LECTURE 3: SIMPLICIAL SETS

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Simplicial sets are the combinatorial model to study algebraic topology, described in a purely categorical way. These are a generalization of the more naive simplicial complexes, that we discuss in first place to give a better idea of what comes next.

They seem scary abstract objects, but you have already treated them (if you took Algebraic Topology I)! Today we will make a more abstract exposition and will highlight the relation between simplicial sets and topological spaces.

The right context to tackle simplicial sets is category theory. I will explicitly state some important results I will use along the talk, but just in case this is your first serious approach to category theory, there is a quick overview in the appendix.

1 Simplicial complexes

We will start describing combinatorial geometric objects called simplicial complexes. The aim of this section is to motivate the world of simplicial sets and its properties.

Definition. An (ordered) **simplicial complex** $(K, \mathcal{P})$ is a (totally ordered) set K together with a family $\mathcal{P}$ of (ordered) non-empty finite subsets of K, called **abstract simplices**, satisfying

(i) Every singleton is an element of $\mathcal{P}$, and they are called **vertices**.

(ii) Every subset of a simplex is a simplex.

If the ordered subset $\{v_0, \ldots, v_n\}$ belongs to $\mathcal{P}$, we will denote this n-simplex by $[v_0, \ldots, v_n]$. We will write $K_n \subset \mathcal{P}$ to the set of n-simplices.

If K is a simplicial complex, we can consider **face maps** $d_i : K_n \longrightarrow K_{n-1}$, $0 \leq i \leq n$, mapping an n-simplex to the "opposite of the i-th vertex",

$$d_i[v_0, \ldots, v_n] := [v_0, \ldots, \widehat{v_i}, \ldots, v_n].$$

Definition. Let K be a simplicial set, and consider the vector space

$$E := \bigoplus_{v \in K} \mathbb{R}v.$$

For every n-simplex $S = [v_0, \ldots, v_n]$, we can consider the linear subspace $\mathbb{R}v_0 \oplus \cdots \oplus \mathbb{R}v_n$, endowed with the topology defined by any norm. We denote by $|S|$ the convex hull of S in the finite-dimensional space, with the subspace topology.

The **geometric realization** of K is the union

$$|K| := \bigcup_{S \in \mathcal{P}} |S| \subset E$$

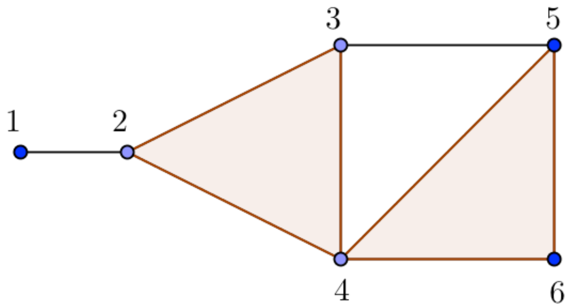
endowed with the final (strong) topology induced by the inclusions $|S| \hookrightarrow |K|$.

Example 1.1 Consider the simplicial complex with

$$K = \{1, 2, 3, 4, 5, 6\}$$

$$\mathcal{P} = \left\{ \begin{array}{c} [2,3,4]\ ,\ [4,5,6],\ [1,2],\ [3,5] \\ \text{and subsets of the} \\ \text{previous simplices} \end{array} \right\}$$

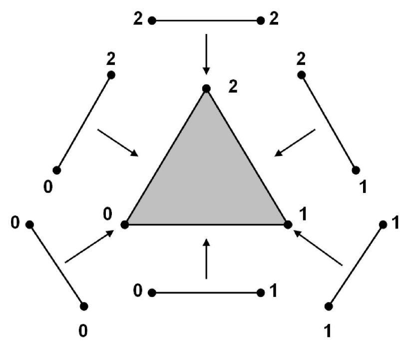
Its geometric realization is the topological space of the picture:



Now the face maps are realized in a geometric way as sending the simplices to one of its faces.

The key point about simplicial complexes is that they are combinatorial objects, and their simplices are completely determined by their vertices; that is, there is at most one n-simplex for a subset of $n+1$ vertices, depending on whether such a subset is or is not in $\mathcal{P}$. In particular, all simplices have the "right dimension", that is, an n-simplex is determined by $n+1$ different vertices. One could go a step further and allow *degenerate* vertices, ie, simplices $[v_0, \ldots, v_n]$ with $v_i \leq v_j$ if $i < j$. Geometrically, this means that we can consider "edges" collapsed into a point still as an edge, for instance. This is the idea behind simplicial sets.

Example 1.2 Consider the simplicial complex with $K = \{1, 2, 3\}$ and $\mathcal{P}$ is the power set of K (except $\emptyset$). There are 3 nondegenerate 1-simplices; but if we also count the degenerate ones, then we get 9. This is depicted in the figure:



In a similar fashion to the face maps, we can consider also degeneracy maps $s_i : K_n \longrightarrow K_{n-1}$, $0 \leq i \leq n$, defined as

$$s_i[v_0, \ldots, v_n] := [v_0, \ldots, v_i, v_i, \ldots, v_n].$$

In other words, we send every n-simplex to a "degenerated" $(n+1)$-simplex. The correct setup where we can formalize all these ideas is the world of simplicial sets.

2 Simplicial sets

Let Δ be the category whose objects are ordered sets $[n] := \{0, 1, \ldots, n\}$ and whose morphisms are order-preserving maps $\alpha : [n] \longrightarrow [m]$, that is, $\alpha(i) \leq \alpha(j)$ if $i \leq j$.

In this category, we have two preferred classes of arrows: let $d^i : [n-1] \longrightarrow [n]$ be the unique order-preserving injection not hitting i; and let $s^i : [n+1] \longrightarrow [n]$ be the unique order-preserving surjection hitting i twice, for $0 \leq i \leq n$. The maps d^i are called **coface** maps; whereas the maps s^i are called **codegeneracy** maps .

Lemma 2.1 *The coface and codegenerary maps satisfy the following **cosimplicial identities**:*

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, & i < j \\ s^j s^i &= s^i s^{j+1}, & i < j \\ s^j d^i &= \begin{cases} \mathrm{Id}, & i = j, j+1 \\ d^i s^{j-1}, & i < j \\ d^{i-1} s^j, & i > j+1 \end{cases} \end{aligned}$$

Proof. This is an easy, painful exercise. $\square$

The importance of the coface and codegeneray maps relies in the following fact:

Lemma 2.2 *Every morphism of Δ can be expressed as a composite of coface and codegeneracy maps.*

Proof. Let $\alpha : [n] \longrightarrow [m]$ an order preserving map. If $k_s < \cdots < k_1$ are the elements of $[m]$ not in $\alpha([n])$, and $p_1 < \cdots < p_r$ are the elements of $[n]$ such that $\alpha(p_i) = \alpha(p_i + 1)$, then $\alpha = d^{k_1} \cdots d^{k_s} s^{p_1} \cdots s^{p_r}$. $\square$

Let us finally give the definition we were chasing:

Definition. A **simplicial set** is a functor $X : \Delta^{op} \longrightarrow \mathsf{Set}$.

