Hopf algebras and the 2-loop polynomial of knots

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Groningen Algebra Seminar - 6 December 2022

Modern knot theory is tied to the area of *quantum topology* that arose in the 1980s after the work of the Fields medallists V. Jones, V. Drinfeld and E. Witten (the three of them in 1990). To construct invariants of objects in algebraic geometry or number theory, one typically only uses the object itself; think of the algebraic K-theory of a ring/scheme, the genus of an algebraic curve, the rank of a rational elliptic curve, etc.

Quantum topology often (and today) lies in the realm of low-dimensional topology, so our analogues of spaces are knots and links in S^3 (ie embeddings of one or several copies of S^1) and smooth manifolds of dimension at most four. Now we want to construct algebraic invariants of one of these objects *associated* to the extra data of an algebraic object $\mathbf{A}$. Some examples are:

- For knots and links, $\mathbf{A}$ can be a ribbon Hopf algebra [Law89], or it can be a certain power series in two variables [LM96], or it can also be a tortile (aka ribbon) category [Tur16].
- For 3-manifolds, $\mathbf{A}$ could be a ribbon Hopf algebra [Hen96], it can be a certain power series in two variables [LMO98], or it can be a modular category (a tortile category with more structure) [Tur16]. This is extremely related with the previous item via the Lickorish-Wallace theorem stating that any (closed, connected, oriented) 3-manifold can be obtained from a link with integers attached to every component by a process called *surgery* (homotopy theorists may know this as the fact that the third oriented cobordism group vanishes, $\Omega_3^{SO} \cong \pi_3(MSO) \cong 0$).
- For 4-manifolds, $\mathbf{A}$ can be a modular category [Tur16].

In this talk we will focus on the two first cases of the first item.

1 The category of tangles

Rather than just talking about knots and links, it is more convenient to generalise a bit and talk instead about "pieces of knots".

An *(oriented, framed) tangle* is an embedding L of finitely many copies of $I \times I$ and $I \times S^1$ into the cube I^3, proper in the sense that $L^{-1}(\partial I^3) = \amalg I \times \partial I$ and the segments $I \times 0$, $I \times 1$ are uniformly distributed along $I \times 1/2 \times \partial I$ with the same orientation. The cores $1/2 \times I$ and $1/2 \times S^1$ are required to be oriented. We regard tangles up to isotopy (ie a homotopy that needs to be an embedding at all times) rel. $\amalg I \times \partial I$.

Figure 1a shows an example of a tangle. Keeping in mind that a strip with a full twist is the same as a strip with a "φ" shape which we only see one of their faces, it is more common to draw only the oriented cores of a tangle with the "blackboard framing", that is, one has to make the strip parallel to the blackboard they have been drawn before. See Figure 1b.

By attaching a sign "$+$" or "$-$" to every endpoint of the tangle strands at the the bottom and top of the cube, depending whether the orientation is up or down respectively, we obtain two sequences of signs, that we call the *source* (at the bottom) and *target* (at the top). This allows us to organise tangles into a (strict) monoidal category $\mathcal{T}$:

- its objects are elements of $\mathrm{Mon}(+,-)$, the free monoid on the set $\{+,-\}$,
- a morphism between elements s and t are isotopy classes of tangles with source s and target t,

(a) (b)

Figure 1

- the composite $L_2 \circ L_1$ consists of stacking L_2 on top of L_1,
- the identity of $w \in \text{Mon}(+,-)$ consists of $|w|$ parallel vertical lines with the orientation being up/down depending whether the sign is $+$ or $-$,
- the monoidal product is given by concadenation at the level of objects and $L_1 \otimes L_2$ consists of placing L_2 to the right of L_1,
- the unit of the monoidal structure is the empty tangle.

2 Ribbon Hopf algebras

The universal tangle invariant is the main idea of this talk, and it is based on the concept of ribbon Hopf algebra.

Recall that a *Hopf algebra* over a ring k is a k-module A endowed with an algebra structure (A, μ, η), a coalgebra structure (A, Δ, ϵ) (compatible in the sense that the coalgebra structure maps are algebra morphisms), and an isomorphism $S : A \longrightarrow A$, called the *antipode*, which is the inverse of Id_A in the convolution monoid $\text{Hom}_k(A, A)$, that is

$$m(S \otimes \text{Id})\Delta = \eta\epsilon = m(\text{Id} \otimes S)\Delta.$$

A *quasi-triangular Hopf algebra* is a Hopf algebra A together with a preferred invertible element $R \in A \otimes A$, called the *universal R-matrix*, satisfying the following properties:

