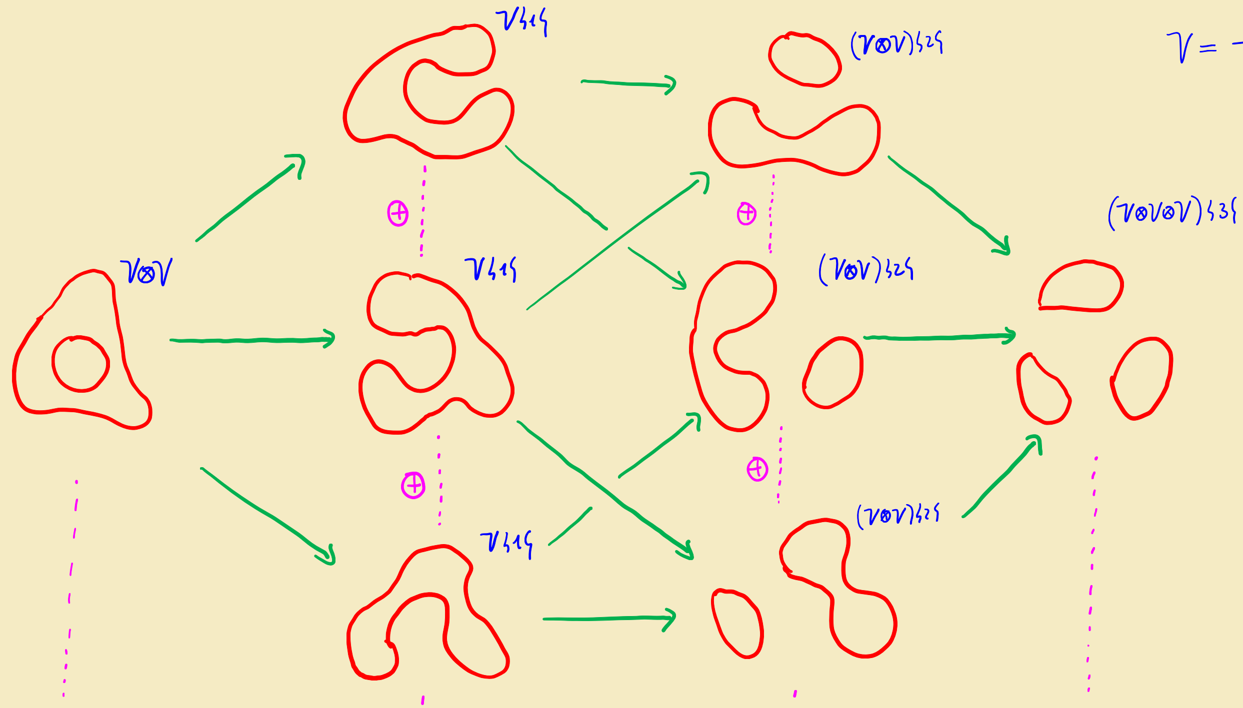
$$V = \frac{\mathbb{Z}/2\,[x]}{(x^2)}, \quad \deg(1) = 1, \quad \deg(x) = -1$$



$\bar{3}_1$

right-handed trefoil

$\rightsquigarrow$



$$C^{0,*}(\bar{3}_1) \xrightarrow{d} C^{1,*}(\bar{3}_1) \xrightarrow{d} C^{2,*}(\bar{3}_1) \xrightarrow{d} C^{3,*}(\bar{3}_1)$$

Example: The Khovanov homology of $\overline{3}_1$ is

$Kh^{i,j}(\overline{3}_1)$

j "q-grading"

$j \backslash i$	-1	0	1	2	3	4
9					$\mathbb{Z}/2$	
8						
7						
6						
5				$\mathbb{Z}/2$		
4						
3		$\mathbb{Z}/2$				
2						
1		$\mathbb{Z}/2$				

$\longrightarrow i$ "homological grading"

and

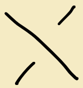
$$\chi_{gr}\left(Kh(\overline{3}_1)\right) =$$

$$= \sum_{i,j} (-1)^i q^j \dim_{\mathbb{Z}/2} Kh^{i,j}(\overline{3}_1)$$

$$= q + q^3 + q^5 - q^9$$

$$= J_{\overline{3}_1}(q)$$

Remark: Khovanov homology is strictly stronger than the Jones polynomial (eg $J_{5_1}(q) = J_{10_{132}}(q)$ but $Kh(5_1) \not\cong Kh(10_{132})$) and it also encodes the properties that $J_K(q)$ satisfies.