Explicitly, it consists of a sequence of sets X_n, $n \geq 0$, and *structure maps* $\alpha^* : X_m \longrightarrow X_n$ for every order-preserving map $\alpha : [n] \longrightarrow [m]$.

If X is a simplicial set, we write

$$\begin{aligned} d_i &:= X(d^i) : X_n \longrightarrow X_{n-1} \\ s_i &:= X(s^i) : X_n \longrightarrow X_{n+1} \end{aligned}$$

and call these the **face** and **degeneracy** maps. From the lemma it follows

Lemma 2.3 *The face and degenerary maps satisfy the following **simplicial identities**:*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, & i < j \\ s_i s_j &= s_{j+1} s_i, & i < j \\ d_i s_j &= \begin{cases} \mathrm{Id}, & i = j, j+1 \\ s_{j-1} d_i, & i < j \\ s_j d_{i-1}, & i > j+1 \end{cases} \end{aligned}$$

In particular, using lemmata 2.2 and 2.3 we get an equivalent definition of simplicial set:

Definition. **(Alternative)** A simplicial set is a sequence of sets X_n, $n \geq 0$, together with face and degeneracy maps

$$d_i : X_n \longrightarrow X_{n-1} \qquad , \qquad s_i : X_n \longrightarrow X_{n+1},$$

$0 \leq i \leq n$, satisfying the simplicial identities.

Definition. Let X, Y be simplicial sets. A **map** of simplicial sets or **simplicial morphism** is a natural transformation of functors $f : X \longrightarrow Y$.

Simplicial sets and simplicial morphisms form a category $\mathsf{sSet} := \mathsf{Set}^{\Delta^{op}}$. One also says that sSet is the category of **presheaves** on Δ, $\mathsf{sSet} = \mathsf{PSh}(\Delta)$.

More generally, for any category $\mathcal{C}$, one can define a **simplicial object** in $\mathcal{C}$ as a functor $X : \Delta^{op} \longrightarrow \mathcal{C}$. The category of simplicial objects and natural transformations is denoted $\mathsf{s}\mathcal{C} := \mathcal{C}^{\Delta^{op}}$.

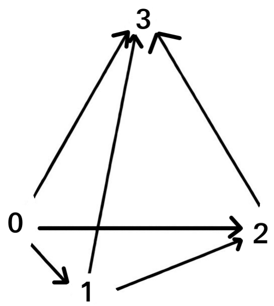
Examples 2.4 (a) The **standard k-simplex** is the simplicial set represented by $[k]$,

$$\underline{\Delta}^k := \mathrm{Hom}_\Delta(-, [k]).$$

Explicitly, for a map $\alpha : [n] \longrightarrow [m]$ we get a map

$$\alpha^* : \underline{\Delta}^k_m \longrightarrow \underline{\Delta}^k_n \qquad , \qquad ([m] \xrightarrow{\beta} [k]) \mapsto ([n] \xrightarrow{\alpha} [m] \xrightarrow{\beta} [k]).$$

So an element of $\underline{\Delta}^k_0 = \mathrm{Hom}_\Delta([0], [k])$ corresponds with picking an element $i \in [n] = \{0, 1, \ldots, n\}$ ("vertices"); an element of $\underline{\Delta}^k_1 = \mathrm{Hom}_\Delta([1], [k])$ corresponds with picking an arrow $i \longrightarrow j$ for $i \leq j$ ("edge"), etc.



(b) Recall from Algebraic Topology I that the first step to define singular homology of a topological space T was to consider the sets of *singular n-simplices*

$$\mathcal{S}(T)_n := \{\text{continuous maps } \sigma : \Delta^n \longrightarrow T\} = \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, T).$$

There is a functor $\Delta \longrightarrow \mathsf{Top}$ sending $[n]$ to the n-simplex Δ^n. This assignment is functorial as for $\alpha : [n] \longrightarrow [m]$ we get a map

$$\alpha_* : \Delta^n \longrightarrow \Delta^m \qquad , \qquad (t_0, \ldots, t_n) \mapsto \left(\sum_{i \in \alpha^{-1}(0)} t_i, \ldots, \sum_{i \in \alpha^{-1}(m)} t_i \right).$$

Composing this with the functor $\mathrm{Hom}_{\mathsf{Top}}(-, T) : \mathsf{Top}^{op} \longrightarrow \mathsf{Set}$ we get the **singular simplicial set**

$$\mathcal{S}(T) : \Delta^{op} \longrightarrow \mathsf{Set}.$$

The face maps $d_i : \mathcal{S}(T)_n \longrightarrow \mathcal{S}(T)_{n-1}$ were used in the course to induce face maps in their A-linearizations

$$d_i : A[\mathcal{S}(T)_n] = C_n(T;A) \longrightarrow C_{n-1}(T;A) = A[\mathcal{S}(T)_{n-1}]$$

(where A is an abelian group) and from there define the singular boundary operator $\partial := \sum_{i=0}^{n}(-1)^i d_i$ which gives rise to singular homology.

(c) If X is a simplicial set and A is an abelian group, we can construct a simplicial abelian group AX by setting $AX_n := A[X_n]$ the A-linearlization of the set of n-simplices of X. The structure maps are the maps induced in the A-linearlization.

This simplicial abelian group has associated to it a chain complex of abelian groups

$$C_*(AX) := (\cdots \xrightarrow{\partial} AX_2 \xrightarrow{\partial} AX_1 \xrightarrow{\partial} AX_0)$$

called **Moore complex**, with $\partial_n := \sum_{i=0}^{n}(-1)^i d_i$. Therefore we can define the **simplicial homology** of a simplicial set as

$$H_n(X;A) := H_n(C_*(AX)).$$

In particular for a space T, the singular homology of T is the simplicial homology of its singular simplicial set $\mathcal{S}(T)$.

Moreover, for a map of topological spaces $f : T \longrightarrow T'$ there is an induced map of simplicial sets $f_* : \mathcal{S}(T) \longrightarrow \mathcal{S}(T')$ given levelwise by $f_*(\sigma) := f \circ \sigma$. Therefore, we can also consider the so-called **singular functor**

$$\mathcal{S} : \mathsf{Top} \longrightarrow \mathsf{sSet},$$

so that singular homology $H_n(-;A)$ arises as the composite

$$\mathsf{Top} \xrightarrow{\mathcal{S}(-)} \mathsf{sSet} \xrightarrow{A[-]} \mathsf{sAbGrp} \xrightarrow{C_*(-)} \mathsf{Ch}_{\geq 0} \xrightarrow{H_n(-)} \mathsf{AbGrp}$$

(d) Let $\mathcal{C}$ be a small category. The **nerve** or **classifying space** of $\mathcal{C}$ is the simplicial set $B\mathcal{C}$ defined by $B\mathcal{C}_n := \mathcal{C}^{[n]}$ where we interpret $[n]$ as a category with objects $\{0,1,\ldots,n\}$ and a unique arrow $i \longrightarrow j$ if $i \leq j$. Explicitly: $B\mathcal{C}_0$ is the set of objects of $\mathcal{C}$; $B\mathcal{C}_1$ is the set of morphisms in $\mathcal{C}$; $B\mathcal{C}_2$ is the set of commutative triangles, etc.

