0: LIE GROUPS AND ALGEBRAS

<u>Definition</u>: A <u>Lie group</u> is a manifold G endowed with a group structure such that both stru[ctures] are compatible, i.e., the map

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ (g,h) & \longmapsto & gh^{-1} \end{array} \qquad \left(\equiv \quad \begin{array}{ccc} G\times G & \longrightarrow & G \\ (g,h) & \longmapsto & gh \end{array} \quad , \quad \begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & g^{-1} \end{array} \right)$$

is $\mathcal{C}^\infty$.

<u>Definition</u>: Let G, H be Lie groups. A <u>Lie group homomorphism</u> $f: G \to H$ is a $\mathcal{C}^\infty$ map that is also [a] group homomorphism. When f is diffeomorphism, f is an isomorphism of Lie groups.

<u>Examples</u>:

1) $(\mathbb{R}, +)$, $(\mathbb{R}^*, \cdot)$, $(S^1 \subset \mathbb{C}, \cdot)$, ...

2) $(S^3 \subset \mathbb{H}, \cdot)$, where $\mathbb{H} := \{x + iy + jz + kt : x, y, z, t\}$ are the <u>quaternions</u>, and the product is $\mathbb{R}$-bilinear and $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$ (note that it is <u>not</u> commutative).

3) $(Gl_n = \{\text{invertible matrices (over } \mathbb{R} \text{ or } \mathbb{C})\}, \cdot)$ is the <u>linear general group</u>

) $O(n) = \{A \in Gl_n(\mathbb{R}) : A \cdot A^t = I\}$ the <u>orthogonal group</u>

) $SO(n) = \{A \in Gl_n(\mathbb{R}) : A \cdot A^t = I, \det A = +1\}$ the <u>special orthogonal group</u>

) $SL(n) = \{A \in Gl_n(\mathbb{R}) : \det A = +1\}$ the <u>special linear group</u>

) $U(n) = \{A \in Gl_n(\mathbb{C}) : A \overline{A}^t = 1\}$ the <u>unitary group</u>

) $SU(n) = \{A \in Gl_n(\mathbb{C}) : A\overline{A}^t = 1, \det A = 1\}$ the <u>special unitary group</u>

) $Sp_k(\mathbb{R}) = \{A \in GL_{2k}(\mathbb{R}) : A^t \begin{pmatrix} 0 & -I_k \\ \cdots \end{pmatrix} A = \begin{pmatrix} 0 & -I_k \\ \cdots \end{pmatrix}$ [illegible] $\begin{pmatrix} 0 & -I_k \\ \cdots \end{pmatrix}$ ...

• How to prove that this things are Lie groups? Subgroups?: Easy. Submanifolds? Well, there will be a theorem that will say "a subgroup $H \subseteq G$ is Lie group $\iff$ H is closed". But so far we can also do it (harder), by using

<u>Definition</u>: Let M, N be manifolds, and $\varphi : M \to N$. We say that $y \in N$ is a <u>regular value</u> of φ if $\varphi_* : T_x M \to T_y N$ is surjective $\forall x \in \varphi^{-1}(y)$.

<u>Theorem (Regular Value)</u>. Let M, N be manifolds and $\varphi : M \to N$. If $y \in N$ is a regular value, then either $\varphi^{-1}(y)$ is empty or $\varphi^{-1}(y)$ is a smooth submanifold of M of dimension

$$\dim \varphi^{-1}(y) = \dim M - \dim N$$

In particular, for $x \in \varphi^{-1}(y)$, for $\varphi_* : T_x M \to T_y N$ it holds that $\underline{i_*(T_x \varphi^{-1}(y)) = \operatorname{Ker} \varphi_*}$ $\subset T_x M$

<u>Definition</u>. Let G be a Lie group and M a manifold. A (left) <u>action</u> of G on M is a "smooth action" of G on M, ie, M is a G-set and the map

$$\begin{aligned} G \times M &\xrightarrow{m} M \\ g, x &\longmapsto gx \end{aligned}$$

is smooth.

If $M = V$ is a vector space, such an action is said to be <u>linear</u> if for each $g \in G$ the map $m(g, -) : V \to V$ is linear. In that case one re-writes the action as the map

$$\begin{aligned} r : G &\longrightarrow GL(V) \\ g &\longmapsto r_g = m(g, -) \end{aligned}$$

and one says that (V, r) (or simply V if r is clear) is a <u>representation</u> of G.

• Ie, a representation of a group is a action, ie, a representation $\rho : G \to \operatorname{End}(V)$ makes V be a G-set: $g * v = \rho(g)(v)$.

• Note that G itself is also a G-set, by the left $g*x = gx$, by the right, $g*x = xg$, and also by conjugation $g*x := gxg^{-1}$.

Definition: The action by conjugation $g*x = gxg^{-1}$ fixed g, ie, $Ad_g : G \to G$, $Ad_g(x) = gxg^{-1}$ is called the adjoint action of g on G

THE LIE ALGEBRA

• The left traslation $L_g : G \to G$, $x \mapsto gx$; and the right traslation $R_g : G \to G$, $x \mapsto xg$, allow us to move from any point $a \in G$ to $e \in G$ (via $L_{a^{-1}}$). In particular, we can move any vector to $T_eG =:$ via the tangent linear map.

Precisely, since $L_a / R_a / Ad_a$ are diffeomorphisms (with inverses $L_{a^{-1}}/R_{a^{-1}}/Ad_{a^{-1}}$) its differential $(L_a)_* : \mathfrak{g} = T_eG \to T_a$ is a isomorphism.

• If $\alpha \in \mathfrak{g}$, it induces a vector field $\alpha^L := \{ \alpha^L_a := (L_a)_* \alpha \in T_aG \}$.

Definition: We will say that a v.f. X is left invariant if $(L_a)_* X_g = X_{ag}$ $\forall a, g \in G$. We will denote by $\mathfrak{X}^{inv}(G)$ the subspace of $\mathfrak{X}(G)$ of left-invariant v.f.

It is straightforward to check that $\alpha^L \in \mathfrak{X}^{inv}(G)$,

Proposition: Let G be a Lie group.

1) The construction $\alpha \longmapsto \alpha^L$ defines a bijection between $\mathfrak{g}$ and $\mathfrak{X}^{inv}(G)$. In consequence, $\mathfrak{X}^{inv}(G)$ is a n-dim vector space.

2) $X, Y \in \mathfrak{X}^{inv}(G) \Rightarrow [X, Y] \in \mathfrak{X}^{inv}(G)$. In particular, there is an unique operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$(\alpha, \beta) \longmapsto [\alpha, \beta]$$

with the property that $[\alpha, \beta]^L = [\alpha^L, \beta^L]$, ie, $[\alpha, \beta]^{(\text{by def})}$ is the unique vector that identifies with $[\alpha^L, \beta^L]$.

) The previous operation on $\mathfrak{g}$ is bilinear, anti-symmetric and satisfies the Jacobi identity

$$[[\alpha, \beta], \gamma] + [[\gamma, \alpha], \beta] + [[\beta, \gamma], \alpha] = 0.$$

In general one can define what a Lie algebra is:

<u>Definition</u>: A <u>Lie algebra</u> is a vector space $\mathfrak{a}$ endowed with an operation

$$[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$$

that verifies:

i) $[\cdot,\cdot]$ is bilinear : $[\lambda e + \mu v, w] = \lambda [e,w] + \mu [v,w]$; $[w, \lambda e + \mu v] = \lambda [w,e] + \mu [w,v]$

ii) $[\cdot,\cdot]$ is anti-symmetric : $[e,v] = -[v,e]$

iii) Jacobi identity : $[[e,v],w] + [[w,e],v] + [[v,w],e] = 0$.

<u>Definition</u>: We will call <u>Lie algebra of the Lie group</u> G to $\mathfrak{g}$ endowed with the bracket $[\cdot,\cdot]$.

<u>Definition</u>: A morphism between two Lie algebras $(\mathfrak{g}, [\cdot,\cdot])$ and $(\mathfrak{g}', [\cdot,\cdot]')$ is any linear map $f : \mathfrak{g} \to \mathfrak{g}'$ which is compatible with the brackets, in the sense that

$$f([\alpha,\beta]) = [f(\alpha), f(\beta)]' .$$

<u>Lemma</u>: The tangent linear map at $e \in H$ of a Lie group homomorphism $H \to G$ is a Lie algebra homomorphism.

<u>Corollary</u>: Let G be a Lie group and H a Lie subgroup, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then $\mathfrak{h}$ is a <u>Lie sub-algebra</u>, i.e., $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ and the Lie bracket of $\mathfrak{h}$ can be computed using the one of $\mathfrak{g}$.

Thanks to the property that $T_x Y^{-1}(y) = \operatorname{Ker} \varphi_x$ in the regular value theorem, one finds that the Lie algebras of the previous Lie groups are Lie subalgebras of $\mathfrak{gl}_n$:

$Gl_n \rightsquigarrow \mathfrak{gl}_n = M_n(\mathbb{R})$

$O(n) \rightsquigarrow \mathfrak{o}(n) = \{X \in \mathfrak{gl}_n : X + X^t = 0\}$

$SO(n) \rightsquigarrow \mathfrak{so}(n) = \mathfrak{o}(n)$

$SL(n) \rightsquigarrow \mathfrak{sl}(n) = \{X \in \mathfrak{gl}_n : \operatorname{tr} X = 0\}$

$U(n) \rightsquigarrow \mathfrak{u}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) : X + X^* = 0\}$

$SU(n) \rightsquigarrow \mathfrak{su}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) : X + X^* = 0, \operatorname{tr} X = 0\}$

$Sp_n \rightsquigarrow \mathfrak{sp}_n(\mathbb{R}) = \{X \in \mathfrak{gl}_{2n} : J_{can} X + X^t J_{can} = 0\}$

• While representations of Lie groups were defined using $GL(V)$, for Lie algebras we'll use $\mathfrak{gl}(V)$:

Definition: A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is any Lie algebras morphism

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

• Note that, because the previous lemma, that any representation $r : G \to GL(V)$ induces a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ after differentiating at the unit.

Definition: Set $Ad_a : G \longrightarrow G$ and its t.l.m. at the unit $(Ad_a)_* = T_e G = \mathfrak{g} \longrightarrow \mathfrak{g} = T_e G$, and let $(Ad_a)_* \overset{not}{=} Ad_a$. The adjoint action of G on its Lie algebra is

$$\begin{aligned} Ad : G &\longrightarrow GL(\mathfrak{g}) \\ a &\longmapsto Ad_a \end{aligned}$$

The corresponding "infinitesimal action" obtained by diff. Ad at the unit

$$\begin{aligned} ad : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ \alpha &\longmapsto ad_\alpha \end{aligned}$$

is called the (infinitesimal) adjoint action of $\mathfrak{g}$ on itself

Proposition: $ad_\alpha(\beta) = [\alpha, \beta]$, i.e., the Lie bracket can be recovered via the adjoint action.

EXPONENTIAL MAP

Proposition: Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $\alpha \in \mathfrak{g}$, the vector field α^L is complete and its flow satisfies $\phi^t_{\alpha^L}(ag) = a\,\phi^t_{\alpha^L}(g)$.