If $\overline{K}$ denotes the mirror image of K (replace  by ), then

$$J_{\overline{K}}(q) = J_K(q^{-1})$$

in other words, $J_K(q)$ detects mirror images (so long as it is not palindromic).

For Kh this property reads

$$Kh^{i,j}(\overline{K}) \cong Kh^{-i,-j}(K).$$

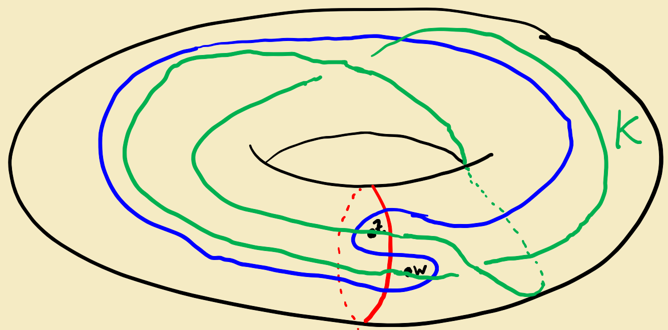
④ Knot Floer homology (P. Ozsváth - Z. Szabó, J. Rasmussen 04)

If $K \subset S^3$ is a knot, one can build a bigraded vector space over $\mathbb{Z}/2$

$$\widehat{HFK}(K) = \bigoplus_{m,s \in \mathbb{Z}} \widehat{HFK}_m(K,s)$$

which is an isotopy invariant of K.

Rough construction. Topological: given a "doubly-pointed Heegaard diagram" for K, build a chain complex over $\mathbb{Z}/2[u,v]$ whose differential "counts pseudo-holomorphic discs".



The major achivement of $\widehat{HFK}$ is that it categorifies the Alexander polynomial:

$$\boxed{\Delta_K(t) = \chi_{gr}\left(\widehat{HFK}(K)\right) = \sum_{m,s} (-1)^m t^s \dim_{\mathbb{Z}/2} \widehat{HFK}_m(K,s)}$$

<u>Example</u>:

$$\widehat{HFK}(\bar{3}_1) =$$

$s \uparrow$ \ $m \rightarrow$	0	1	2
1			$\mathbb{Z}/2$
0		$\mathbb{Z}/2$	
-1	$\mathbb{Z}/2$		

and $\chi_{gr}\left(\widehat{HFK}(\bar{3}_1)\right) = t^{-1} - 1 + t = \Delta_{\bar{3}_1}(t)$.

$\widehat{HFK}$ is a strictly stronger invariant than Alexander

(eg $\Delta_{11n34}(t) = \Delta_{11n42}(t)$ but $\widehat{HFK}(11n34) \not\cong \widehat{HFK}(11n42)$),

and not only encodes the properties of Δ_K but strengthens them!

$\Delta_K(t) = a_0 + \sum_{s \geq} a_s (t+t^{-1})$	$\widehat{HFK}$
Δ_K gives a lower bound for the knot genus: $g(K) \geq \frac{1}{2} \deg \Delta_K(t) = \max\{s : a_s \neq 0\}$	$\widehat{HFK}$ detects the knot genus: $g(K) = \max\{s : \widehat{HFK}(K,s) \neq 0\}$
If K is fibred $\Rightarrow$ Δ_K is monic, ie $a_{g(K)} = \pm 1$	$\widehat{HFK}$ detects fibreness: K fibred $\Longleftrightarrow$ $\widehat{HFK}(K, g(K)) \cong \mathbb{Z}/2$

Thanks for your attention

Slides can be found on my website: sites.google.com/view/becerra

References:

[1] Bar-Natan, D. – On Khovanov categorification of the Jones polynomial

[2] Ozsváth, P & Szabó, Z. – An overview of Knot Floer homology.

[3] Turner, P. – Five Lectures on Khovanov homology