$$(\Delta \otimes \text{Id})R = R_{13} \cdot R_{23} \tag{1}$$
$$(\text{Id} \otimes \Delta)R = R_{13} \cdot R_{12} \tag{2}$$
$$\tau\Delta = R \cdot \Delta(-) \cdot R^{-1} \tag{3}$$

where $R_{12} := R \otimes \eta$, $R_{13} := (\text{Id} \otimes \tau)R_{12}$ and $R_{23} := \eta \otimes R$, where we put $\tau(x \otimes y) = y \otimes x$. We will write $R = \sum_i \alpha_i \otimes \beta_i$ for the universal R-matrix and $R^{-1} = \sum_i \overline{\alpha}_i \otimes \overline{\beta}_i$ for its inverse.

A *ribbon Hopf algebra* is a quasi-triangular Hopf algebra A together with a preferred invertible element $\kappa \in A$, called the *balancing element* satisfying

$$\Delta(\kappa) = \kappa \otimes \kappa \tag{4}$$
$$\epsilon(\kappa) = 1 \tag{5}$$
$$\kappa^2 = u \cdot S(u^{-1}) \tag{6}$$
$$S^2 = \kappa \cdot (-) \cdot \kappa^{-1} \tag{7}$$

where $u := \mu(S \otimes \text{Id}_A)R_{21} = \sum_i S(\beta_i) \cdot \alpha_i$ is called the *Drinfeld element* and $u^{-1} := \mu(\text{Id}_A \otimes S^2)R_{21} = \sum_i \beta_i \cdot S^2(\alpha_i)$ is its inverse. Here we wrote $R_{21} = \tau(R)$.

An important property of a ribbon Hopf algebra is that the universal R-matrix satisfies the so-called *Yang-Baxter equation*,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{8}$$

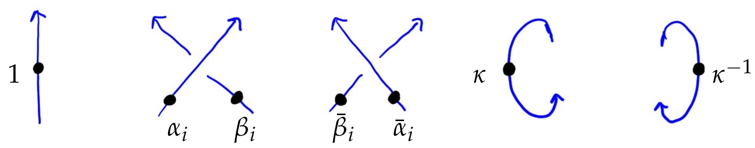


Figure 2

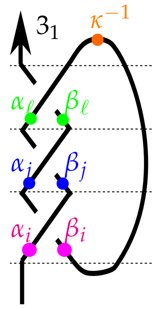


Figure 3: We have $Z_A(3_1) = \sum_{i,j,\ell} \alpha_i\, \beta_j\, \alpha_\ell\, \kappa^{-1}\, \beta_i\, \alpha_j\, \beta_\ell.$

3 The universal invariant

Let L be an open, upwards tangle, that is, a tangle with no closed components (embeddings of $I \times S^1$) and that it is an element of $\mathrm{Hom}_{\mathcal{T}}(+^n, +^n)$. Suppose besides that L has n components and that they are ordered.

Given a ribbon Hopf algebra A as in the previous section, we associate to each such L an element $Z_A(L) \in A^{\otimes n}$ as follows: we place beads representing the elements $1, R^{\pm 1}$ and $\kappa^{\pm 1}$ in a diagram of L according to Figure 2. Now for $1 \le i \le n$, let $Z_A(L)_{(i)}$ be the (formal) word given by writing from left to right the labels of the beads in the i-th component according to the orientation of the strand. Then put

$$Z_A(L) := \sum Z_A(L)_{(1)} \otimes \cdots \otimes Z_A(L)_{(n)} \in A^{\otimes n}$$

where the summation runs through all subindices in $R^{\pm 1}$. Figure 3 shows an example for the trefoil knot 3_1.

Theorem 3.1 *Z_A is an isotopy invariant of tangles.*

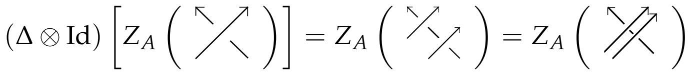
Proof sketch. Any two isotopic tangles are related by a sequence of moves called the *Reidemeister moves* (also called Turaev moves in this context). Some of these moves are the shown in Figure 4a and 4b. These two pairs of tangles get same invariant because of the invertibility of R and the Yang-Baxter equation (8), respectively. See [Oht02b] for a full proof. □

Remark 3.2 The universal invariant Z_A can be made into a strong monoidal functor, even into a monoidal natural transformation in a more flexible context where tangles are not embedded in the cube but instead they are viewed as graphs, see [Bec22c].