In particular, the flow can be reconstructed from what it does at time $t=1$ at the identity: from the so called exponential map

$$\begin{aligned} \exp : \mathfrak{g} &\longrightarrow G \\ \alpha &\longmapsto \exp(\alpha) := \phi^1_{\alpha^L}(e). \end{aligned}$$

y the formula

$$\boxed{\phi_{\alpha^L}^t(a) = a \exp(t\alpha)}$$, i.e., $\phi^t_{\alpha^L}(a) = a \cdot \phi^1_{(t\alpha)^L}(e)$

Moreover, the exp is a local diff. around the origin (i.e., t.l.m is $Id: \mathfrak{g} \to \mathfrak{g}$) at $e \in G$

"Recall" that $[X,Y]_p = \frac{d}{dt}\Big|_{t=0} (\phi_X^{-t})_* Y_{\phi_X^t(p)}$

Proposition: For a Lie group G with Lie algebra $\mathfrak{g}$, it holds

$$[\alpha,\beta] = \frac{d}{dt}\Big|_{t=0} Ad_{\exp(t\alpha)}(\beta).$$

Lemma:

1) $\exp((t+s)\alpha) = \exp(t\alpha)\exp(s\alpha)$

2) $\exp(-t\alpha) = \exp(t\alpha)^{-1}$

Theorem: Let $H \leq G$ subgroup, G Lie group.

$$H \text{ is a Lie group} \iff H \text{ is closed}.$$

ACTIONS OF A LIE GROUP

Definition: An infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold M is a Lie algebra morphism

$$a : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$$

Definition: Let $\mu: M \times G \to M$ be a smooth right action of a Lie group G on a manifold M.
The infinitesimal generator induced by $\alpha \in \mathfrak{g}$ is the vector field $\alpha_M \in \mathfrak{X}(M)$ whose flow is given by

$$\phi^t_{\alpha_M}(x) = x \cdot \exp(t\alpha) \equiv \mu(x, \exp(t\alpha))$$

Equivalently, α_M can be also obtain as follows: if $\mu^x : G \to M$, $a \mapsto x\cdot a = \mu(x,a)$, then

$$(\alpha_M)_x = \mu^x_* \alpha \in T_x M \qquad (\mu^x_* \text{ is in } e \in G)$$

Proposition:
$$a: \mathfrak{g} \longrightarrow \mathfrak{X}(M), \quad \alpha \longmapsto \alpha_M$$
is an infinitesimal action of $\mathfrak{g}$ on M.

- We could have also defined the inf. gen. with a left action, but we had had a sign changing: with $\mu: G\times M \to M$, it should be

$$\Phi^t_{\alpha_M}(x) = \exp(-t\alpha) * x$$

$$(\alpha_M)_p = -\mu^R_{*,e}\alpha \,, \quad \text{or} \quad (\alpha_M)_x = \frac{d}{dt}\Big|_{t=0} \exp(-t\alpha)\cdot x \,.$$

Corollary: Let V a finite-dim v.s. The Lie algebras morphism

$$i: \mathfrak{gl}(V) \longrightarrow \mathfrak{X}(V)$$
$$T \longmapsto i(T) = \left\{ i(T)_v := \frac{d}{dt}\Big|_{t=0} \exp(-tT)(v) \right\}$$

is injective. This identifies $\mathfrak{gl}(V)$ with the sub-Lie algebra of $\mathfrak{X}(V)$ of "linear vector fields" (vector fields on V whose coeff. are lin. funct.).

- Recall that an action G on M gives a quotient space M/G, where $[x]=[x'] \Leftrightarrow G\cdot x = G\cdot x'$, and we have $\pi: M \to M/G$ and one endows M/G with the quotient top, i.e., the largest top s.t. $\pi: M \to M/G$ is continuous. But M/G is not manifold easily. When the problem is easy? We'll restrict the fac.

Definition. A (left, eg) Lie group action of G on M is

a) free when $g\cdot x = h x$ for some $x \Rightarrow g = h$, i.e., $I_x = \{e\}$ $\forall x \in M$.

b) proper if the map

$$G\times M \xrightarrow{f} M\times M$$
$$(g, x) \longmapsto (gx, x)$$

verifies that $f^{-1}(K)$ is compact $\forall K \subset M\times M$ compact, (and it is said that f is proper).

Lemma: If an action of G on M is free, then

$$\alpha_x : \mathfrak{g} \longrightarrow T_x M$$
$$\alpha \longmapsto \alpha_M(x)$$

is injective $\forall x \in M$.

Definition: An infinitesimal action $a : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ with such property is called infinitesimally free action.

Theorem: If G is a Lie group that acts freely and properly (e.g. if G is compact) on a manifold M, then the quotient M/G admits a unique smooth structure with the property that the quotient projection $\pi : M \longrightarrow M/G$ is a submersion. Moreover, $\dim M/G = \dim M - \dim G$.

INVARIANT ELEMENTS

Definition: Let G act on M. A k-form $\omega_k \in \Omega^k(M)$ is said to be G-invariant if

$$(\Psi_g)^* \omega_k = \omega_k \quad , \forall g \in G.$$

in AHM $\mathrm{Diff}(M) \ni \Psi_g : M \to M$

(Here $\Psi_g : M \to M$, $x \mapsto \Psi_g(x) = g * x$)

It is also said that ω_k is invariant w.r.t. an action of G on M.

Definition: Let G act on M and let $\mathfrak{g}$ be the Lie algebra of G. A k-form $\omega_k \in \Omega^k(M)$ is $\mathfrak{g}$-invariant if

$$\mathcal{L}_{a(\alpha)} \omega_k = 0 \qquad \forall \alpha \in \mathfrak{g}.$$

($a : \mathfrak{g} \to \mathfrak{X}(M)$, $\alpha \mapsto a(\alpha) = \alpha_M$)

It is also said that ω_k is invariant w.r.t. an infinitesimal action of $\mathfrak{g}$ on M.

Proposition: Let G act on M with inf. action $a : \mathfrak{g} \to \mathfrak{X}(M)$.

1) $\omega_k \in \Omega^k(M)$ G-invariant $\Longrightarrow$ ω_k $\mathfrak{g}$-invariant.

2) If G is connected, then ω_k G-invariant $\Longleftrightarrow$ ω_k $\mathfrak{g}$-invariant.

Definition: let G act on M with inf. action $a: \mathfrak{g} \to \mathfrak{X}(M)$. A k-form $\omega_k \in \Omega^k(M)$ is called G-basic if

i) ω_k is G-invariant

ii) ω_k is horizontal, ie, $i_{a(x)} \omega_k = 0 \quad \forall x \in \mathfrak{g}$.

Theorem: Let G act on M freely and properly, so M/G is a manifold in a unique way and $\pi: M \to M/G$ is submersion. Then for a k-form $\omega_k \in \Omega^k(M)$ one has

$$\omega_k \ G\text{-basic} \iff \exists\, \overline{\omega}_k \in \Omega^k(M/G) : \omega_k = \pi^* \overline{\omega}_k ,$$

ie, there's a bijection (via π^*)

$$\Omega^k(M/G) \xrightarrow{\pi^*} \Omega^k(M)_{G\text{-basic}}$$

I : SYMPLECTIC MANIFOLDS

Definition : A symplectic manifold (M, ω) is a manifold M together with a symplectic form,
a 2-form $\omega \in \Omega^2(M)$ which is

i) closed, $d\omega = 0$

ii) ω_x is non-degenerated $\forall x \in M$

Definition : A symplectomorphism $\varphi : (M, \omega) \to (M', \omega')$ is a diffeomorphism s.t. $\omega = \varphi^* \omega'$.

Eg: $(\mathbb{R}^{2n}, x_1 \dots x_n, y_1 \dots y_n)$, $\omega_{can} := \sum_{i=1}^{n} dx_i \wedge dy_i$ is symplectic. In every manifold, indeed, this is the only example.

Theorem (Darboux) : A symplectic manifold (M, ω) can be covered by coordinate charts $(U; x_1, \dots, x_n, y_1 \dots y_n)$ in which $\omega_{|U}$ takes the form $\sum dx_i \wedge dy_i$.

(Beginning of symp. geometry): If M is a manifold, T^*M has a canonical symplectic structure, given by

$$\boxed{\omega_{can} := -d\theta_L},$$

where θ_L is a canonical 1-form in T^*M, called the Liouville 1-form, defined by

$$\left.\begin{array}{l} \pi : T^*M \to M, \\ \pi_* : T_{\omega_p}(T^*M) \to T_p M \\ \pi^* : T_p^* M \to T^*_{\omega_p}(T^*M) \end{array}\right\} \quad \boxed{(\theta_L)_{\omega_p} := \pi^* \omega_p}$$

If $(T^*U; x_1 - x_n, y_1 - y_n)$ are the usual coord. of that chart, then $\omega_{can} = \sum dx_i \wedge dy_i$.

Not every manifold admits a symplectic structure, viz. :

- dim M = even if ω is non-deg.
- M must be orientable (because ω non-deg $\Rightarrow$ $\omega^n \neq 0$)
- if M compact, $[\omega^n] \neq 0$ in $H^n_{dR}(M)$ $\Rightarrow$ $[\omega] \neq 0$ in $H^2_{dR}(M)$, ie, $H^2_{dR}(M) \neq 0$

HAMILTONIAN VECTOR FIELDS

Let (M,ω) symplectic. We have a $\mathcal{C}^\infty(M)$-module isomorphism (bc ω is non-deg)

$$\begin{array}{ccc} \mathfrak{X}(M) & \overset{\phi}{=\!=\!=} & \Omega^1(M) \\ X & \longmapsto & i_X\omega \end{array}$$

Definition: We will call Hamiltonian vector field associated to $f\in\mathcal{C}^\infty(M)$ to the unique v.f. $X_f\in$ such that $\underline{i_{X_f}\omega = df}$, i.e., is the only v.f. X_f that correspond with df via ϕ.

If a chart $(U; x_1, \dots x_n, y_1 - y_n)$ with $\omega = \sum dx_i \wedge dy_i$, then

$$X_f = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\partial_{x_i} - \frac{\partial f}{\partial x_i}\partial_{y_i}\right).$$

Definition: We say that a v.f. $X\in\mathfrak{X}(M)$ is symplectic if it holds $\mathcal{L}_X\omega = 0 \iff i_X\omega$ is close

Lemma: Hamiltonian v.f $\Rightarrow$ symplectic v.f

Note that the condition $\mathcal{L}_X\omega=0$ means that the flow of X is a symplectomorphism.

Definition: Let (M,ω) symplectic. We define the Poisson bracket of $f,g\in\mathcal{C}^\infty(M)$ as

$$\{f,g\} := -\omega(X_f, X_g). \qquad (= X_f g = -X_g f)$$

Proposition: The Poisson bracket of a sympl. mfld verifies:

1) $\{\cdot,\cdot\}$ bilinear
2) $\{\cdot,\cdot\}$ anti-symmetric
3) $\{\cdot,\cdot\}$ Jacobi
4) $\{\cdot,\cdot\}$ is a derivation in the second argument.

In Darboux coor. $\omega = \sum dx_i \wedge d$

$$\boxed{\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}}$$

This properties form ~~an~~ structure itself: a Poisson manifold (t.b.c.)

