Three faces of the 2-loop polynomial of knots

BYMAT 2022

9 Nov

Jorge Becerra (Groningen)

Setup:

Knot theory $\cap$ Quantum topology

Goal: Describe a strong knot polynomial invariant, called the 2-loop polynomial $\Theta_K(t) \in \mathbb{Z}[t+t^{-1}] \subset \mathbb{Z}[t, t^{-1}]$ from three different points of view.

$\boxed{\text{Take 1}}$ (Rozansky, Ohtsuki)

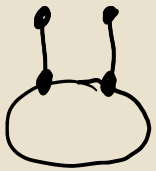
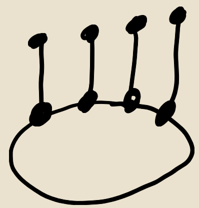
There is a (functorial) knot invariant, called the Kontsevich invariant

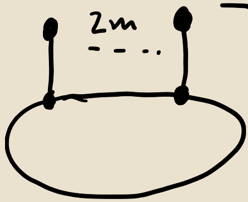
$$Z : \mathcal{T}_q \longrightarrow \hat{\mathcal{A}} \qquad \text{(Le – Murakami)}$$

$\cup$

knots

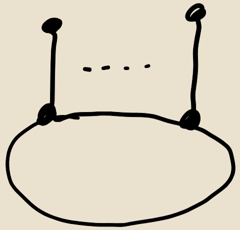
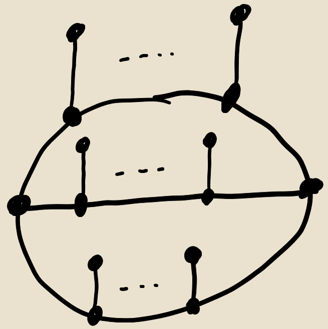
such that $Z(K)$ is an infinite formal linear combination of uni-trivalent graphs, eg

$$Z(\bigcirc) = \phi + \frac{1}{48}$$  $$+ \frac{1}{23040}$$  $$+ \cdots$$

$$= \exp_{\sqcup}\Big[\sum_{m\geq 1} b_{2m}$$ 

$$\Big]$$ (Bar-Natan, Le, Thurston)

In general (Kriker - Rozansky)

$Z(K) = \exp_{\sqcup}\Big[\sum \lambda_i \cdot$  $+ \sum \mu_i \cdot$  $+ \cdots \Big]$

tantamount to the celebrated Alexander polynomial Δ_K of K

gives rise to a knot polynomial invariant $\Theta_K(t) \in \mathbb{Z}[t+t^{-1}]$

$\boxed{\text{Take 2}}$ (Rozansky , Melvin-Morton)

$$sl_2 = sl_2(\mathbb{C}) \rightsquigarrow U(sl_2) \rightsquigarrow U_h(sl_2) \rightsquigarrow \mathrm{Mod}_{U_h(sl_2)}$$

a Lie algebra over $\mathbb{C}$

an algebra over $\mathbb{C}$

a (top) algebra over $\mathbb{C}[[h]]$

its category of representations

<u>Fact</u>. $U_h(sl_2)$ has a unique rank n irreducible representation V_n, $n \geq 1$.

* For any knot K, one can associate an isotopy invariant $J_K^n(q) \in \mathbb{Z}[q, q^{-1}]$ cooked up from V_n (as a knot-theoretical analogue of a TQFT).

Theorem (Rozansky): For any knot K, there exist knot polynomial invariants

$$P_K^i \in \mathbb{Z}[t+t^{-1}] \quad , \quad i \geq 0$$

such that

$$J_K^n(q) = \sum_{i=0}^{\infty} \frac{P_K^i(q^n)}{\Delta_K^{2i+1}(q^n)} (q-1)^i \in \mathbb{Q}[[q-1]]$$

where Δ_K is the celebrated Alexander polynomial of K.

Fact: $\Theta_K(t) = P_K^1(t)$ for any knot K.