(a) (b)

Figure 4

Now we want to look at how the Hopf algebra structure maps behave with respect to the tangle under the universal invariant. Indeed (1) can be rephrased as



which suggests that, under the universal invariant Z_A, the comultiplication doubles the strand. This is the beginning of a rough dictionary

OPERATION	HOPF ALGEBRA	TANGLES
μ	multiplication	strand merging
η	unit	trivial strand
Δ	comultiplication	strand doubling
ε	counit	strand removal
S	antipode	"tweaked" strand orientation reversal

which can be made precise in terms of the universal invariant Z_A being a monoidal natural transformation map between algebras over the PROP for Hopf algebras, see [Bec22c].

Example 3.3 Let us describe three examples of ribbon Hopf algebras:

1. **Boring example:** $A = k[G]$ the group algebra of some group G with its usual Hopf algebra structure trivial ribbon structure, $R = 1 \otimes 1$ and $\kappa = 1$. This is cocommutative Hopf algebra and then it can only produce trivial knot invariants. Another boring example is $U(\mathfrak{g})$ the universal enveloping algebra of a finite-dimensional semisimple Lie algebra.

2. **Classical example:** $A = U_h(\mathfrak{g})$ a "quantisation" of $U(\mathfrak{g})$, typically called a *quantum group*. The word "quantisation" means that $U_h(\mathfrak{g})/hU_h(\mathfrak{g}) \cong U(\mathfrak{g})$ as $\mathbb{C}$-algebras and that $U_h(\mathfrak{g}) \cong U(\mathfrak{g})[[h]]$ as $\mathbb{C}[[h]]$-modules. This is ribbon in a highly non-trivial way, but too high: in practice the calculation turns out to be not feasible: one has to perform infinite many non-commutative multiplications and there is no known way of doing this in an efficient way.

3. **Desired example:** $A = \mathbb{D}$ a (topological) ribbon Hopf algebra over the ring $\mathbb{Q}_\varepsilon[[h]]$, where $\mathbb{Q}_\varepsilon := \mathbb{Q}[\varepsilon]$. As a topological $\mathbb{Q}_\varepsilon[[h]]$-module, $\mathbb{D} \cong \mathbb{Q}_\varepsilon[x_1, x_2, x_3, x_4][[h]]$, that is, it is a (topologically) free $\mathbb{Q}_\varepsilon[[h]]$-module. This Hopf algebra arises from a construction called *Drinfeld double* [Kas95] and it is related to $U_h(\mathfrak{sl}_2)$. As it will turn out, this one has the right balance between being sufficiently non-trivial to produce strong knot invariants and having a structure that allows to compute efficiently. More precisely, in the algebra $\mathbb{D}$ the parameter ε plays a crucial role: it allows us to truncate the universal invariant leaving us with a "simpler" invariant.

Claim 3.4 ([BNvdV19, BNvdV21]) *For any $N > 0$, the universal invariant $Z_{\mathbb{D}}(L) \pmod{\varepsilon^N}$ can be effectively computed (in polynomial time).*

The main ingredient of this is the so-called *Gaussian calculus*, which roughly speaking consists of turning a linear map into a perturbed exponential expression representing a power series, with rules to perform the composite of the linear maps using the exponential expressions. We refer the reader to [BNvdV21] and [Bec22a].

For the following key theorem we need one definition. Given a knot K, let K_0 and K_1 be the embeddings of $0 \times I$ and $1 \times I$. The *framing* of K is the integer

$$\mathrm{fr}(K) := \frac{1}{2} \sum_{p \in K_0 \cap K_1} \mathrm{sign}(p)$$

(p refers to the crossing) where $\mathrm{sign}(p) = 1$ if the crossing is positive and $\mathrm{sign}(p) = -1$ if it is negative.

Theorem 3.5 ([BNvdV21]) *For any 0-framed knot K, there exist knot polynomial invariants*

$$\rho_K^{k,j} \in \mathbb{Q}[t, t^{-1}],$$

$k \geq 0$, $0 \leq j \leq 2k$, such that

$$Z_{\mathbb{D}}(K) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{2k} h^{k+j} \frac{\rho_K^{k,j}(T)}{\Delta_K^{2k+1-j}(T)} w^j \right) \varepsilon^k$$

where $T := e^{-hx_1}$ and w are central elements.

In the statement above, $\Delta_K(t) \in \mathbb{Z}[t + t^{-1}]$ is the *Alexander polynomial* of K (as unframed knot). This is a classical knot polynomial invariant with a well-understood topological interpretation [Lic97] as well as algebraic, in terms of representation theory.