WEINSTEIN – MARSDEN REDUCTION

• It is a general technique to construct sym. mufld

<u>Definition</u>: let a Lie group G act on a sym mufd (M,ω). We will say that the action of G on M is a <u>symplectic action</u> when ω is G-invariant, i.e, when $\psi_g \in Sympl(M)$ $\forall g \in G$ $(\psi_g(x) = g * x)$

<u>Definition</u>: Consider a Lie group acting on itself by conjugation: $Ad_g : G \to G$, $x \mapsto g x g^{-1}$.
By diff. $(Ad_g)_* \overset{not}{=} Ad_g$ at the unit, we have a action of G on $\mathfrak{g}$: $G \times \mathfrak{g} \to \mathfrak{g}$, $(g, v) \mapsto Ad_g(v) = \frac{d}{dt}|_{t=0}\, g \exp(tv) g^{-1}$, called the <u>adjoint action</u> of G on its Lie algebra. Moreover, we will call the <u>co-adjoint action</u> of G on $\mathfrak{g}^*$ to the one obtained by "dualizing" the adjoint action,
$Ad^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$, $Ad^*_g(\xi) := \xi \circ Ad_{g^{-1}}$ (we use g^{-1} so that the action is a left action)

• If $G \circlearrowright (M,\omega)$ sym. action, then the induced inf. action $a: \mathfrak{g} \to \mathfrak{X}(M)$ satisfies that $i_{a(v)}\omega$ is closed $\forall v \in \mathfrak{g}$. Q: Is also exact? We want $i_{a(v)}\omega = d\mu_v$, with $\mu_v \in \mathcal{C}^\infty(M)$ linear in v, i.e, a map $\mu : M \to \mathfrak{g}^*$, $x \mapsto \mu(x)$. (but also $\mu : \mathfrak{g} \to \mathcal{C}^\infty(M)$, $v \mapsto \mu_v$, and $\mu_v(x) = \mu(x)(v)$)

<u>Definition</u>: Let G be a group and X, Y G-sets. A map $f: X \to Y$ is said to be <u>G-equivariant</u> or a <u>G-sets morphism</u> if $f(g\cdot x) = g \cdot f(x)$.

<u>Definition</u>. A <u>G-Hamiltonian space</u> is a triple (M, ω, μ) where

i) (M,ω) is a symp. mufd endowed with a sympl. action G

ii) A G-equivariant map $\mu : M \to \mathfrak{g}^*$ called the <u>momentum map</u> s.t. $i_{a(v)}\omega = d\mu_v$. $\forall v \in \mathfrak{g}$

• There is also an infinitesimal counterpart:

<u>Definition</u>: Let $\mathfrak{g}$ be a Lie algebra. A <u>$\mathfrak{g}$-Hamiltonian space</u> is a triple (M, ω, μ) where

i) (M,ω) is a sympl. mufd endowed with an inf. action $a: \mathfrak{g} \to \mathfrak{X}(M)$.

ii) A map $\mu : M \to \mathfrak{g}^*$, called <u>momentum map</u>, satisfying $i_{a(v)}\omega = d\mu_v$ and $a(u)(\mu_v) = \mu_{[u,v]}$.

• How to construct a symplectic mnfld?

<u>Definition</u>: Let (M,ω,μ) be a G-Ham. space, and $\xi \in \mathfrak{g}^*$. For simplicity, assume $\xi = 0$. If it verifies that

a) 0 is a regular value of μ (so $\mu^{-1}(0)$ is submnfld of M)

b) The action of G on $\mu^{-1}(0)$ is free & proper (so $\mu^{-1}(0)/G$ is mnfld & the projection submersion)

then we will call the <u>reducted space at $\xi = 0$</u> to

$$M//G := \mu^{-1}(0)/G .$$

<u>Theorem</u>: The 2-form $\omega|_{\mu^{-1}(0)} \in \Omega^2(\mu^{-1}(0))$ is G-basic, hence it comes from a 2-form $\omega_0 \in \Omega^2(M//G)$.

In particular, $(M//G, \omega_0)$ is a symplectic manifold.

· In general, one can reduct at arbitrary regular values $\xi \in \mathfrak{g}^*$, but now the action must come from isotropy subgroups (or stabilizer group) $G_\xi = I_\xi = \{g \in G : Ad^*_g(\xi) = \xi\}$ that act freely and properly on $\mu^{-1}(\xi)$. The result is the same: $\mu^{-1}(\xi)/G_\xi$ has a sym. structure ω_ξ characterized by $\pi^* \omega_\xi = \omega|_{\mu^{-1}(\xi)}$.

<u>Lemma</u>: In the hypothesis of the theorem, we have $\ker a_x = T_x(G * x)$ and

$$T_{[x]} M//G = \frac{T_x \mu^{-1}(0)}{T_x(G*x)} .$$

II: POISSON MANIFOLDS

<u>Definition</u>: A <u>Poisson manifold</u> is a manifold M endowed with a Lie algebra structure on $\mathcal{C}^\infty(M)$

$$\{\cdot,\cdot\}: \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$$
$$(f \quad , g) \longmapsto \{f,g\}.$$

such that $\{f,\cdot\}$ is a derivation for all $f \in \mathcal{C}^\infty(M)$, ie, it holds

i) <u>Bilinearity</u>: $\{f, \lambda g + \mu h\} = \lambda\{f,g\} + \mu\{f,h\}$, $\{\lambda g + \mu h, f\} = \lambda\{g,f\} + \mu\{h,f\}$.

ii) <u>Anti-symmetry</u>: $\{f,g\} = -\{g,f\}$

iii) <u>Jacobi</u>: $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$

iv) <u>Leibniz</u>: $\{f, gh\} = \{f,g\}\cdot h + g\{f,h\}$.

We will say that $\{\cdot,\cdot\}$ is the <u>Poisson bracket</u>

<u>Definition</u>: Given $(M,\{\cdot,\cdot\})$, $(M',\{\cdot,\cdot\}')$ Poisson manifolds, we will say that a smooth map $\varphi: M \to M'$ is a <u>Poisson map</u> if the pull-back $\varphi^*: \mathcal{C}^\infty(M') \to \mathcal{C}^\infty(M)$ is a Lie algebra homomorphism, ie, if

$$\{\varphi^* f, \varphi^* g\} = \varphi^*\{f,g\}' \qquad \forall f,g \in \mathcal{C}^\infty(M').$$

- In $(\mathbb{R}^{2n}, x_1 \dots x_n, y_1 \dots y_n)$, the <u>canonical Poisson bracket</u> is

$$\{f,g\} := \sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}\right)$$

- By I, every symplectic manifold is a Poisson manifold

Definition: Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. The Hamiltonian vector field associated to $f \in \mathcal{C}^\infty(M)$ is the v.f. that as derivation is $X_f := \{f, -\}$, and it is usually said that f is the Hamiltonian function. If $X_f = 0$, f is said to be a Casimir function.

Lemma: The map $\mathcal{C}^\infty(M) \longrightarrow \mathfrak{X}(M)$, $f \longmapsto X_f$, is a Lie algebra morphism, i.e., $X_{\{f,g\}} = [X_f, X_g]$.

Proposition: Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and fix some Hamiltonian function $H \in \mathcal{C}^\infty(M)$.

1) f first integral of $X_H \iff \{H, f\} = 0$.

2) H is a first integral of X_H.

3) f, g first integrals of $X_H \Rightarrow \{f,g\}$ first integral of X_H.

Lemma: Let $\varphi : (M, \{\cdot,\cdot\}) \to (M', \{\cdot,\cdot\}')$ be a Poisson map, and $H \in \mathcal{C}^\infty(M')$. It holds

$$X_H = \varphi_* X_{\varphi^* H}$$

Lemma: $\operatorname{supp}\{f,g\} \subset \operatorname{supp} f \cap \operatorname{supp} g$

- The lemma says that we can define a Poisson bracket in any open set $U \subset M$, denoted by $\{\cdot,\cdot\}_U$, determined by

$$\{f,g\}_{|U} = \{f_{|U}, g_{|U}\}_U,$$

so in an open subset U we don't need to distinguish them. In particular,

$$\{f,g\}_U = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

- This expression suggest that $\{\cdot,\cdot\}$ can be also expressed as a bivector $\pi = \sum \pi_{ij}\, \partial_{x_i} \wedge \partial_{x_j}$. And then another point of view for $\{\cdot,\cdot\}$ arises.

POISSON STRUCTURE AS BIVECTORS

- Recall that one can see $X \in \mathfrak{X}(M)$ as

  1. Section of $\pi : TM \to M$, $X \in \Gamma(TM)$
  2. Tensor $(0,1)$, i.e., map $\Omega^1(M) \to \mathcal{C}^\infty(M)$
  3. Derivation $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$.

(1) **Theorem**: There's a 1-1 correspondence

$$\mathfrak{X}^k(M) = \text{Mult.der}_{\text{skew-sym}}(\mathcal{C}^\infty(M), \overset{k}{\dots}, \mathcal{C}^\infty(M); \mathcal{C}^\infty(M))$$

$$T^k \longmapsto D^k(f_1 \dots f_k) := T^k(df_1, \dots, df_n)$$

$$T^k = \{(T^k)_p\}_{p \in M} \longleftarrow D^k$$

$$(T^k)_p(d_p f_1, \dots, d_p f_n) := D^k(f_1 \dots f_n)(p)$$

$d_p : \mathcal{O}_x \to T_x^* M$ surj.

So one can also see bivectors, i.e., skew-symmetric $(0,2)$ tensors, as

1. Section of $\Lambda^2 TM := \coprod_{x \in M} \Lambda^2 T_x M$, $\pi \in \Gamma(\Lambda^2 TM)$

2. Skew-sym. tensor $(0,2)$

3. Skew-sym "multiderivation" $\pi : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, i.e, it is a derivation in each argue

D

!! $\{;\}$ is precisely a multiderivation!!

Note that as multiderivation is $\overset{\text{der}}{\pi(f,g)} = \overset{\text{tensor}}{\pi(df, dg)}$

- Set $\overset{\text{not}}{\mathfrak{X}^k(M)} := \{$ skew-symmetric tensors $(0,k)\} \subseteq \mathcal{T}_0^k(M)$
- Given $(M, \{;\})$, then we define a bivector: $\pi(df, dg) := \{f, g\}$. Conversely, given $\pi \in \mathfrak{X}^2(M)$ $\{f,g\} := \pi(df, dg)$ is a "bracket"

We want to define a Poisson structure from π. Jacobi? We will firstly extend the Lie bracket:

GO TO 9-BIS

Theorem: There exists a unique operator, called Nijenhuis - Schouten bracket

$$[\cdot,\cdot] : \mathfrak{X}^{k+1}(M) \times \mathfrak{X}^{l+1}(M) \longrightarrow \mathfrak{X}^{k+l+1}(M) \qquad k, l \geq -1$$

satisfying:

1) For $k = l = 0$, it is the Lie bracket.

2) For $k = 0$, $l = -1$, $[X, f] = Xf$; and $k = -1$, $l = -1$, $[f, g] = 0$

3) It is graded skew-symmetric:

$$[\xi_2, \xi_1] = -(-1)^{kl} [\xi_1, \xi_2]$$

4) It is an anti-derivation:

$$[\xi, \xi_1 \wedge \xi_2] = [\xi, \xi_1] \wedge \xi_2 + (-1)^{k(l+1)} \xi_1 \wedge [\xi, \xi_2].$$

Explicitly, it must be

$$[X_1\wedge\cdots\wedge X_k,\ Y_1\wedge\cdots\wedge Y_\ell] = \sum_{i,j} [X_i, Y_j]\ X_1\wedge \overset{i}{\overset{\downarrow}{\cdots}} \wedge X_k \wedge Y_1 \wedge \overset{j}{\overset{\downarrow}{\cdots}} \wedge Y_\ell$$

• As a multiderivation, $[\xi_1,\xi_2] = \xi_1\circ\xi_2 - (-1)^{k\ell}\,\xi_2\circ\xi_1$, where

$$\xi_1\circ\xi_2 : C^\infty(M)\times\overset{k+\ell+1)}{\cdots}\times C^\infty(M) \longrightarrow C^\infty(M),$$

$$(\xi_1\circ\xi_2)(f_0,\dots,f_{k+\ell}) := \sum_{\substack{\sigma \\ (\ell+1,k)\text{ shuffle}}} \mathrm{sgn}(\sigma)\, \xi_1\Big(\xi_2(f_{\sigma(0)},\dots,f_{\sigma(\ell)}),\ f_{\sigma(\ell+1)},\dots,f_{\sigma(k+\ell)}\Big)$$

<u>Proposition</u>: The NS bracket satisfies the following "Jacobi" identity: if $\xi_1\in\mathfrak{X}^k$, $\xi_2\in\mathfrak{X}^\ell$, $\xi_3\in\mathfrak{X}^m$,

$$(-1)^{km}[[\xi_1,\xi_2],\xi_3] + (-1)^{\ell k}[[\xi_2,\xi_3],\xi_1] + (-1)^{m\ell}[[\xi_3,\xi_1],\xi_2] = 0.$$

If $\pi\in\mathfrak{X}^2(M)$ whose associated bracket is $\{f,g\} = \pi(df,dg)$, it holds

$$\frac{1}{2}[\pi,\pi](df_1,df_2,df_3) = \{\{f_1,f_2\},f_3\} + \{\{f_3,f_1\},f_2\} + \{\{f_3,f_1\},f_2\},$$

so ...