The face maps are given by

$$d_i : [C_0 \longrightarrow \cdots \longrightarrow C_n] := [C_0 \longrightarrow \cdots \longrightarrow \widehat{C_i} \longrightarrow \cdots \longrightarrow C_n]$$

where the notation $\widehat{C_i}$ means that C_i is ommited for $i = 0, n$ and replaced by the composition $C_{i-1} \longrightarrow C_{i+1}$ for $1 \leq i \leq n-1$. The degeneracy maps s_i insert the identity $\mathrm{Id} : C_i \longrightarrow C_i$ in the i-th position.

In particular, if G is a group, we can interpret G as a category with a single object $*$ and an arrow for every element of the group G. The **classifying space** of a group G is the classifying space of G viewed as a category.

For simplicial sets, the Yoneda lemma has straightforward consequences. From A.3 it follows

Corollary 2.5 *There are natural bijections*

$$\mathrm{Hom}_{\Delta}([n],[m]) \cong \mathrm{Hom}_{\mathsf{sSet}}(\underline{\Delta}^n, \underline{\Delta}^m)$$

This is the categorical version of the geometric fact that affine maps between simplices are determined by the image of the vertices.

The more robust version of the Yoneda lemma A.4 implies

Corollary 2.6 *For any simplical set X, there are bijections*

$$X_n \cong \mathrm{Hom}_{\mathsf{sSet}}(\underline{\Delta}^n, X).$$

More precisely, if $\iota_n = \mathrm{Id}_{[n]} \in \underline{\Delta}^n_n$, then the bijection associates the n-simplex $f_n(\iota_n) \in X_n$ for every simplicial map $f : \underline{\Delta}^n \longrightarrow X$.

This means that an n-simplex $x \in X_n$ can be viewed as a map $x : \underline{\Delta}^n \longrightarrow X$.

3 Geometric realization

In section 1, we built a topological space out of a simplicial complex. We want to pursue the same goal for simplicial sets. The key difference is that for simplicial complexes, all abstract simplices had the "right" dimension, that is, they were determined by an array of different points. In simplicial sets we deal with degenerate simplices, so we have to polish the way we treated that before.

The first observation is a merely categorical consequence of the Density theorem A.8:

Lemma 3.1 *Let X be a simplicial set. There is a natural isomorphism*

$$X \cong \operatorname{colim}\left(\int X \xrightarrow{U} \Delta \hookrightarrow \mathsf{sSet}\right).$$

This motivates the following

Definition. Let X be a simplicial set. Its **geometric realization** is the topological space

$$|X| := \operatorname{colim}\left(\int X \xrightarrow{U} \Delta \longrightarrow \mathsf{Top}\right).$$

The category Top is complete and cocomplete (ie, it admits all small limits and colimits), and in particular we can give a more down-to-Earth description of the above topological space:

$$|X| = \frac{\coprod_{n\geq 0} X_n \times \Delta^n}{(\alpha^*(x), t) \sim (x, \alpha_*(t))},$$

for all $x \in X_n$, $t \in \Delta^m$ and $\alpha : [n] \longrightarrow [m]$. Here we consider every X_n with the discrete topology. The relations $(d_i(x), t) \sim (x, d^i(t))$ identify common edges; whereas the relations $(s_i(x), t) \sim (x, s^i(t))$ suppress degenerate simplices, ie, images of the degeneracy maps ("simplices which do not have the right dimension").

In particular, any simplicial map $f : X \longrightarrow Y$ induces a natural map of spaces $|f| : |X| \longrightarrow |Y|$, showing the geometric realization as a functor $|\cdot| : \mathsf{sSet} \longrightarrow \mathsf{Top}$.

Theorem 3.2 *The geometric realization functor is left-adjoint to the singular functor,*

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "|\cdot|"] \arrow[r, phantom, "\perp"] & \mathsf{Top} \arrow[l, bend left, "\mathcal{S}"] \end{tikzcd}$$

Proof. We have the following chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Top}}(|X|, T) &\cong \mathrm{Hom}_{\mathsf{Top}}(\operatorname{colim} \Delta^n, T)\\ &\cong \lim \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, T)\\ &\cong \lim \mathrm{Hom}_{\mathsf{sSet}}(\underline{\Delta}^n, \mathcal{S}(T))\\ &\cong \mathrm{Hom}_{\mathsf{sSet}}(X, \mathcal{S}(T)).\end{aligned}$$

The first isomorphism is the definition; the second is A.6.(ii); the third is 2.6 for $X = \mathcal{S}(T)$; and the fourth is again A.6.(ii) together with 3.1. □

Definition. (a) Given X, Y simplicial sets, its **product** $X \times Y$ is the simplicial set with $(X \times Y)_n := X_n \times Y_n$ and structure maps $\alpha^* \times \alpha^* : X_m \times Y_m \longrightarrow X_n \times Y_n$.

(b) A **subsimplicial set** of a simplicial set X is a simplicial set Y such that $Y_n \subset X_n$ for all n and its structure maps coincide with the restriction of the ones of X.

(c) If Y is a subsimplicial set of X, the **quotient** simplicial set X/Y is the result of identifying levelwise simplices in Y together.

We find preferred subsimplicial sets in $\underline{\Delta}^n$:

Definition. (a) The **boundary** of $\underline{\Delta}^n$ is the smallest subsimplicial set $\partial\underline{\Delta}^n$ of $\underline{\Delta}^n$ which contains the faces $d_j\iota_n$, for all $j = 0, \ldots, n$.

(b) The **k-th horn** of $\underline{\Delta}^n$ is the smallest subsimplicial set Λ^n_k of $\underline{\Delta}^n$ which contains the faces $d_j\iota_n$, for all $j = 0, \ldots, k-1, k+1, \ldots, n$.

As suggested by the name, we have

Lemma 3.3 *$|\underline{\Delta}^n| \cong \Delta^n$, $|\partial\underline{\Delta}^n| \cong \partial\Delta^n \cong S^{n-1}$, and $|\underline{\Delta}^n/\partial\underline{\Delta}^n| \cong \Delta^n/\partial\Delta^n \cong S^n$.*
More generally, for $Y \subset X$ simplicial sets, $|X/Y| \cong |X|/|Y|$.