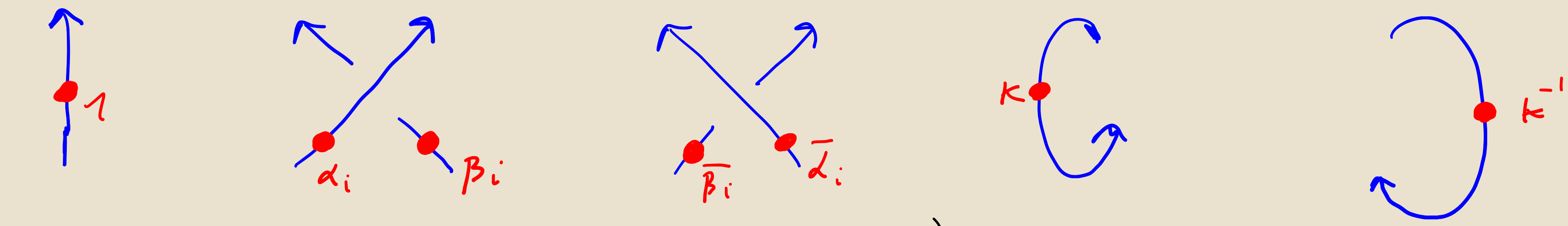
$\boxed{\text{Take 3}}$ (Bar-Natan, van der Veen, B.)

If A is a ribbon Hopf algebra, ie a (Hopf) algebra with preferred invertible elements

$$R = \sum_i \alpha_i \otimes \beta_i \in A \otimes A \quad , \quad \kappa \in A$$

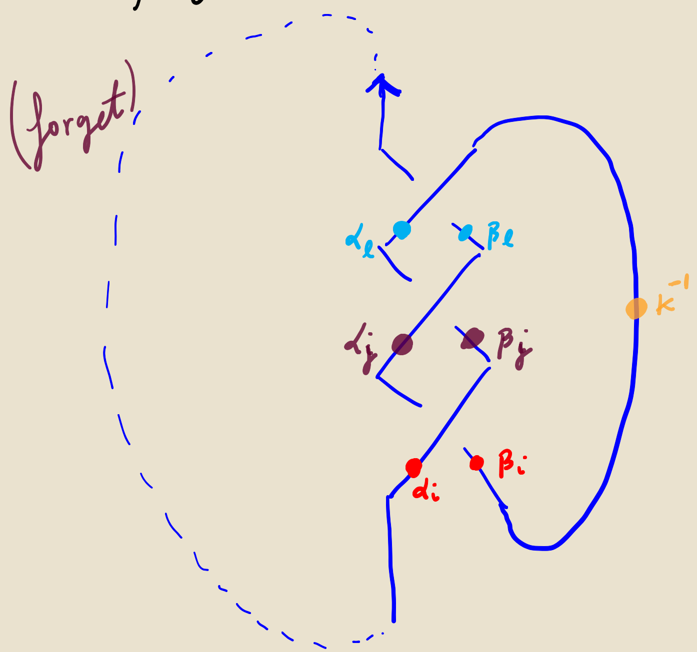
(satisfying axioms), then one can define the so-called <u>universal</u> tangle <u>invariant</u> associated to A :

Place beads in a diagram of K as follows:



$$\left(R^{-1} = \sum \bar{\alpha}_i \otimes \bar{\beta}_i \right)$$

and multiply along the strand, eg



$$Z_A(K) = \sum_{i\,j\,\ell} \alpha_i \, \beta_j \, \alpha_\ell \, \kappa^{-1} \beta_i \, \alpha_j \, \beta_\ell \quad \in A$$

Take $A = U_h(gl_{2,\varepsilon})$ a ribbon Hopf algebra over $\mathbb{Q}[\varepsilon][[h]]$.

<u>Theorem</u>. For any knot K, there exist knot polynomial invariants

$$\rho_K^{i,j}(t) \in \mathbb{Q}[t,t^{-1}] \qquad i \geq 0, \quad 0 \leq j \leq i,$$

such that

$$Z_{U_h(gl_{2,\varepsilon})}(K) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} h^{i+j} \cdot \frac{\rho_K^{i,j}(T)}{\Delta_K^{2j+1}(T)} \cdot w^j \right) \varepsilon^i$$

where $T, w \in U_h(gl_{2,\varepsilon})$ are central elements.

Conjecture (BNV) : For any knot K,

$$\rho_K^{1,0}(t) = \Theta_K(t) = P_K^1(t)$$

Theorem (B, 2022). If K is a genus ≤ 1 knot, ie, if K bounds a compact, connected, oriented surface of genus 1 in $\mathbb{R}^3$, then the conjecture holds.

Thank you

for your attention.

Slides available at bit.do/jbecerra

- J. Becerra – A Hopf-algebraic construction of the 2-loop polynomial for genus one knots (in preparation)

- D. Bar-Natan, R. van der Veen – Perturbed gaussian generating functions for universal knot invariants, arXiv.2109.02057