<u>Theorem</u>: There exists a 1-1 correspondence

$$\left\{\begin{array}{c}\text{Poisson brackets}\\ \text{on } M\end{array}\right\} \;=\!=\!=\; \left\{\begin{array}{c}\text{Bivectors } \pi\in\mathfrak{X}^2(M)\\ \text{s.t. } [\pi,\pi]=0\end{array}\right\}$$

$$\{\,,\,\} \longleftrightarrow \pi$$

$$\boxed{\{f,g\} = \pi(df,dg)}$$

<u>Theorem:</u> There exists a unique $\mathbb{R}$-bilinear operator

$$[\cdot,\cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \longrightarrow \mathfrak{X}^{k+l-1}(M), \qquad (k,l \geq 0)$$

called the <u>Nijenhuis-Schouten bracket</u>, satisfying:

1) For $k=0=l$, $[f,g]=0$

2) For $k=1$, $l=0$, $[X,f] = Xf$.

3) For $k=1=l$, $[X,Y]$ is the usual Lie bracket

4) It is graded skew-symmetric:

$$[\xi_k, \xi_l] = -(-1)^{(k-1)(l-1)} [\xi_l, \xi_k] \qquad , \xi_k \in \mathfrak{X}^k(M), \xi_l \in \mathfrak{X}^l(M)$$

5) It is a graded anti-derivation:

$$[\xi_k, \xi_l \wedge \xi_m] = [\xi_k, \xi_l] \wedge \xi_m + (-1)^{(k-1)l} \xi_l \wedge [\xi_k, \xi_m]$$

Moreover, although (1)-(5) determine $[\cdot,\cdot]$ uniquely, it also satisfies the following graded Jacobi identity

$$6)\ (-1)^{(k-1)(m-1)}[[\xi_k, \xi_l], \xi_m] + (-1)^{(l-1)(k-1)}[[\xi_l, \xi_m], \xi_k] + (-1)^{(m-1)(l-1)}[[\xi_m, \xi_k], \xi_l] = 0$$

<u>Remarks:</u> 1) To describe $[\xi_k, \xi_l]$ it is enough to check 'pure wedges', and for these $[\cdot,\cdot]$ is given by

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \overset{\overset{i}{\downarrow}}{\cdots} \wedge X_k \wedge Y_1 \wedge \overset{\overset{j}{\downarrow}}{\cdots} Y_l$$

2) As a skew-sym multiderivation, $[\xi_k, \xi_l]$ is given by

$$[\xi_k, \xi_l] = \xi_k \circ \xi_l - (-1)^{(k-1)(l-1)} \xi_l \circ \xi_k \,,$$

where we set $\left(\begin{array}{l}\text{calling } \underline{(p,q)\text{-shuffle}} \text{ to a permutation } \sigma \in S_{p+q} \text{ s.t. } \sigma(1) < \cdots \sigma(p) \text{ and } \sigma(p+1) < \cdots < \sigma(p+q) \\ \text{One can show that } \#\{(p,q)\text{-shuffles}\} = \binom{p+q}{p}\end{array}\right.$

$$(\xi_\ell \circ \xi_k)(f_1 \cdots f_{\ell+k-1}) = \sum_{\sigma\ (k,\ell-1)\text{ shuffle}} \operatorname{sgn}\sigma\ \xi_\ell\left(\xi_k(f_{\sigma(1)}, \dots, f_{\sigma(k)}), f_{\sigma(k+1)}, \dots, f_{\sigma(k+\ell-1)}\right).$$

• The following might be helpful to memorize the signs:

<u>Definition</u>: A <u>$\mathbb{Z}$-graded vector space</u> is a v.s which is direct sum of vector spaces, $V = \bigoplus_{n \in \mathbb{Z}} V_n$. We say that elts of V_n have <u>degree</u> n.

<u>Definition</u>: A <u>$\mathbb{Z}$-graded Lie algebra</u> is a $\mathbb{Z}$-graded v.s $V = \bigoplus V_n$ with a Lie bracket $[\cdot,\cdot]$ such that $[V_n, V_m] \subseteq V_{n+m}$.

<u>Definition</u>: A <u>super $\mathbb{Z}$-graded Lie algebra</u> is a $\mathbb{Z}$-graded vector space with a bracket $[\cdot,\cdot]$ that satisfies $[V_n, V_m] \subseteq V_{n+m}$ and the following super versions of the Lie bracket:

i) <u>$\mathbb{R}$-bilinear</u>

ii) <u>Super skew-symmetric</u>: $[v,w] = -(-1)^{(\deg v)(\deg w)}[w,v]$

iii) <u>Super graded Jacobi</u>:

$$(-1)^{(\deg v)\deg(z)}[[v,w],z] + (-1)^{(\deg w)(\deg v)}[[w,z],v] + (-1)^{(\deg z)(\deg w)}[[z,v],w] = 0.$$

<u>Proposition</u>: The Schouten bracket makes $\mathfrak{X}^\bullet(M)$ into a super $\mathbb{Z}$-graded Lie algebra, with a shift in the degree of a multivector field, $\deg \mathfrak{X}^k(M) = k-1$. The property 5) shows that $[\xi_k, -]$ acts as a graded anti-derivation on the exterior algebra (with a shift of ξ_k, but not in the exterior algebra).

<u>Proposition</u>: 1) $[\xi_k, f] = \iota_{df}\xi_k$ (so for a Poisson manifold, $[\pi, f] = \iota_{df}\pi = X_f$)

2) $\mathcal{L}_X \xi_k = [X, \xi_k]$

• Therefore, we can give an eq. def of Poisson manifold:

Definition: A Poisson manifold is a manifold M together with a bivector field $\pi \in \mathfrak{X}^2(M)$ such that $[\pi, \pi] = 0$. Such bivector is also called a Poisson structure.

• (Linear Poisson structure on a vector space): In a finite dim v.s. V, we have $V^* \subseteq C^\infty(V)$. We say that a Poisson structure on V is linear if $\{V^*, V^*\} \subseteq V^*$. Such structure induces a bracket in V^*:

$$[\cdot,\cdot] : V^* \times V^* \longrightarrow V^*$$
$$(\omega, \omega') \longmapsto [\omega, \omega'] := \{\omega, \omega'\}$$

Let endow V^* with a Lie algebra structure. And one can recover $\{\cdot,\cdot\}$ from $[\cdot,\cdot]$ via

$$\{f, g\}(v) = [d_v f, d_v g](v).$$

Theorem: There's a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Linear Poisson} \\ \text{structures on } V \end{array} \right\} = \left\{ \begin{array}{c} \text{Lie algebra} \\ \text{structures on } V^* \end{array} \right\}$$

"linear Poisson geometry" = "Lie Theory"

It is very usual to take $V = \mathfrak{g}^*$, $\mathfrak{g}$ Lie algebra, and for $f \in C^\infty(\mathfrak{g}^*)$ to see $d_\xi f : T_\xi \mathfrak{g}^* = \mathfrak{g}^* \to \mathbb{R}$ as an elmt of $\mathfrak{g}$, then define $\{f, g\}(\xi) = [d_\xi f, d_\xi g](\xi)$, and

Corollary:

$$\left\{ \begin{array}{c} \text{Linear Poisson} \\ \text{structures on } \mathfrak{g}^* \end{array} \right\} = \left\{ \begin{array}{c} \text{Lie algebra} \\ \text{structures on } \mathfrak{g} \end{array} \right\}$$

SYMPLECTIC VS. POISSON GEOMETRY

Given $\pi \in \mathfrak{X}^2(M)$, we have $\pi^\# : \Omega^1(M) \to \mathfrak{X}(M)$, $\omega \longmapsto i_\omega \pi$ (and pointwise $\pi^\# : T^*M \to TM$, $\omega_p \mapsto i_{\omega_p}\pi_p$)

and $\omega \in \Omega^2(M)$, also $\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M)$, $X \longmapsto i_X \omega$

Definition: A bivector field $\pi \in \mathfrak{X}^2(M)$ is non degenerate when $\pi^\#$ is a $C^\infty(M)$-modules isomorphism.

Proposition: There's a 1-1 correspondence between non-deg bivectors and non-deg. 2-forms:

$$\left\{ \begin{array}{c} \text{non-deg.} \\ \text{bivectors } \pi \in \mathfrak{X}^2(M) \end{array} \right\} = \left\{ \begin{array}{c} \text{non-deg} \\ \text{2-forms } \omega \in \Omega^2(M) \end{array} \right\},$$

$$\pi \longmapsto \omega(X,Y) \equiv \omega(i_\alpha \pi, i_\beta \pi) := -\pi(\alpha,\beta)$$

$$\pi(\alpha,\beta) \equiv \pi(i_X\omega, i_Y\omega) := -\omega(X,Y) \longleftarrow \omega$$

And in particular it holds $i_{i_\alpha \pi}\omega = \alpha$, $i_{i_X\omega}\pi = X$ and

$$[\pi,\pi](df_1, df_2, df_3) = -2\, d\omega(\pi^\#(df_1), \pi^\#(df_2), \pi^\#(df_3))$$

Theorem: There's a 1-1 correspondence between non-deg Poisson struct. and sympl. structures:

$$\boxed{\text{Symplectic manifold} = \text{Non-deg. Poisson manifold}}$$

If in an open chart $(U; u_1, \dots u_n)$ (n even) $\pi = \sum_{i<j} \pi_{ij}\, \partial_{u_i} \wedge \partial_{u_j}$ with π non-deg, the matrix (π_{ij}) is invertible and if $\omega_{ij} := (\pi_{ij})^{-1}$ then $\omega = \sum_{i<j} \omega_{ij}\, du_i \wedge du_j$ is the corresponding sympl. form

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Definition: A vector field $X \in \mathfrak{X}(M)$ is said to be a Poisson vector field if $\mathcal{L}_X \pi = 0$

Proposition: Let (M,π) be a Poisson manifold. Then the Lie derivative of π w.r.t. $X \in \mathfrak{X}(M)$ is

$$(\mathcal{L}_X \pi)(df, dg) = X(\{f,g\}) - \{Xf, g\} - \{f, Xg\}.$$

Corollary: Hamiltonian v.f $\Longrightarrow$ Poisson v.f.

III THE SYMPLECTIC FOLIATION

<u>Definition</u>: Let (M, π) be a Poisson manifold and define the following equivalence relation:

$$x \equiv y \iff \exists f_1, \dots, f_k \in \mathcal{C}^\infty(M) : y = \varphi^1_{X_{f_1}} \circ \cdots \varphi^1_{X_{f_k}} (x) ,$$

(ie, $x \equiv y$ iff one can travel from x to y through flows of Hamiltonian v.f). The equivalence classes of $\equiv$ are the <u>symplectic leaves</u>.