As the explicit description of the geometric realization suggested,

Proposition 3.4 *$|X|$ is a CW-complex for each simplicial set X.*

Proof. Define the n-skeleton sk_nX of X as the smallest subcomplex of X containing all simplices of degree $\leq n$. Then

$$X = \bigcup_{n\geq 0} sk_nX.$$

The proof consists in showing that

$$|sk_0X| \subset |sk_1X| \subset |sk_2X| \subset \cdots \subset |X|$$

is a CW decomposition, arising by pushout diagrams.

$$\begin{tikzcd} J_n \times \partial\Delta^n \arrow[r] \arrow[d, hook] & |sk_{n-1}X| \arrow[d] \\ J_n \times \Delta^n \arrow[r] & |sk_{n-1}X| \end{tikzcd}$$

where $J_n \subset X_n$ is the subset of *non-degenerate simplices*, that is, simplices $x \in X_n$ that are not in the image of the degeneracy maps. To see this, observe that we have pushout diagrams

$$\begin{array}{ccc} J_n \times \partial\underline{\Delta}^n & \longrightarrow & sk_{n-1}X \\ \big\downarrow & & \big\downarrow \\ J_n \times \underline{\Delta}^n & \longrightarrow & sk_{n-1}X \end{array}$$

where for every $x \in J_n \subset X_n$, the attaching maps $\partial\underline{\Delta}^n \longrightarrow sk_{n-1}X$ are the composites $\partial\underline{\Delta}^n \hookrightarrow \underline{\Delta}^n \xrightarrow{x} X$, where the last map is the one arising in 2.6. The composite lands in $sk_{n-1}X$ because of the explicit description of $\partial\underline{\Delta}^n$: $\partial\underline{\Delta}^n_k \subset \underline{\Delta}^n_k$ are formed by $(k+1)$-tuples with not all elements $0, 1, \ldots, n$ on it.

Since the geometric realization functor is left-adjoint, it commutes with colimits, in particular pushouts; giving rise to the desired diagram. □

In particular, the geometric realization of a simplicial set is a compactly generated Hausdorff space, so actually $|\cdot| : \mathsf{sSet} \longrightarrow \mathsf{CGHaus}$. We shall interpret the geometric realization as such a functor. Here is the reason:

Proposition 3.5 *The functor* $|\cdot| : \mathsf{sSet} \longrightarrow \mathsf{CGHaus}$ *preserves finite products, that is,*

$$|X \times Y| \cong |X| \times |Y|.$$

If we consider $|\cdot| : \mathsf{sSet} \longrightarrow \mathsf{Top}$, *then*

$$|X \times Y| \cong |X| \times_k |Y|,$$

where $\times_k$ *is the Kelley product (the k-fication of the usual product).*

Example 3.6 Let us see that $|\underline{\Delta}^1 \times \underline{\Delta}^1| \cong |\underline{\Delta}^1| \times |\underline{\Delta}^1|$. We will adopt the following notation (making explicit the reminiscence with simplicial complexes): if $\alpha : [k] \longrightarrow [n]$ is an order-preserving map with $\alpha(j) = i_j$, we will denote α as $[i_0, \ldots, i_k]$. As discussed before, we only have to take care of the non-degenerate simplices.

In dimension 0, we have

$$(\underline{\Delta}^1 \times \underline{\Delta}^1)_0 := \{([0],[0]), ([0],[1]), ([1],[0]), ([1],[1]), \},$$

and there is no degenerate simplices. For dimension 1, we see that $(\underline{\Delta}^1 \times \underline{\Delta}^1)_1$ has 9 elements as $(\underline{\Delta}^1)_1$ has 3 elements. There are 5 non-degenerate 1-simplices

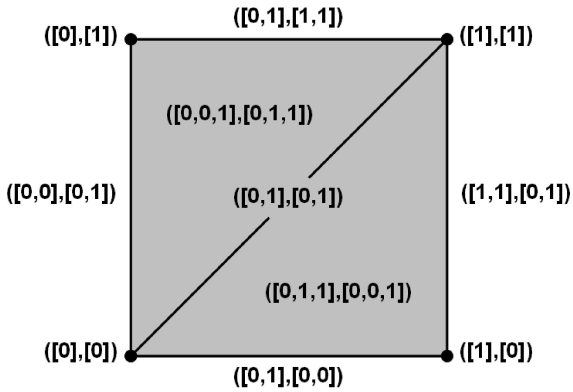
$$([0,1],[0,1])\ ,\ ([0,0],[0,1])\ ,\ ([0,1],[0,0])\ ,\ ([1,1],[0,1])\ ,\ ([0,1],[1,1]),$$

and the other 4 are degenerated, for instance

$$([0,0],[1,1]) = (s_0([0], s_0[1]) = s_0([0],[1]).$$

For dimension 2, we have that $(\underline{\Delta}^1 \times \underline{\Delta}^1)_2$ has 16 elements. Arguing in a similar fashion and taking into account the simplicial identities, we get that there are 2 non-degenerate 2-simplices $([0,0,1],[0,1,1])$ and $([0,1,1],[0,0,1])$. A further computation in this manner shows that all k-simplices for $k \geq 3$ are degenerated.

Its geometric realization is indeed the square,



4 Simplicial homotopy theory

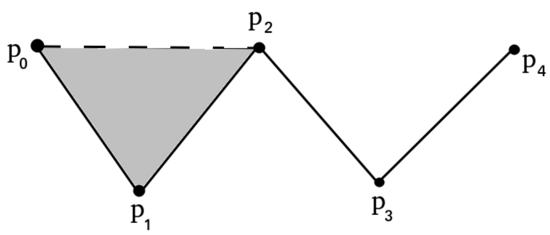
At this stage, the natural question to ask is: why do we care about simplicial sets? As we will show, it provides a combinatorial model to study homotopy theory of spaces, this is the goal to achieve.

The next natural question is: are there preferred classes of simplicial sets? The answer is positive, and the reason is the same as the one why we prefer to work with CW-complexes instead of arbitrary spaces in Algebraic Topology: CW pairs have the HEP, the inclusion of a subcomplex is a cofibration, we have tools to compute homology and cohomology (cellular (co)homology), we have relations between the higher homotopy groups of a CW-complex and its structure, etc.

These preferred simplicial sets are called *Kan complexes*, and this is the reason:

Example 4.1 (Motivating) Let X be a simplicial set. We say that two 0-simplices $p, q \in X_0$ are in the same *path-component* if there is a sequence of 0-simplices $p = p_0, p_1, \ldots, p_n = q$ and simplicial morphisms $\sigma_1, \ldots, \sigma_n : \underline{\Delta}^1 \longrightarrow X$ with $\sigma_i[0] = p_{i-1}$ and $\sigma_i[1] = p_i$.