The main takeaway from this theorem is that knowing the universal invariant $Z_A(K)$ is equivalent to knowing $\Delta_K(t)$ and the collection $\rho_K^{k,j}(t)$. So the question I would like to address then is: what can we say about these $\rho_K^{k,j}$'s? First off, it turns out that half of them are trivial:

Proposition 3.6 ([Bec22b]) *For a 0-framed knot K, we have $\rho_K^{k,j}(t) = 0$ for all $j > k > 0$.*

One would expect the collection of polynomials $\rho_K^{k,j}$ to be brand new invariants (or at least a subset of these). However, both theoretical and experimental results suggest that the first nontrivial polynomial $\rho_K^{1,0}$ (it is a fact that $\rho_K^{0,0} = 1$ for any knot K) coincides with a not-very-well-understood, hard to compute knot polynomial invariant Θ_K, called the *2-loop polynomial*, that arises from the most powerful knot invariant up to date, that we describe in the next section.

Conjecture 3.7 ([BNvdV19, BNvdV21]) *For a 0-framed knot K, the polynomial $\rho_K^{1,0}$ coincides with the 2-loop polynomial Θ_K of K.*

In the following section we will briefly sketch how this polynomial arises.

4 The Kontsevich invariant and the 2-loop polynomial

The Kontsevich invariant [Kon93] is a tangle invariant that depends on a choice of *Drinfeld associator*, a certain element $\varphi \in \mathbb{Q}\langle\langle X, Y\rangle\rangle$, the ring of formal power series in two noncommutative variables X, Y (although for knots and links the invariant is independent of this choice). The Kontsevich invariant is (one of) the strongest knot invariant(s) we know; in fact it is a conjecture that the Kontsevich invariant distinguishes all (oriented) knots [Oht02a]. Moreover, it is known that it dominates a large family of knot invariants, called the *Reshtikhin-Turaev invariants*, that have been extensively studied for the last 30 years.

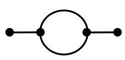
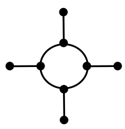
The invariant can be arranged as a strong monoidal functor [LM96, Oht02b, HM21]

$$Z = Z_\varphi : \mathcal{T}_q \longrightarrow \widehat{\mathcal{A}}$$

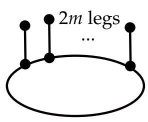
where

- $\mathcal{T}_q$ is the *nonstrictification* of the category of tangles $\mathcal{T}$, which is a nonstrict monoidal category monoidally equivalent to $\mathcal{T}$. Its objects are the elements of $\mathrm{Mag}(+,-)$, the free unital magma on the set $\{+,-\}$ (that is parenthesised sequences of signs). There is an obvious map $U : \mathrm{Mag}(+,-) \longrightarrow \mathrm{Mon}(+,-)$ that forgets parentheses. A map $w \longrightarrow w'$ in $\mathcal{T}_q$ is by definition a map $Uw \longrightarrow Uw'$ in $\mathcal{T}$.
- $\widehat{\mathcal{A}}$ is the (degree-completion of the) category of *Jacobi diagrams*

We will not go into how the category $\widehat{\mathcal{A}}$ is defined. The important bit is that for a (long) knot K, the value $Z(K)$ is an infinite linear combination with rational coefficients of unitrivalent graphs. For instance, one can show that

$Z(\text{unknot}) = \varnothing + \frac{1}{48}$  $+ \frac{1}{23040}$  $+ \cdots.$

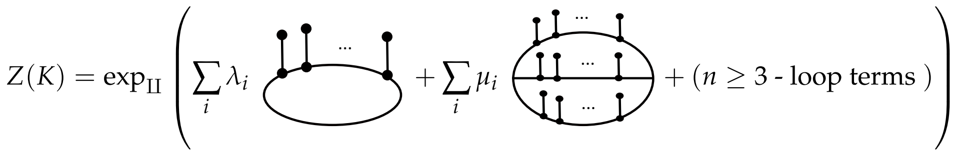
In fact the previous expression is actually given by an exponential [BNLT03]:

$Z(\text{unknot}) = \exp_{\sqcup}\left(\sum_{m\geq 1} b_{2m} \right.$ 

$\left. \right)$

where for a unitrivalent graph D, $\exp_{\amalg}(D) := \emptyset + D + \frac{1}{2}D \amalg D + \cdots$, extended linearly; and b_{2m} are the *modified Bernoulli numbers*, $\sum_{m\geq 1} b_{2m}X^{2m} := \frac{1}{2}\log\left(\frac{\sinh(X/2)}{X/2}\right) \in \mathbb{Q}[[X]]$.