<u>Definition</u>: Let $i: N \hookrightarrow M$ an injective immersion. We say that i is an <u>initial submanifold</u> if $\forall f: P \to M$: $f(P) \subset i(N)$ the induced map $P \xrightarrow{f} i(N) \xrightarrow[\sim]{i^{-1}} N$ is smooth.

Note that here we endow $i(N)$ with the topology & smooth struc. given by i.

<u>Proposition</u>: Every initial submanifold has a unique smooth structure for which the inclusion is an immersion.

<u>Definition</u>: The <u>rank</u> of a Poisson structure π at $x \in M$ is rank $\pi^\#_x = \dim \operatorname{Im} \pi^\#_x$.

<u>Lemma</u>: $\operatorname{rk} \pi$ = constant along each symplectic leaf.

<u>Theorem</u> (Weinstein-Darboux Splitting Thm): Let (M, π) a Poisson manifold and $m \in M$. There exists a coordinate open $(U; \phi(p_1 \cdots p_n, q_1 \cdots q_n, x_1, \dots, x_s))$ centered at m (ie, $\phi(m) = 0$), where $\operatorname{rk} \pi_m = 2n$, s.t.

$$\pi_{|U} = \sum_{i=1}^{n} \partial_{p_i} \wedge \partial_{q_i} + \frac{1}{2} \sum_{a,b=1}^{s} \theta_{a,b}(x) \, \partial_{x_a} \wedge \partial_{x_b} ,$$

where $\theta_{a,b}$ are smooth functions of $(x_1 \dots x_s)$ verifying $\theta_{a,b}(0) = 0$.

Note that for non-deg Poisson struct. = symplectic structure, splitting thm = Darboux thm.

• By shrinking U, one can always suppose that $\phi(U) = V \times W$, $V \subset \mathbb{R}^{2n}$, $W \subset \mathbb{R}^s$, and if $\pi_{can}|_V = \sum \partial_{p_i} \wedge \partial_{q_i} \in \mathfrak{X}^2(V)$, $\theta = \frac{1}{2}\sum \theta_{ab}(x)\, \partial_a \wedge \partial_b \in \mathfrak{X}^2(W)$, then $(U, \pi) \xrightarrow{\sim} (V, \pi_{can}|_V) \times (W, \theta)$, (and θ is a Poisson str. !). This arises the following

<u>Definition</u>: A <u>splitting chart of rank $2n$</u> on (M, π) is a chart $\phi : U \xrightarrow{\sim} V \times W$, with $V \subset \mathbb{R}^{2n}$, $W \subset \mathbb{R}^s$ (with $\dim M = 2n+s$), with V connected, such that

$$\phi : (U, \pi|_U) \xrightarrow{\sim} (V, \pi_{can}|_V) \times (W, \theta)$$

is a Poisson diffeomorphism (and θ some Poisson str. on $\mathbb{R}^s$).

<u>Lemma</u>: let (M, ω) be P. manifold, and let S be a symplectic leaf with rk $\pi = 2n$ along S. If ϕ is a splitting chart as before, then $S \cap U = \phi^{-1}(V \times \Lambda)$ for a subset $\Lambda \subset \{w \in W : \theta_w = 0\}$. Moreover, any Hamiltonian flow which is contained in $S \cap U$ will be contained in $\phi^{-1}(V \times \{\lambda\})$ for a unique $\lambda \in \Lambda$.

<u>Lemma</u>: there is a differentiable structure on S s.t. S is an initial submanifold of M.

<u>Theorem</u>: The symplectic leaves of a Poisson manifold (M, π) are connected and initial submanifolds of M. Moreover, if S is a symplectic leaf, then

1) $T_x S = \operatorname{Im} \pi_x^\#$,

2) S has a symplectic structure given by

$$\omega_{S_x}(\pi_x^\# \alpha, \pi_x^\# \beta) := -\pi_x(\alpha, \beta), \qquad x \in S;\ \alpha, \beta \in T_x^* M.$$

<u>Definition</u>: The <u>symplectic foliation</u> of a Poisson manifold (M, π) is the family of sympl. leaves

$$\mathcal{S} = \{(S, \omega_S) : S \text{ sympl. leaf}\}$$

• If I know $\mathcal{S}$, I know π: the sympl. foliation determines the Poisson structure, and also

$$M = \coprod_{S} (S, \omega_S)$$

<u>Definition</u>: A <u>cotangent</u> or <u>Poisson path</u> on a Poisson manifold (M,π) is a pair (γ, a) consisting of a path $\gamma : I \longrightarrow M$ on M and a path $a : I \longrightarrow T^*M$ sitting above γ, i.e., $pr \circ a = \gamma$, such that

$$\pi^\#(a(t)) = \frac{d\gamma}{dt}(t)$$

$$\begin{tikzcd} I \arrow[r, "a"] & T^*M \arrow[r, "\pi^\#"] \arrow[d, "pr"] & TM \\ & M \arrow[ur, dashed, "\frac{d}{dt}\big|_{t=0}"'] & \end{tikzcd}$$

<u>Proposition</u>: let (M,π). Two points $x, y \in M$ belong to the same symplectic leaf $\Longleftrightarrow \exists (\gamma, a)$ cotangent path such that $\gamma(0) = x$, $\gamma(1) = y$.

<u>Proposition</u>: let (M,π) Poisson, and let $i : N \hookrightarrow M$ be a connected, immersed submanifold satisfying

$$T_x N = \operatorname{Im} \pi^\#_x \quad , \quad x \in N.$$

Then N is an open subset of a <u>single</u> symplectic leaf.

<u>Corollary</u>: let (M,π) Poisson and consider $\mathcal{N} = \{N\}$ a partition (i.e. disjoints) of M of immersed connected submanifolds satisfying

$$T_x N = \operatorname{Im} \pi^\#_x \quad , \quad \forall x \in N, \quad \forall N.$$

Then $\mathcal{N}$ is the symplectic foliation of M.

IV. POISSON TRANSVERSALS

<u>Definition</u>: Let (M,ω) symplectic. The <u>symplectic complement</u> or <u>ω-orthogonal</u> of $V \subset T_xM$ is

$$V^{\omega,\perp} := \{ w \in T_xM : (i_w \omega_x)_{|V} = \omega_x(w,V) = 0 \}.$$

<u>Definition</u>: Let (M,π) Poisson, $x \in M$ and (S,ω_S) the sympl. leaf passing through x. We'll say that the <u>π-orthogonal</u> of $V \subset T_xM$ is

$$V^{\pi,\perp} := (V \cap T_xS)^{\omega_S,\perp}.$$

<u>Lemma</u>: $V^{\pi,\perp} = \pi_x^{\#}(V^{\circ})$.

<u>Definition</u>: Let (M,π) Poisson. A <u>Poisson transversal</u> is an embedded submanifold $X \subset M$ such that

$$T_xM = T_xX + T_x^{\pi,\perp}X \qquad \forall x \in X.$$

Recall from Linear Algebra: $E = V \oplus W \Rightarrow E^* = V^{\circ} \oplus W^{\circ}$.

<u>Definition</u>: Let (M,π) Poisson and (S,ω_S) a symplectic leaf. A <u>complementary transversal</u> through $x \in S$, an embedded submanifold $X \subset M$ passing through x such that

$$T_xM = T_xX \oplus T_xS$$

<u>Lemma</u>: If $X \subset (M,\pi)$ is a complementary transversal through $x \in S$, then there is an open nbh $Y \subset X$ of x such that Y is a Poisson transversal. If π_Y is the induced Poisson str. in Y, then $\pi_{Y,x} = 0$.

If $\phi : (U, \pi_{|U}) \xrightarrow{\sim} (V, \pi_{can|V}) \times (W, \pi_W)$ is a splitting chart around x, then $Y := \phi^{-1}(\{0\} \times W)$ is a complementary Poisson transversal, and ϕ restricts to a Poisson diffeomorphism $\phi_{|Y} : (Y, \pi_Y) \xrightarrow{\sim} (W, \pi_W)$

Theorem: Let (M, π) Poisson and (S, ω_S) a sympl. leaf. If $X_1, X_2 \subset M$ are two complementary transversals through $x_1, x_2 \in S$ respectively, then there exist open nbhd's $Y_1 \subset X_1$, $Y_2 \subset X_2$ of x_1 and x_2 which are Poisson transversals and which are Poisson diffeomorphic

$$\Psi : (Y_1, \pi_{Y_1}) \xrightarrow{\sim} (Y_2, \pi_{Y_2})$$

and $\Psi(x_1) = x_2$.

Proposition: Let $X \subset (M, \pi)$ be an embedded submanifold. The following are equivalent:

1) X is a Poisson transversal

2) $T_x M = T_x X \oplus T_x^{\pi,\perp} X$

3) π in the conormal bundle of X, $\sigma_{X,x} : T_x^0 X \times T_x^0 X \to \mathbb{R}$, $\sigma_{X,x}(\alpha, \beta) = \pi_x(\alpha, \beta)$, is non-degenerated.

4) For every symplectic leaf S, "X intersects S transversally and symplectically":
   - X is transverse with S, $T_x M = T_x X + T_x S$ $\forall x \in X \cap S$, and we write $X \pitchfork S$.
   - $X \cap S$ is symplectic in (S, ω_S), i.e., $\omega_S|_{X \cap S}$ is non-deg.

Recall from Linear Algebra: if $E = V \oplus W$, then $\Lambda^2 E = \Lambda^2 V \oplus (V \otimes W) \oplus \Lambda^2 W$.

So $\pi|_X \in \Lambda T_X M = \Lambda^2 TX \oplus (TX \otimes T^{\pi,\perp} X) \oplus \Lambda^2 T^{\pi,\perp} X$ decomposes in 3 components, the the middle one is 0, so $\pi|_X = \pi_X + \sigma_X$.

Proposition: With the previous notation, π_X is a Poisson structure on X.

Remark: In general, it is not true that the inclusion $i : (X, \pi_X) \hookrightarrow (M, \pi)$ is a Poisson map unless X is an open set in M).

Proposition: Let $\phi : (M_1, \pi_1) \to (M_2, \pi_2)$ Poisson and $X_2 \subset M_2$ a Poisson transversal. Then ϕ is transverse to X_2 (ie, $T_y M_2 = T_y X_2 + \phi_*(T_x M_1)$) and $X_1 := \phi^{-1}(X_2)$ is a Poisson transversal in M_1. Moreover, $\phi|_{X_1} : (X_1, \pi_{X_1}) \to (X_2, \pi_{X_2})$ is a Poisson map.

We sum up everything:

Theorem: A Poisson transversal $X \subset (M, \pi)$ has a canonical Poisson structure π_X. The symplectic leaves of (X, π_X) are the connected components of the intersections $(X \cap S, \omega_S|_{X \cap S})$ where (S, ω_S) is a symplectic leaf of (M, π).

V : DIRAC GEOMETRY

DIRAC LINEAR ALGEBRA

Definition: Let V be a real v.s. of dimension n. We will denote

$$\mathbb{V} := V \oplus V^*$$

In $\mathbb{V}$ we have a canonical symmetric metric $S(v+\alpha, w+\beta) := \alpha(w) + \beta(v)$.

Lemma: S has signature $(p=n, q=n)$.

Definition: A vector subspace $L \subseteq \mathbb{V}$ is said to be *Lagrangian* if

i) $S_{|L} = 0$ (ie, rad $L = L$)

ii) $\dim L = n$.