In analogy with spaces, one may wonder whether in this situation one can find an only "simplicial path" $\sigma : \underline{\Delta}^1 \longrightarrow X$ with $\sigma[0] = p$ and $\sigma[1] = q$. In general, the answer in negative, but it is true if our simplicial set satisfies the property that "every horn can be filled"; with precision: every map $\Lambda^2_k \longrightarrow X$ extends to a map $\underline{\Delta}^2 \longrightarrow X$.



Indeed, if $\sigma_1, \sigma_2 : \underline{\Delta}^1 \longrightarrow X$ are paths from p_0 to p_1 and from p_1 to p_2 respectively, then let $f : \Lambda^2_1 \longrightarrow X$ be σ_1 in $[0,1]$ and σ_2 in $[1,2]$. By the condition, f extends to a map $\widehat{f} : \underline{\Delta}^2 \longrightarrow X$, and the composite

$$\underline{\Delta}^1 \hookrightarrow \underline{\Delta}^2 \xrightarrow{\widehat{f}} X$$

is the desired path, where the inclusion maps $[0,1]$ to $[0,2]$. Inductively this gives a simplicial path from p_0 to p_2.

This motivates the following

Definition. A simplicial set X satisfies the **Kan condition** if every map $\Lambda^n_k \longrightarrow X$, $0 \leq k \leq n$, extends to a map $\underline{\Delta}^n \longrightarrow X$ ("every horn has a filler"),

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \underline{\Delta}^n \arrow[ur, dashed] & \end{tikzcd}$$

In that case, we say that X is a **Kan complex** (after DANIEL KAN) or a fibrant simplicial set.

Examples 4.2 (a) For any topological space T, the singular simplicial set $\mathcal{S}(T)$ is a Kan complex. Indeed, by the adjunction 3.2, the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathcal{S}(T) \\ \underline{\Delta}^n \arrow[ur, dashed] & \end{tikzcd}$$

in sSet corresponds to the lifting problem

$$\begin{tikzcd} |\Lambda^n_k| \arrow[r] \arrow[d, hook] & T \\ |\underline{\Delta}^n| \arrow[ur, dashed] & \end{tikzcd}$$

in Top. But a topological horn is a deformation retract of the standard n-simplex $|\underline{\Delta}^n| \cong \Delta^n$, so the lift exists in the diagram in Top. Again by the adjunction, it corresponds with a map $\underline{\Delta}^n \longrightarrow X$, which extends the map $\Lambda^n_k \longrightarrow X$, since the composite $\Lambda^n_k \hookrightarrow \underline{\Delta}^n \longrightarrow \mathcal{S}(T)$ and the original $\Lambda^n_k \longrightarrow \mathcal{S}(T)$ correspond with the same map by the naturality of the adjunction.

(b) $\underline{\Delta}^0$ is a Kan complex. Indeed, any map $\Lambda^n_k \longrightarrow \underline{\Delta}^0$ is levelwise constant, so it admits as extension the levelwise constant simplicial map $\underline{\Delta}^n \longrightarrow \underline{\Delta}^0$.

(c) $\underline{\Delta}^n$, $n > 0$, is not a Kan complex.

(d) The classifying space $B\mathcal{C}$ of a category is not, in general, a Kan complex. The reason is that only "inner" horns, ie, horns Λ^n_k for $0 < k < n$, have fillers. In this case we say that the simplicial set is an $\infty-$**category.**

If $\mathcal{C}$ is a **groupoid** (ie, all arrows are invertible), then $B\mathcal{C}$ is a Kan complex. For instance, for a group G, BG is a Kan complex.

Simplicial paths

Definition. Let X be a simplicial set. A **path** in X is a simplicial map $\sigma : \underline{\Delta}^1 \longrightarrow X$. We say that $\sigma[0], \sigma[1] \in X_0$ are the **initial** and **end** points, respectively.

We say that two 0-simplices $p, q \in X_0$ are in the same **path-component** of X if there is a path with initial point p and end point q.

Observe that to give a path is the same thing as to give a 1-simplex in X.

Proposition 4.3 *If X is a Kan complex, then the path-component relation is an equivalence relation.*

Proof. *Reflexivity:* if $p \in X_0$, then the 1-simplex $s_0(p)$ does the job.

Transitivity: Let σ_1 be a path from p to q and let σ_2 be a path from q to r. As in the motivating example, define $f : \Lambda^2_1 \longrightarrow X$ to be σ_1 in $[0,1]$ and σ_2 in $[1,2]$. By the Kan condition, f extends to a simplicial map $\widehat{f} : \underline{\Delta}^2 \longrightarrow X$ and the composite

$$\underline{\Delta}^1 \hookrightarrow \underline{\Delta}^2 \xrightarrow{\widehat{f}} X$$

is the desired path, where the inclusion maps $[0,1]$ to $[0,2]$.

Simmetry: Let σ be a path from p to q. Let $f : \Lambda^2_0 \longrightarrow X$ be the simplicial map $f[0,1] = \sigma[0,1]$ and $f[0,2] = s_0(p)$. We see that

$$\begin{aligned} f[0] &= fd_1[0,2] = d_1 f[0,2] = d_1 s_0(p) = p, \\ f[1] &= fd_0[0,1] = d_0 f[0,1] = d_0\sigma[0,1] = q, \\ f[2] &= fd_0[0,2] = d_0 f[0,2] = d_0 s_0(p) = p, \end{aligned}$$

by the simplicial identities. By the Kan condition, f extends to a simplicial map $\widehat{f} : \underline{\Delta}^2 \longrightarrow X$ and the composite

$$\underline{\Delta}^1 \hookrightarrow \underline{\Delta}^2 \xrightarrow{\widehat{f}} X$$

is the desired path, where the inclusion maps $[0,1]$ to $[1,2]$. $\square$

We write $\pi_0(X)$ for the set of equivalence classes under this relation.

Simplicial homotopies of maps

For $i = 0, 1$, the maps $j_i : [0] \longrightarrow [1]$ mapping 0 to i induce maps

$$j_i : \underline{\Delta}^0 \longrightarrow \underline{\Delta}^1 \qquad , \qquad ([m] \xrightarrow{\beta} [0]) \mapsto ([m] \xrightarrow{\beta} [0] \xrightarrow{j_i} [1])$$

that we think as "inclusion of endpoints".

Definition. We say that two simplicial maps $f_0, f_1 : X \longrightarrow Y$ are **homotopic** if there is a map $H : X \times \underline{\Delta}^1 \longrightarrow Y$ such that f_i is given by the composite

$$X \xrightarrow{\cong} X \times \underline{\Delta}^0 \xrightarrow{\mathrm{Id} \times j_i} X \times \underline{\Delta}^1 \xrightarrow{H} Y.$$

Extending the argument given for path-connectedness we get

Theorem 4.4 *If Y is a Kan complex, then homotopy of maps $X \longrightarrow Y$ is an equivalence relation.*

Definition. A **pair** of simplicial sets (X, A) is a simplicial set X together with a subsimplicial set A. If both X and A are Kan complexes, we say that (X, A) is a **Kan pair**.