In fact these two features of the value of the Kontsevich invariant of the unknot, namely the exponentiation and the legs, are common to all knots. More precisely, a connected unitrivalent graph is called *n-loop* if its Euler characteristic is $\chi(D) = 1 - n$. It is known [Kri00, GK04] that the Kontsevich invariant of has a loop expansion



The first summand, that is the 1-loop part, can be shown to be tantamount to the Alexander polynomial Δ_K of the knot. The second summand, ie the 2-loop part, can be shown to be tantamount to a fully symmetric Laurent polynomial $\Theta'(t_1, t_2) \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}]$, which is called the *(unreduced) 2-loop polynomial*. Here "fully symmetric" means that $\Theta'(t_1, t_2) = \Theta'(t_i^{\pm 1}, t_j^{\pm 1})$ for $\{i, j\} = \{1, 2\}$.

Remark 4.1 It turns out that this unreduced 2-loop polynomial $\Theta'(t_1, t_2)$ is a strong polynomial invariant. More specifically, it is able to detect *mutation* [Mor15], a operation on knots consisting on removing a piece of the knot, rotate it and glue it back to create a new knot. This process creates pairs of knots that are indistinguishable from the point of view of classical knot polynomial invariants (they have the same HOMFLY-PT and Kauffman polynomials, and hence the same Alexander and Jones polynomials).

The polynomial appearing in the conjecture above is a reduced version of this one, by letting the second variable be 1,

$$\Theta(t) := \Theta'(t, 1) \in \mathbb{Q}[t + t^{-1}] \subset \mathbb{Q}[t, t^{-1}].$$

5 Merging of the two worlds

Recently we have given a positive answer to the conjecture for a particular class of knots. Recall that a knot in S^3 has *genus* ≤ 1 if it bounds a compact, connected, oriented surface of genus one in S^3.

Theorem 5.1 ([Bec22a]) *The Conjecture 3.7 holds for knots of genus* ≤ 1*, that is,*

$$\rho_K^{1,0}(t) = \Theta_K(t)$$

if $\mathrm{genus}(K) \leq 1$.

The proof is quite technical and heavily relies on the so-called Gaussian calculus. Developing this tool would take a full mini-course, so here we would only like to give an idea of the argument.

The main point is that every knot K of genus ≤ 1 arises as the *thickening* of an open, upwards 2-component tangle L, as depicted in Fig 5. Closing up the open component allows to see the genus 1 surface that bounds $K = Th(L)$.

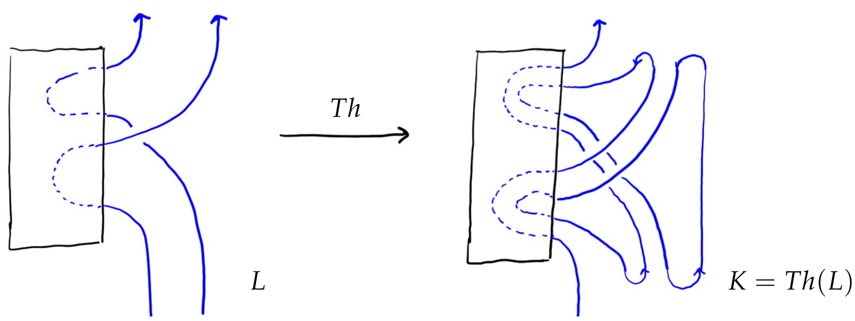


Figure 5

Ohtsuki [Oht07] has given a closed expression for Θ_K for such a genus ≤ 1- knot K in terms of well-known integral invariants of L; namely the linking matrix of L and Vassiliev invariants of order 2 and 3 of L [Oht02b].

It turns out that there is an algebraic analogue of the thickening map

$$\breve{Th} : \mathbb{D} \otimes \mathbb{D} \longrightarrow \mathbb{D}$$

such that

$$Z_{\mathbb{D}}(K) = Z_{\mathbb{D}}(Th(L)) = \breve{Th}(Z_{\mathbb{D}}(L))$$

(this uses the Hopf algebra structure maps), hence all one has to study is

$$Z_{\mathbb{D}}(L) \in \mathbb{D} \otimes \mathbb{D} \cong \mathbb{Q}_{\varepsilon}[x_1, x_2, x_3, x_4, x'_1, x'_2, x'_3, x'_4][[h]].$$

The key point in the argument is that this element depends exactly on the same integral invariants as Ohtsuki's expression for the 2-loop polynomial, and the algebraic thickening map $\breve{Th}$ rearranges these terms giving the equality in the theorem.

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