We will denote as $\mathrm{Lag}(\mathbb{V})$ the set of Lagrangian subspaces of $\mathbb{V}$.

Proposition: $L \subseteq \mathbb{V}$ is Lagrangian $\iff L^\perp = L$.

Examples: $V = V \oplus 0 \in \mathrm{Lag}(\mathbb{V})$, $V^* = 0 \oplus V^* \in \mathrm{Lag}(\mathbb{V})$, $W \oplus W^0 \in \mathrm{Lag}(\mathbb{V})$ $\forall W \subseteq$

Definition: We will say that $W \subseteq \mathbb{V}$ is *transverse* to V^* (resp. V) if $\mathbb{V} = W \oplus V^*$ (resp. $\mathbb{V} = V \oplus W$).

*)

Lemma: Let V be a finite dim v.s. There's an isomorphism

$$\begin{aligned} \Lambda_2 V &= \mathrm{Hom}_{\text{skew-sym}}(V, V^*) \\ \omega &\longmapsto T : V \to V^* \\ &\qquad\quad v \mapsto i_v\omega \end{aligned}$$

And analogously

$\Lambda^2 V = \mathrm{Hom}_{\text{skew-sym}}(V^*, V)$

And also

$\mathcal{M}(V) = S_2 V = \mathrm{Hom}_{\text{sym}}(V, V^*)$

$\mathcal{M}(V^*) = S^2 V = \mathrm{Hom}_{\text{sym}}(V^*, V)$

($S_2 V \subset \mathcal{T}_2^0(V)$ symmetric, $S^2 V \subseteq \mathcal{T}_0^2(V)$ sym)

where $T : V \to V^*$ being skew-symmetric means $T^* = -T$, and symmetric $T^* = T$

Definition: Given $\omega \in \Lambda_2 V$ and $\pi \in \Lambda^2 V$, we will denote

$$V^\omega := \{ v + i_v \omega \; : \; v \in V \}$$

and $$\mathbb{V}^{*^{\pi}} := \{ i_\alpha \pi + \alpha \;:\; \alpha \in V^* \}.$$

It can be easily checked that $\mathbb{V}^{\omega}, \mathbb{V}^{*^{\pi}} \in Lag(\mathbb{V})$.

Theorem: Let $L \in Lag(\mathbb{V})$.

1) L is transverse to V^* $\iff$ $L = \mathbb{V}^{\omega}$ for some $\omega \in \Lambda_2 V$

2) L is transverse to V $\iff$ $L = \mathbb{V}^{*^{\pi}}$ for some $\pi \in \Lambda^2 V$.

Given $\omega \in \Lambda_2 V$, the map $E_\omega : \mathbb{V} \longrightarrow \mathbb{V}$, $v + \alpha \longmapsto v + \alpha + i_v \omega$ is a isometry, then isomorphism, and therefore it preserves the Lagrangian subspaces:

Definition: The gauge transformation of L by $\omega \in \Lambda_2 V$ is $L^{\omega} := E_\omega(L) = \{ v + \alpha + i_v \omega \;:\; v + \alpha \in L \} \in Lag(\mathbb{V})$

Note that this notion coincides with the one given for $\mathbb{V}$.

Lemma: The gauge transformation of $W \oplus W^{\circ}$ by $\tilde{\omega} \in \Lambda_2 V$ only depends on the restriction of $\tilde{\omega}$ to W.

If $\tilde{\omega}_{|W} = \omega$, we will write $L(W, \omega) := (W \oplus W^{\circ})^{\tilde{\omega}}$.

Theorem: If $L \in Lag(\mathbb{V})$, there exists a unique vector subspace $W \subseteq V$ and a unique 2-form $\omega \in \Lambda_2 W$ such that $L = L(W, \omega)$. In particular, this elements are

$$W := \pi_1(L) \qquad , \quad \omega(v_1, v_2) = \alpha_1(v_2) = -\alpha_2(v_1), \qquad v_1 + \alpha_1, v_2 + \alpha_2 \in L$$

Definition: Let U another v.s. and consider $\mathbb{U} := U \oplus U^*$. If $T : V \longrightarrow U$ is a linear map, we will call push-forward along T of a v. subspace $L_1 \subseteq \mathbb{V}$ to

$$T_!(L_1) := \{ Tv + \beta \;:\; v + T^*\beta \in L_1 \} \subseteq \mathbb{U},$$

---

*) **Proposition**: Let $E = V \oplus W$ be a v.s. decomposed as direct sum of two subspaces, and let $L \subseteq E$.

1) $E = L \oplus W$ $\iff$ the first projection restricts to an isomorphism $\pi_1|_L : L \xrightarrow{\sim} V$

2) $E = V \oplus L$ $\iff$ second $\pi_2|_L : L \xrightarrow{\sim} V^*$

and the pull-back along T of a linear subspace $L_2 \subseteq \mathbb{W}$ is

$$T^!(L_2) := \{ v + T^*\beta \ : \ Tv + \beta \in L_2 \} \subseteq \mathbb{V}$$

Lemma: If $L_1 \in Lag(\mathbb{V})$, $L_2 \in Lag(\mathbb{W}) \implies T_!(L_1) \in Lag(\mathbb{W})$, $T^!(L_2) \in Lag(\mathbb{V})$.

DIRAC STRUCTURES

Definition: Let M be a n-dim manifold. We will denote

$$\mathbb{T}M := TM \oplus T^*M = \coprod_{p \in M} T_pM \oplus T_p^*M .$$

We also have the canonical metric S defined in every fibre.

- Note that $\Gamma(\mathbb{T}M) = \mathfrak{X}(M) \oplus \Omega^1(M)$.

Definition: The Dorfman bracket is the following bracket in the sections of $\mathbb{T}M$:

$$[X+\omega, Y+\xi] := [X,Y] + \mathcal{L}_X \xi - i_Y d\omega .$$

$$\left(\begin{aligned} &= [X,Y] + \mathcal{L}_X \xi - \mathcal{L}_Y \omega + d(S(\omega, Y)) \\ &= [X,Y] + i_X d\xi - i_Y d\omega + d(S(\xi, X)) \end{aligned} \right)$$

Proposition (Properties): Let $a_1, a_2, a_3 \in \Gamma(\mathbb{T}M)$, and $f \in \mathcal{C}^\infty(M)$; and denote $\pi_1 : \Gamma(\mathbb{T}M) \to \mathfrak{X}(M)$, $X_p + \omega_p \longmapsto X_p$. The Dorfman bracket satisfies the following relations:

1) It is the Lie bracket on $\mathfrak{X}(M)$.

2) Leibniz: $[a_1, f a_2] = \pi_1(a)(f) \cdot a_2 + f \cdot [a_1, a_2]$

3) Skew-symmetry up to an exact 1-form: $[a_1, a_2] + [a_2, a_1] = d(S(a_1, a_2))$

4) "Jacobi-id." : $[a_1, [a_2,a_3]] = [[a_1,a_2], a_3] + [a_2, [a_1, a_3]]$.

5) Metric preserving : $\pi_1(a_1)(S(a_2,a_3)) = S([a_1,a_2], a_3) + S(a_2, [a_1,a_3])$.

We need the concept of vector bundle to continue:

VECTOR BUNDLES

Definition: Let M be a manifold. A vector bundle of rank K over M is a smooth manifold E together with a smooth surjective map $\pi: E \to M$ s.t.

i) $E_p := \pi^{-1}(p)$ is a vector space of dim. K, $\forall p \in M$, called the fiber of E over p.

ii) E is locally trivial: every point $p \in M$ has a nbhd $U \subset M$: $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{R}^K$ and the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi", "\sim"'] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^K \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

We say that E is the total space, M is the basis, π the projection and the isomorphism ϕ a local trivialization.

We say that E is a trivial bundle if there exists a local trivialization of the entire manifold M.

Examples: The trivial bundle $E = M \times \mathbb{R}^K$; TM ; T^*M, ...

Definition: A section of E is a section of π, ie, a smooth map $s: M \to E$: $\pi \circ s = Id_M$.

We will denote as $\Gamma(E)$ the sections of E.

Definition. Let $\pi: E \to M$ be a vector bundle. A vector subbundle of E is a submanifold $V \subset E$ such that $\pi_{|V}: V \to M$ is a vector bundle, ie,

i) $V_p := (\pi_{|V})^{-1}(p) = V \cap \pi^{-1}(p)$ is a linear subspace of E_p, $\forall p$

$V \cap E_p$

ii) Every point $p\in M$ has a nbhd $U\subset M$:

$$\begin{tikzcd}
(\pi_{|V})^{-1}(U) \arrow[rr, "\phi", "\sim"'] \arrow[dr, "\pi"'] & & U\times\mathbb{R}^k \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

<u>Definition</u>: A <u>Lagrangian subbundle</u> is a vector subbundle $L\subset \mathbb{T}M$ such that for every $p\in M$, L_p is a Lagrangian subspace of $\mathbb{T}_pM = T_pM\oplus T_p^*M$. In other words, $S(e,e')=0$ $\forall e,e'\in L_p$, $\forall p\in M$, and $\operatorname{rank}(L)=\dim M=n$. Ie, $L=L^\perp$, ie, $L_p=L_p^\perp$ $\forall p\in M$.

<u>Definition</u>: A <u>Dirac structure</u> is a Lagrangian subbundle $L\subset\mathbb{T}M$ which is <u>involute</u>, ie, $[a_1,a_2]\in\Gamma(L)$ $\forall a_1,a_2\in\Gamma(L)$. We will call <u>Dirac manifold</u> to a pair (M,L).

- In a Dirac manifold, $(\Gamma(L),[\cdot,\cdot])$ is a Lie Algebra (bc the involutive property).
- For a Lagrangian subbundle we can consider the following 3-form on L, ie, $\Upsilon_L\in\Gamma(\Lambda_3 L)$

$$\begin{aligned}
\Upsilon_L : \Gamma(L)^3 &\longrightarrow \mathcal{C}^\infty(M)\\
(a_1,a_2,a_3) &\longmapsto \Upsilon_L(a_1,a_2,a_3) := S([a_1,a_2],a_3)
\end{aligned}$$

<u>Proposition</u>: L is Dirac $\iff \Upsilon_L=0$.

Note that a Lagrangian subbundle $L\subseteq\mathbb{T}M$ is transverse to T^*M (resp. TM) $\iff L=TM^\omega$ (resp. $L=T^*M^\pi$) for $\omega\in\Omega^2(M)$ (resp. $\pi\in\mathfrak{X}^2(M)$).

<u>Theorem</u>: Let $L=TM^\omega$ transverse to T^*M and $L'=T^*M^\pi$ transverse to TM. (L Lagrangian subbundle)

1) L Dirac $\iff$ ω closed
2) L' Dirac $\iff$ π Poisson.

Definition: Any $\omega \in \Omega^2(M)$ induces a metric-preserving bundle isomorphism

$$E_\omega : \mathbb{T}M \longrightarrow \mathbb{T}M$$
$$v+\alpha \longmapsto v+\alpha+i_v\omega_p \qquad , \qquad p=\pi(v+\alpha)\in M.$$

called gauge transformation.

Lemma: E_ω preserves the Dorfman bracket (i.e. $E_\omega[a,b]=[E_\omega a, E_\omega b]$) $\Leftrightarrow d\omega=0$.

Corollary: L Lagrangian subbundle, then L^ω Dirac $\Leftrightarrow \omega$ closed.