A **pointed** simplicial set is a simplicial pair $(X, < x_0 >)$ where $x_0 \in X_0$ and we take the subsimplicial set $< x_0 >$ generated by the 0-simplex. We will denote this simplicial pair as (X, x_0).

A map of pairs $f : (X, A) \longrightarrow (Y, B)$ is a simplicial map $X \longrightarrow Y$ such that the image of A is levelwise contained in B.

Definition. Let (X, A) be a simplicial pair and let Y be a simplicial complex. We say that two simplicial maps $f_0, f_1 : X \longrightarrow Y$ are **homotopic** (rel. A) if f_0, f_1 are homotopic via $H : X \times \underline{\Delta}^1 \longrightarrow Y$ and moreover the following diagram commutes as well,

$$\begin{array}{ccc} A \times \underline{\Delta}^1 & \xrightarrow{\mathrm{pr}_1} & A \\ {\scriptstyle i\times \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ X \times \underline{\Delta}^1 & \xrightarrow{H} & Y \end{array}$$

where $f' = f_0 \circ i = f_1 \circ i$.

One can also show that relative homotopy of simplicial sets is also an equivalence relation when Y is Kan.

Another crucial result we should mention by its analogy to the topological world is

Theorem 4.5 (HEP for simplicial sets) *Let (X, A) be a simplicial pair and let Y be a Kan complex. Given a simplicial map $f : X \longrightarrow Y$ and a simplicial homotopy $F : A \times \underline{\Delta}^1 \longrightarrow Y$ such that $F_{|A\times 0} = f_{|A}$, there is an extension $H : X \times \underline{\Delta}^1 \longrightarrow Y$ such that $H_{|A\times\underline{\Delta}^1} = F$ and $H_{|X\times 0} = f$.*

Simplicial homotopy groups

Definition. Let (X, x_0) be a pointed Kan complex. For $n > 0$, we write

$$\pi_n(X, x_0) := [(\underline{\Delta}^n, \partial\underline{\Delta}^n), (X, x_0)]$$

for the set of homotopy (rel. $\partial\underline{\Delta}^n$) classes of maps of pairs $(\underline{\Delta}^n, \partial\underline{\Delta}^n) \longrightarrow (X, x_0)$.

As one would expect, we can define group structures on these sets: for $[\alpha], [\beta] \in \pi_n(X, x_0)$, there is well defined simplicial map

$$\begin{tikzcd} \Lambda^{n+1}_n \arrow[r, "{(x_0,\dots,x_0,\alpha,\beta)}"] \arrow[d, hook] & X \\ \underline{\Delta}^{n+1} \arrow[ur, dashed, "\omega"'] & \end{tikzcd}$$

which extends, by the Kan condition, to a simplicial map $\omega : \underline{\Delta}^{n+1} \longrightarrow X$. If $\delta^n : \underline{\Delta}^n \longrightarrow \underline{\Delta}^{n+1}$ is the inclusion induced by $d^n : [n] \longrightarrow [n+1]$, then

Theorem 4.6 *There is a well-defined map*

$$\pi_n(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0) \qquad , \qquad ([\alpha], [\beta]) \mapsto [\omega \circ \delta^n]$$

which induces a group structure in $\pi_n(X, x_0)$ for $n \geq 1$. In particular, $\pi_n(X, x_0)$ is an abelian group for $n \geq 2$.

There are three more important results that here we will only mention:

Theorem 4.7 (Dold-Kan) *There is an equivalence of categories*

$$\mathsf{sAbGrp} \xrightarrow{\sim} \mathsf{Ch}_{\geq 0}$$

between the category of simplicial abelian groups and the category of chain complexes of abelian groups.

Moreover, under this map, simplicial homotopy corresponds to simplicial homology; and homotopic simplicial maps correspond to chain homotopic maps.

Theorem 4.8 *Let X be a Kan complex. The canonical map $X \longrightarrow \mathcal{S}(|X|)$ (corresponding with the identity of $|X|$ by the adjunction) is a weak equivalence, ie, it induces isomorphisms in all homotopy groups.*

Theorem 4.9 *Let T be a topological space. The canonical map $|\mathcal{S}(T)| \longrightarrow T$ (corresponding with the identity of $\mathcal{S}(T)$ by the adjunction) is a weak homotopy equivalence, ie, it induces isomorphisms in all homotopy groups.*

In particular, if T is a CW-complex, it is a homotopy equivalence.

Corollary 4.10 *For a Kan complex X, there are isomorphisms*

$$\pi_n(X, x_0) \cong \pi_n(|X|, x_0) \qquad , \qquad n \geq 1.$$

Proof. We have

$$\begin{aligned}
\pi_n(X, x_0) &\cong \pi_n(\mathcal{S}(|X|), x_0) \\
&\cong [(\underline{\Delta}^n, \partial\underline{\Delta}^n), (\mathcal{S}(|X|), x_0)] \\
&\cong [(\underline{\Delta}^n/\partial\underline{\Delta}^n, \bar{s}_0), (\mathcal{S}(|X|), x_0)] \\
&\cong [(|\underline{\Delta}^n/\partial\underline{\Delta}^n|, \bar{s}_0), (|X|, x_0)] \\
&\cong [(S^n, \bar{s}_0), (|X|, x_0)] \\
&\cong \pi_n(|X|, x_0).
\end{aligned}$$

□

Example 4.11 (Construction of the Eilenberg-MacLane spaces) We will show now how we can give rise to Eilenberg-MacLane spaces out of the geometric realization of a simplicial set.

In first place, recall from 2.4.(c) how the homology groups of a simplicial set X were defined. In a similar fashion, as for topological spaces, we can augment the Moore chain complex by considering the augmentation $\varepsilon : AX_0 \longrightarrow A[*] \cong A$ induced by the unique map $X_0 \longrightarrow *$.

We will write $\widetilde{A}X$ for the simplicial abelian group with $\widetilde{A}X_k := AX_k$ for $k > 0$ and $\widetilde{A}X_0 := \operatorname{Ker}\varepsilon \cong AX_0/A$. Its Moore complex is

$$C_*(\widetilde{A}X) := (\cdots \xrightarrow{\partial} AX_2 \xrightarrow{\partial} AX_1 \xrightarrow{\partial} \operatorname{Ker}\varepsilon).$$

Let $n \geq 1$ and let A be an abelian group, and consider the simplicial abelian group $\widetilde{A}(\underline{\Delta}^n/\partial\underline{\Delta}^n)$. Then its geometric realization $|\widetilde{A}(\underline{\Delta}^n/\partial\underline{\Delta}^n)|$ is a $K(A,n)$. Indeed, we have

$$\pi_k(|\widetilde{A}(\underline{\Delta}^n/\partial\underline{\Delta}^n)|) \overset{4.10}{\cong} \pi_k(\widetilde{A}(\underline{\Delta}^n/\partial\underline{\Delta}^n)) \overset{4.7}{\cong} H_k(C_*(\widetilde{A}(\underline{\Delta}^n/\partial\underline{\Delta}^n))) = \widetilde{H_k}(\underline{\Delta}^n/\partial\underline{\Delta}^n; A) \cong \begin{cases} A, & k = n, \\ 0, & k \neq n. \end{cases}$$

The last isomorphism is due to the explicit computation of the homology of the augmented Moore complex of $\underline{\Delta}^n/\partial\underline{\Delta}^n$.

A Category theory

Definition. A **category** $\mathcal{C}$ is the data of:

(i) A collection of objects $|\mathcal{C}|$,

(ii) A collection $\operatorname{Hom}_{\mathcal{C}}(A, B)$ of morphisms or arrows from A to B for objects A, B in $\mathcal{C}$,

(iii) A composition law

$$\circ : \operatorname{Hom}_{\mathcal{C}}(B, C) \times \operatorname{Hom}_{\mathcal{C}}(A, B) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(A, C)$$

for every triple A, B, C in $\mathcal{C}$,

(iv) An identity morphism $\mathrm{Id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ for every A in $\mathcal{C}$

such that

- $\mathrm{Id}_B \circ f = f = f \circ \mathrm{Id}_A$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$,
- $h \circ (g \circ f) = (h \circ g) \circ f$ for f, g, h composable arrows.

Example A.1 Set, Grp, Ring Vect_K, Mod_R, Alg_K, Top, Top_*, ...

Definition. Given a category $\mathcal{C}$, its **opposite** category $\mathcal{C}^{op}$ is the category with same objects as $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A)$ with composition $g^{op} \circ f^{op} := (f \circ g)^{op}$.

Definition. Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of

(i) An assignment $A \mapsto F(A)$ sending an object A of $\mathcal{C}$ to an object $F(A)$ of $\mathcal{D}$.

(ii) An assignment $\mathrm{Hom}_{\mathcal{C}}(A, B) \mapsto \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ for every pair A, B in $\mathcal{C}$.

such that

- $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$,
- $F(g \circ f) = F(g) \circ F(f)$.

Functors $F : \mathcal{C} \longrightarrow D$ are clasically called *covariant*, and functors $F : \mathcal{C}^{op} \longrightarrow D$ *contravariant*.

Example A.2 $\mathsf{Grp} \longrightarrow \mathsf{Set}$, mapping to the underlying set; $H_n(-; A) : \mathsf{Top} \longrightarrow \mathsf{Grp}$ the singular homology functor ; $\pi_1 : \mathsf{Top}_* \longrightarrow \mathsf{Grp}$, the fundamental group; $-^*$: $\mathsf{Vect}_K^{op} \longrightarrow \mathsf{Vect}_K$, taking the dual vector space, etc.

Definition. Let $\mathcal{C}$ be a category, and let A be an object. The functor **represented** by A is the functor $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \longrightarrow \mathsf{Set}$. Similarly we obtain another functor $\mathrm{Hom}_{\mathcal{C}}(-, A) : \mathcal{C}^{op} \longrightarrow \mathsf{Set}$

Definition. Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A **natural transformation** $\eta : F \Longrightarrow G$ is a collection of maps $\eta_A : F(A) \longrightarrow G(A)$ in $\mathcal{D}$, for all A in $\mathcal{C}$, such that for every $f : A \longrightarrow B$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

We say that η is a **natural isomorphism** if η_A is an isomorphism for all A in $\mathcal{C}$, and we say that F and G are **naturally isomorphic**.

The importance of representable functors is encoded in the central theorem of category theory:

Lemma A.3 (Yoneda, weak) *Let $\mathcal{C}$ be a category. The **Yoneda functor***

$$h : \mathcal{C} \longrightarrow \mathsf{Set}^{\mathcal{C}^{op}} \qquad , \qquad C \mapsto \mathrm{Hom}_{\mathcal{C}}(-, C)$$

is fully faithful, that is, for every $C, C' \in \mathcal{C}$ there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(C, C') \cong \mathrm{Hom}_{\mathsf{Set}^{\mathcal{C}^{op}}}(\mathrm{Hom}_{\mathcal{C}}(-, C), \mathrm{Hom}_{\mathcal{C}}(-, C')).$$

In other words: given two objects C, C', if for all $B \in \mathcal{C}$ there are bijections $\mathrm{Hom}_{\mathcal{C}}(B, C) \cong \mathrm{Hom}_{\mathcal{C}}(B, C')$ natural in B, then there is an isomorphism $C \cong C'$. Of course, the lemma is also true if we consider covariant representable functors instead.

A slightly more general version of this is the following:

Lemma A.4 (Yoneda, strong) *Let $\mathcal{C}$ be a category, let $C \in \mathcal{C}$ and let $F : \mathcal{C}^{op} \longrightarrow \mathsf{Set}$. Then there is a natural bijection*

$$\mathrm{Hom}_{\mathsf{Set}^{\mathcal{C}^{op}}}(\mathrm{Hom}_{\mathcal{C}}(-, C), F) \cong F(C).$$

The assignment maps a natural transformation $\eta : \mathrm{Hom}_{\mathcal{C}}(-, C) \Rightarrow F$ to the element $\eta_C(\mathrm{Id}_C)$.

Definition. Let $\mathcal{C}$ be a category. An **initial** object is an object A such that for any B in $\mathcal{C}$, $\#\mathrm{Hom}_{\mathcal{C}}(A, B) = 1$. Similarly, we say that A is a **terminal** object if for any B in $\mathcal{C}$, $\#\mathrm{Hom}_{\mathcal{C}}(B, A) = 1$.

Definition. Let $F : \mathcal{C} \longrightarrow \mathsf{Set}$ be a functor. The **category of elements** of F is the category $\int F$ whose objects are $x \in FA$ for A in $\mathcal{C}$; and an arrow between $x \in FA$ and $y \in FB$ is a map $f : A \longrightarrow B$ such that $Ff(x) = y$.

Observe that there is a canonical forgetful functor $U : \int F \longrightarrow \mathcal{C}$ sending $x \in FA$ to A.

Now consider a **small** category $\mathcal{J}$ (that is, the collections of objects and arrows are sets) and a **locally small**[1] category $\mathcal{C}$ (that is, the classes Hom are sets).

Definition. Let A be an object of $\mathcal{C}$ and denote as $\mathrm{const}_A : \mathcal{J} \longrightarrow \mathcal{C}$ the constant functor to A, sending every morphism of $\mathcal{J}$ to Id_A.

Let $F : \mathcal{J} \longrightarrow \mathcal{C}$ be a functor (we say that it is a **diagram of shape** $\mathcal{J}$).

(a) A **cone over** F **with appex** A is a natural transformation $\eta : \mathrm{const}_A \Longrightarrow F$.