DIRAC MAPS & SUBMANIFOLDS

Definition: Let (N,L_N), (M,L_M) be two Dirac manifolds. A smooth map $f:N\to M$ is called forward Dirac if

$$L_{M,f(p)}=(f_{*,p})_! L_{N,p} \qquad \forall p\in N$$

and backward Dirac if

$$L_{N,p}=(f_{*,p})^! L_{M,f(p)} \qquad \forall p\in N.$$

Lemma: 1) $f:(N,\pi_N)\to(M,\pi_M)$ is a Poisson map $\Leftrightarrow f$ is forward Dirac

2) $f:(N,\omega_N)\to(M,\omega_M)$ is "symplectic map $\Rightarrow f$ is backward Dirac.

If $f:N\to M$ and L is a Dirac structure on M, then we can consider the pull back of L:

$$f^!(L):=\{(f_{*,p})^! L_{f(p)} \in Lag(\mathbb{T}_pN) : p\in N\}$$

In general $f^!(L)$ will ~~not~~ not be a smooth subbundle of $\mathbb{T}N$. But, if it is, then it is a Dirac structure!!

Theorem: Let $f:N\to M$ be smooth & let L be a Dirac structure on M. If $f^!(L)$ is a smooth subbundle of $\mathbb{T}N$, then $f^!(L)$ is a Dirac structure on N.

· Do we have a "easy" way to know if $f^!(L)$ is smooth?

THE PRESYMPLECTIC FOLIATION

· If L is a Dirac structure, then we saw that $L_p = L(W,\omega) = (W \oplus W^\circ)^{\tilde{\omega}}$ for some $W \subset T_pM$ and $\omega \in \Lambda_2 W$.

<u>Theorem</u> (Presymplectic foliation): A Dirac manifold (M,L) has a unique foliation

$$(M,L) = \coprod_{\lambda \in \Lambda} (F_\lambda, \omega_\lambda)$$

into connected, initial submanifolds F_λ, called <u>the presymplectic leaves</u> of the Dirac structure, which are endowed with closed 2-forms $\omega_\lambda \in \Omega^2(F_\lambda)$, called <u>presymplectic forms</u>, that satisfy

$$L_p = L(T_pF_\lambda, \omega_{\lambda,p})$$

<u>Theorem</u>: Let $f: N \to (M,L)$ be a smooth map and L a Dirac structure on M. If the family of vector spaces

$$f_{*,p}(T_pN) + \pi_1(L_{f(p)}) \subseteq T_{f(p)}M \qquad (p \in N)$$

have constant dimension, then $f^!(L)$ is smooth, and therefore a Dirac structure.

Moreover, for each presymplectic leaf $(F_\lambda, \omega_\lambda)$, we have that $f^{-1}F_\lambda$ is an initial submanifold of N, and the presymplectic leaves of $(N, f^!(L))$ are the connected components of the presymplectic manifolds $(f^{-1}F_\lambda, f^*\omega_\lambda)$

DIRAC TRANSVERSALS

<u>Definition</u>: A <u>Dirac transversal</u> for a Dirac manifold (M,L) is an embedded submanifold N whose inclusion map is transverse to L:

$$T_pN + \pi_1(L_p) = T_pM$$

Corollary: A Dirac transversal $N \subset (M, L)$ has an induced Dirac structure $L_N = i^! L$ for which $i : (N, L_N) \longrightarrow (M, L)$ is backward Dirac.

Definition: A Poisson transversal in a Dirac manifold (M, L) is a Dirac transversal $N \subset M$ whose induced Dirac structure is the graph of a Poisson structure, $L_N = T^*M^{\pi}$ with π Poisson.

Definition: A Dirac submanifold in a Dirac manifold (M, L) is an immersed submanifold $i : N \hookrightarrow M$ such that $\pi_1(L_p) \subset T_p N$.

Proposition: Every Dirac submanifold $N \subset (M, L)$ has an induced Dirac structure $L_N = i^! L$ for which the inclusion is both forward and backward Dirac.

Going back to Poisson manifolds,

Definition: Let (M, π) Poisson. A Poisson submanifold is an immersed submanifold $N \subset M$ s.t.

$$\operatorname{Im} \pi_p^{\#} \subset T_p N \qquad \forall p \in N$$

Proposition: Every Poisson submanifold N has a Poisson structure π_N such that the inclusion

$$i : (N, \pi_N) \hookrightarrow (M, \pi)$$

is a Poisson map.

VI : SYMPLECTIC REALIZATIONS

- Can we lift Poisson geometry up to sympl. geometry?

<u>Definition</u>: Let (M,π) be a Poisson mnfld. A <u>symplectic realization</u> of (M,π) is a symplectic mnfld (S,ω) together with a surjective submersion

$$\mu : (S,\omega) \longrightarrow (M,\pi)$$

that is a Poisson map.

We say that the symplectic realization is ...

a) <u>weak</u> if μ is submersion almost everywhere

b) <u>proper</u> if μ is proper (ie, fibres are compact)

c) <u>complete</u> if μ is complete (ie, $X_f \in \mathfrak{X}(M)$ Ham. v.f is complete $\Rightarrow X_{\mu^* f} \in \mathfrak{X}(S)$ complete).

<u>Lemma</u>: Proper $\Rightarrow$ Complete.

<u>Proposition</u>: Let $\phi : (M,\pi_M) \to (N,\pi_N)$ a smooth map between Poisson mnfld. The following are equivalent:

1) ϕ is a Poisson map

2) $\phi_* \pi_M = \pi_N$

3) For every $p \in M$, $q = \phi(p) \in N$, the following diagram commutes:

$$\begin{tikzcd}
T_pM \arrow[d, "\phi_*"'] & T_p^*M \arrow[l, "\pi_M^\#"'] \\
T_qN & T_q^*N \arrow[l, "\pi_N^\#"'] \arrow[u, "\phi^*"']
\end{tikzcd}$$

Lemma: If (S, ω) is a sympl. realization of (M, π), then

$$\dim S \geq 2 \dim M - \operatorname{rank}_x \pi \qquad , \forall x \in M.$$

In particular, if (M, π) has a fixed point $(x \in M : \pi_x = 0)$, then $\dim S \geq 2 \dim M$.

Examples:

1) $(M = \mathbb{R}^n, \pi = 0)$: There exist sympl. realiz.: e.g. take $(\mathbb{R}^{2n}, \omega_{can} = \sum dx_i \wedge dy_i)$, and $\mu : (\mathbb{R}^{2n}, \omega_{can}) \to (\mathbb{R}^n, 0)$ the projection of first n-coord.

Definition: An integrable system is a symplectic manifold (S, ω) together with functions $f_1, \dots, f_n \in C^\infty(S)$ (where $\dim S = 2n$) such that

i) $\{f_i, f_j\} = 0 \quad \forall i,j$ (Involution)

ii) (Independence) $df_1 \wedge \dots \wedge df_n \neq 0$ almost everywhere.

It follows that Integrable systems $=$ weak sympl. realizations of $(\mathbb{R}^n, \pi = 0)$.

2) $(M, \pi = 0)$ There exist sympl. reali.: eg take $(S = T^*M, \omega_{can})$, and μ the canonical projection $T^*M \to M$.

Sympl. realizations are related with "Lagrangians":

Definition Let (E, Ω) be a symplectic vector space of dim $2n$. A vector subspace $L \subseteq E$ is said to be Lagrangian if

i) $\Omega|_L = 0$ (ie, rad $L = L$),

ii) $\dim L = n$

In other words, if $L = L^{\Omega}$.

Proposition: Let (E, Ω) be a sympl. v.s.. If L is a Lagrangian subspace, then (E, Ω) is symplectomorphic to the space $(L \oplus L^*, \Omega_0)$, where $\Omega_0(e + \alpha, v + \beta) = \beta(e) - v(\alpha)$.

Definition: Let (M,ω) be a symplectic manifold of dim $2n$. A submanifold $L \subseteq M$ is said to be **Lagrangian** if $\forall p \in L$ T_pL is a Lagrangian vector subspace of T_pM, ie, if $\omega_{|L} = 0$ and $\dim L = n$.

Definition: A **Lagrangian fibration** of a manifold M (not neces. Poisson) is a symplectic manifold (S,ω) together with a surjective

$$\mu : (S,\omega) \longrightarrow M$$

such that the fibers of μ are connected Lagrangian submanifolds of (S,ω).

Lemma: The Lagrangian fibration with base M (dim n) are precisely the symplectic realizations of $(M, \pi = 0)$ of dimension $2n$.

3) $(V, \pi \text{ constant})$, ie, $\pi \in \Lambda^2 V$. As with 2-form, $\exists$ basis of V $\{e_1 - e_n, f_1 - f_n, g_1 - g_k\}$: $\pi = \sum_{i=1}^{n} e_i \wedge f_i$ $(\dim V = 2n+k)$. Therefore $(V, \pi_{const}) \simeq (\mathbb{R}^{2n}, \omega_{can}^{-1}) \times (\mathbb{R}^k, 0)$ and have a sympl. realization is $(\mathbb{R}^{2n}, \omega_{can}) \times (\mathbb{R}^{2k}, \omega_{can})$

4) $(\mathfrak{g}^*, \pi_{linear})$. That's not trivial.

Note that if G is a Lie group integrating $\mathfrak{g}$, then $T^*G = \coprod_{g \in G} T_g^*G \simeq \coprod_{g\in G} \mathfrak{g}^* \simeq G \times \mathfrak{g}^*$.

There is a canonical map $\mu : T^*G \simeq G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ projection up to the isomorphism,

Theorem:

$$\mu : (T^*G, \omega_{can}) \longrightarrow (\mathfrak{g}^*, \pi_{lin})$$
$$\xi_g \longmapsto l_g^* \xi \qquad , \quad l_g = (L_g)_{*,e}$$

is a symplectic realization.

Remark: The proof is just to notice that according to the first argument, if (S,ω) is a $\mathfrak{g}$-Hamiltonian space, then $\mu : (S,\omega) \longrightarrow (\mathfrak{g}^*, \pi_{lin})$ is a Poisson map, and that (other argument)

$\mu : (T^*G, \omega_{can}) \longrightarrow (\mathfrak{g}^*, \pi_{lin})$ can be made into $\mathfrak{g}$-Ham. space.

• Note that if $\mu : (S,\omega) \to (M,\pi)$ is a sym. realization, π is determined uniquely

because $\pi^\# = \mu_* \circ \pi_\omega^\# \circ \mu^*$

Q: Given $\mu : (S,\omega) \to M$ submersion, does there exist a Poisson structure on M : μ is a sym. real?

~~Theorem. Let $\mu : (S,\omega) \to M$ be a submersion and assume that the μ-fibres are connected.~~

~~Theo~~

Definition. Let M be a manifold and let $L \subset TM$ be a subbundle. Then we say that L is involutive if for $X, Y \in \Gamma(L) \Rightarrow [X,Y] \in \Gamma(L)$.

Theorem: Let $\mu : (S,\omega) \to M$ be a submersion and assume that the μ-fibres are connected.

$$\exists\, \pi \in \mathfrak{X}^2(M) \text{ Poisson} : \mu \text{ is a sympl. realization} \iff \mathcal{F}_\mu = \operatorname{Ker} \mu_* \text{ is involutive.}$$

VII. GEOMETRY BEHIND SYMP. REALIZATIONS

RECOVERING G FROM $\mathfrak{g}$

• If G is a Lie group, we consider its Lie algebra $\mathfrak{g} = \mathfrak{X}^L(G) \cong T_e G$:

$$\begin{array}{ccc} \text{Lie Groups} & \longrightarrow & \text{Lie Algebras} \\ G & \longmapsto & \mathfrak{g} \\ G \xrightarrow{F} H & \longmapsto & \mathfrak{g} \xrightarrow{F_*} \mathfrak{h} \end{array}$$

is the Lie functor

• Given $\mathfrak{g}$, a Lie algebra, can we recover G?