(b) A **cone under** F **with nadir** A is a natural transformation $\eta : F \Longrightarrow \mathrm{const}_A$.

For any diagram $F : \mathcal{J} \longrightarrow \mathcal{C}$ there are functors

$$\mathrm{Cone}(-, F) : \mathcal{C}^{op} \longrightarrow \mathsf{Set} \qquad , \qquad \mathrm{Cone}(F, -) : \mathcal{C} \longrightarrow \mathsf{Set}$$

sending $A \in \mathcal{C}$ to the set of cones over and under A.

Definition. Let $F : \mathcal{J} \longrightarrow \mathcal{C}$ be a diagram of shape $\mathcal{J}$.

(a) A **limit** of F is a terminal object in the category $\int \mathrm{Cone}(-, F)$.

Unravelling the definition, this is an object $\lim F \in \mathcal{C}$ together a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(-, \lim F) \cong \mathrm{Cone}(-, F).$$

(b) A **colimit** of F is an initial object in the category $\int \mathrm{Cone}(F, -)$.

Unravelling the definition, this is an object $\mathrm{colim}\, F \in \mathcal{C}$ together a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, F, -) \cong \mathrm{Cone}(F, -).$$

Example A.5 Direct products, equalizers, pullbacks, inverse limits, kernels, etc are examples of limits.

On the other hand, direct sums, disjoint unions, free products, pushouts, amalgamated products (Van Kampen!), direct limits, coequalizers, cokernels, etc are examples of colimits.

[1]These are just technical reasons we avoid to discuss here.

Theorem A.6 *Let $\mathcal{C}$ be a locally small category, and let $C \in \mathcal{C}$.*

(i) Covariant representable functors $\mathrm{Hom}_{\mathcal{C}}(C,-)$ *preserve all limits that exist in $\mathcal{C}$, that is, there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(C, \lim_{\mathcal{J}} F) \cong \lim_{\mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(C, F(-))$$

for any diagram $F : \mathcal{J} \longrightarrow \mathcal{C}$ whose limit exists.

(ii) Contravariant representable functors $\mathrm{Hom}_{\mathcal{C}}(-,C)$ *carry colimits in $\mathcal{C}$ to limits in* Set*, that is, there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{J}} F, C) \cong \lim_{\mathcal{J}^{op}} \mathrm{Hom}_{\mathcal{C}}(F(-), C)$$

for any diagram $F : \mathcal{J} \longrightarrow \mathcal{C}$ whose colimit exists.

Definition. Let $\mathcal{C}, \mathcal{D}$ be locally small categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ be functors. We say that F and G are **adjoints** (more precisely, F is **left-adjoint** to G or G is **right adjoint** to F) if the functors

$$\mathrm{Hom}_{\mathcal{D}}(F(-), -)\ ,\ \mathrm{Hom}_{\mathcal{C}}(-, G(-)) : \mathcal{D}^{op} \times \mathcal{C} \longrightarrow \mathsf{Set}$$

are naturally isomorphic, and we write

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

In other words, F and G are adjoints if for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$ we have bijections

$$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, G(D))$$

which are natural in C and D.

Examples A.7 Lots of universal properties of some objects express an adjunction:

1. The universal property of the free abelian group says that $R(-)$ and the forgetful functor are adjoints, $\mathrm{Hom}_{\mathsf{AbGrp}}(\mathbb{Z}[S], A) \cong \mathrm{Hom}_{\mathsf{Set}}(S, A)$. Same with free R-module, free group, free monoid, free... .

2. For an R-module M, the functor $M \otimes_R -$ is left adjoint to the functor $\mathrm{Hom}_{R-\mathsf{mod}}(M, -)$,

$$\mathrm{Hom}_{R-\mathsf{mod}}(M \otimes_R M', M'') \cong \mathrm{Hom}_{R-\mathsf{mod}}(M', \mathrm{Hom}_{R-\mathsf{mod}}(M, M'')).$$

3. In the subcategory CGHaus_* of pointed compactly generated Hausdorff spaces, the reduced suspension functor $\Sigma-$ is left-adjoint to the loop functor $\Omega-$,

$$\mathrm{Hom}_{\mathsf{CGHaus}_*}(\Sigma X, Y) \cong \mathrm{Hom}_{\mathsf{CGHaus}_*}(X, \Omega Y).$$

We will also use the following technical result of category theory:

Theorem A.8 (Density) *Let $\mathcal{C}$ be a locally small category.*

(i) Any functor $F : \mathcal{C} \longrightarrow$ Set *is naturally isomorphic to the colimit of the diagram*

$$\left(\int F\right)^{op} \xrightarrow{U} \mathcal{C}^{op} \xhookrightarrow{h} \mathsf{Set}^{\mathcal{C}}.$$

(ii) Any functor $F : \mathcal{C}^{op} \longrightarrow$ Set *is naturally isomorphic to the colimit of the diagram*

$$\int F \xrightarrow{U} \mathcal{C} \xhookrightarrow{h} \mathsf{Set}^{\mathcal{C}^{op}}.$$

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PROBLEMS

1. Write the map $\alpha : [3] \longrightarrow [4]$, $\alpha = [0, 1, 1, 2]$, as composite of coface and codegeneracy maps.
2. Check the cosimplicial identities.
3. Show that the simplicial set $\underline{\Delta}^n$ has a unique non-degenerate n-simplex, $\mathrm{Id} : [n] \longrightarrow [n]$.
4. Show that every (ordered) simplicial complex can be made into a simplicial set by adjoining degenerate simplices.
5. Show the weak Yonneda lemma A.3 (you can skip the naturality statement if you want).
6. How many k-simplices has the simplicial set $\underline{\Delta}^n$ for $n \geq 0$?
7. Show that $\underline{\Delta}^2$ is not a Kan complex. (*Hint:* Define a map $\Lambda^2_0 \longrightarrow \underline{\Delta}^1$ on the non-degenerate 1-simplices so that it cannot be extended in a order-preserving way to $\underline{\Delta}^2$).
8. Give a counter-example which shows that $B\mathcal{C}$ is not a Kan complex in general.
9. Show that $|\underline{\Delta}^0| \cong *$ the one-point space and compute $H_n(\underline{\Delta}^0; A)$ for all $n \geq 0$.
10. Argue whether $\underline{\Delta}^1 \times \underline{\Delta}^1$ is isomorphic or not (as simplicial sets) to $\underline{\Delta}^2$.
11. Show that if a map $(\partial\underline{\Delta}^{n+1}, s_0) \longrightarrow (X, x_0)$ extends to a map $(\underline{\Delta}^{n+1}, s_0) \longrightarrow (X, x_0)$, then it is homotopic to the constant map $\partial\underline{\Delta}^{n+1} \longrightarrow x_0$.

Hand-in exercises: 1, 5, 6, 7, 9.