<u>Theorem</u>: For every finite-dimensional Lie algebra $\mathfrak{g}$, there exists a Lie group G integrating $\mathfrak{g}$, i.e., such that $\mathfrak{g}$ is the Lie algebra of G. Moreover, if we ask that the Lie group G is <u>1-connected</u> (connected and simply-connected), then it exists and it is unique (up to isomorphism).

<u>Definition</u>: Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. The <u>Maurer-Cartan</u> 1-form is the $\mathfrak{g}$-valued 1-form $\theta \in \Omega^1(G, \mathfrak{g})$ that in every point $g \in G$ is:

$$\begin{aligned} \theta_g : T_g G &\longrightarrow \mathfrak{g} \\ v &\longmapsto \theta_g(v) := (R_{g^{-1}})_{*,g}(v). \end{aligned}$$

<u>Proposition</u>: Let $X, Y \in \mathfrak{X}(G)$ and let $\theta \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan 1-form. It holds

$$d\theta(X,Y)(g) = [\theta_g(X_g), \theta_g(Y_g)]_{\mathfrak{g}}.$$

<u>Lemma</u>: Let M be a manifold and $g = g(\varepsilon, t): I \times I \longrightarrow M$ a smooth map. If $\omega \in \Omega^1(M)$, then

$$d\omega\left(\frac{dg}{d\varepsilon}, \frac{dg}{dt}\right) = \frac{d}{d\varepsilon}\left(\omega\left(\frac{dg}{dt}\right)\right) - \frac{d}{dt}\left(\omega\left(\frac{dg}{d\varepsilon}\right)\right)$$

or more explicitly,

$$(d\omega)_{g(\varepsilon,t)}\left(\frac{dg}{dt}(\varepsilon,t),\frac{dg}{dt}(\varepsilon,t)\right)=\frac{d}{d\varepsilon}\left(\omega_{g(\varepsilon,t)}\left(\frac{dg}{dt}(\varepsilon,t)\right)\right)-\frac{d}{dt}\left(\omega_{g(\varepsilon,t)}\left(\frac{dg}{d\varepsilon}(\varepsilon,t)\right)\right).$$

It is clear that can be also applied for a 1-form valued in a v.s.

After thinking a lot, one arrives to the correct notion of homotopy in $\mathfrak{g}$ that makes the Th:

<u>Definition</u>: Two paths $a_0, a_1 \in P(\mathfrak{g})$ are called <u>$\mathfrak{g}$-homotopic</u> if there exists a naive homotopy $(a_\varepsilon)_{\varepsilon\in[0,1]}$ (ie, $H: I\times I \to \mathfrak{g}$) such that the unique $b: I\times I \to \mathfrak{g}$, $b = b(\varepsilon, t)$ satisfying

$$\begin{cases} \dfrac{da}{d\varepsilon}-\dfrac{db}{dt}=[b,a] \\ b_\varepsilon(0)=0 \quad \forall\varepsilon \end{cases}$$

also satisfies $b(\varepsilon,1)=0$. $\forall\varepsilon$.

<u>POISSON GEOMETRY ON SYMPLECTIC REALIZATIONS</u>

Consider $\mu:(S,\omega)\to(M,\pi)$ a symplectic realization, and let $\mathcal{F}_\mu = Ker(\mu_*: TS \to TM)$ and $\mathcal{F}_\mu^\perp$. <u>Note</u> that they may intersect non trivially in general ((E,ω) symp. v.s., $V\subset E$, then V sympl. $\Rightarrow 0 = V\cap V^\perp \Leftrightarrow E = V\oplus V^\perp$.)

<u>Definition</u>: An <u>infinitesimal action</u> associated to a SR $\mu:(S,\omega)\to(M,\pi)$ is a bundle map

$$\rho: \mu^*(T^*M) \longrightarrow TS$$

(ie, $\rho_p: T^*_{\mu(p)}M \to T_pS$ $\forall p\in S$) defined by the equation $i_{\rho(\alpha)}\omega = \mu^*(\alpha)$.

<u>Proposition</u>: $\rho: \mu^*(T^*M) \xrightarrow{\simeq} \mathcal{F}_\mu^\perp$ is an isomorphism of vector bundles, ie, $\rho: T^*_{\mu(p)}M \to \mathcal{F}_{\mu,p}^\perp$ is isomorphism $\forall p\in S$.

• Given a Poisson manifold (M, π), there's a canonical bracket on $\Omega^1(M)$:

Definition: Let (M,π) be Poisson. There's a bracket of 1 forms

$$[\cdot,\cdot]_\pi : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^1(M)$$
$$(\alpha\ ,\ \beta) \longmapsto \boxed{[\alpha,\beta]_\pi := \mathcal{L}_{\pi^\# \alpha}\beta - \mathcal{L}_{\pi^\#\beta}\alpha - d(\pi(\alpha,\beta))}$$

Proposition (Properties):

1) $[df, dg]_\pi = d\{f,g\}$.

2) $[\alpha, f\beta]_\pi = (\pi^\#\alpha)f \cdot \beta + f\cdot[\alpha,\beta]_\pi$

3) $\pi^\#([\alpha,\beta]_\pi) = [\pi^\#\alpha, \pi^\#\beta]$

4) The properties 1) and 2) determine $[\cdot,\cdot]_\pi$; and 3) is equivalent to Jacobi.

Definition: Let (M,π) be Poisson, let $x\in M$ and consider $\pi_x^\# : T_x^*M \longrightarrow T_xM$. The isotropy algebra of (M,π) at x, is

$$\mathfrak{g}_x(\pi) := \operatorname{Ker} \pi_x^\#$$

which is endowed with the unique Lie bracket $[\cdot,\cdot]_{\mathfrak{g}_x}$ such that $\boxed{[\alpha_x, \beta_x]_{\mathfrak{g}_x} = ([\alpha,\beta]_\pi)_x}$, for all $\alpha,\beta\in\Omega^1(M)$ s.t. $\alpha_x, \beta_x \in \mathfrak{g}_x$. In particular, $[\alpha_x,\beta_x]_{\mathfrak{g}_x} = [d_xf, d_xg]_{\mathfrak{g}_x} = d_x\{f,g\}$.

Note that if S is the sympl leaf passing through $x\in M$, then $\mathfrak{g}_x(\pi) = \operatorname{Ker}\pi_x^\# = (\operatorname{Im}\pi_x^\#)^\circ = T_x^\circ S$.

Corollary: ρ restricts to isomorphisms (pointwise)

$$\rho_p : \mathfrak{g}_x(\pi) \xrightarrow{\sim} \mathcal{F}_{\mu,p} \cap \mathcal{F}_{\mu,p}^\perp$$

Proposition: The map induced on sections by ρ,

$$\rho : \Omega^1(M) \longrightarrow \mathfrak{X}(S),$$

is a Lie algebra map.

__Definition__: Given a symplectic realization $\mu:(S,\omega)\to(M,\pi)$, and a cotangent (Poisson) path (γ,a) of (M,π), a __lift of (γ,a) to S__ is any path $u: I\to S$ sitting above γ, ie, $\mu\circ u=\gamma$, and satisfying

$$i_{\frac{du}{dt}(t)}\omega=\mu^* a(t) \iff \rho(a(t))=\frac{du}{dt}(t)$$

__Lemma__: A path $u: I\to S$ is the lift of a Poisson path $\iff \frac{du}{dt}(t)\in\widetilde{\mathcal{F}}_\mu^{\perp}\ \forall t$.

__Theorem__: Let $\mu:(S,\omega)\to(M,\pi)$ be a __complete__ SR, and let (γ,a) be a Poisson path on M with $\gamma(0)=x_0$, $\gamma(1)=x_1$.

For any $u_0\in\mu^{-1}(x_0)$ $\exists$ lift $u: I\to S$ of (γ,a) : $u(0)=u_0$.

• Therefore, if $P_\pi(M,x_0,x_1)=\{$Poisson paths from x_0 to $x_1\}$, then we obtain an "action"

$$P_\pi(M,x_0,x_1)\times\mu^{-1}(x_0)\longrightarrow\mu^{-1}(x_1)$$
$$((\gamma,a)\ ,\ u_0)\longmapsto u(1)\quad,\ u=\text{lift of }(\gamma,a)\text{ with }u(0)=u_0$$

how does this "action" depend on the Poisson path? More concretely, of the "homotopy class". If we propose the naive homotopy $(\gamma_0,a_1)\sim(\gamma_1,a_1)\iff\exists(\gamma_\varepsilon,a_\varepsilon)$ interpolating we failure bc nothing guarantees that $u_\varepsilon(1)=$ constant. We need a extra condition!!! The thing is that we control $\frac{du}{dt}$ with a, but nothing relates to $\frac{du}{d\varepsilon}$. Solution: put it.

Proposition: Consider a triple (γ, a, b) where

$\gamma = \gamma(\varepsilon, t) : I \times I \to M$ "base square"

$a = a(\varepsilon, t) : I \times I \to T^*M$ "Poisson speed in t-direction"

$b = b(\varepsilon, t) : I \times I \to T^*M$ "Poisson speed in ε-direction"

and satisfying

$$\pi^\#(a) = \frac{d\gamma}{dt} \quad , \quad \pi^\#(b) = \frac{d\gamma}{d\varepsilon} \quad , \quad \gamma(\varepsilon, 0) = \text{const} \quad , \quad b(\varepsilon, 0) = 0 \quad \forall \varepsilon .$$

Then the following are equivalent:

1) For any coordinate chart (U, x), writing

$$a = \sum a^i dx_i \quad , \quad b = \sum b^i dx_i \quad , \quad \pi = \frac{1}{2} \sum \pi_{ij} \, \partial_{x_i} \wedge \partial_{x_j}$$

one has

$$\frac{da^i}{d\varepsilon} - \frac{db^i}{dt} = - \sum_{k} \frac{\partial \pi_{jk}}{\partial x_i} (\gamma(\varepsilon, t)) \, a^j b^k$$

2) Writing $a(\varepsilon, t) = \alpha_{\varepsilon,t}(\gamma(\varepsilon,t))$, $b(\varepsilon,t) = \beta_{\varepsilon,t}(\gamma(\varepsilon,t))$ for $\alpha_{\varepsilon,t}, \beta_{\varepsilon,t} \in \Omega^1(M)$ (ε,t)-dependent,

$$\frac{d\alpha}{d\varepsilon}(x) - \frac{d\beta}{dt}(x) = [\beta, \alpha]_\pi (x) \quad , \quad x = \gamma(\varepsilon, t).$$

3) For $X \in \mathfrak{X}(M)$,

$$\frac{d}{d\varepsilon} \langle X, a \rangle - \frac{d}{dt} \langle X, b \rangle = [\pi, X](a, b)$$

Moreover, if $\mu : (S, \omega) \to (M, \pi)$ is a complete SR, we also have the equivalence

4) If for any ε, $u_\varepsilon : I \to S$ is a lift of $(\gamma_\varepsilon, a_\varepsilon)$ with $u_\varepsilon(0) = \text{const}$, then for any t, $u_t : I \to S$ is a lift of (γ_t, b_t).

Definition: A Poisson homotopy is a triple (γ, a, b) as in the previous proposition which satisfies the equivalence conditions,

$$\frac{da}{d\varepsilon} - \frac{db}{dt} + [a,b]_\pi = 0 .$$

Lemma: Any naive homotopy $(\gamma_\varepsilon, a_\varepsilon)$ can be completed to a Poisson homotopy (γ, a, b). Moreover, b is uniquely determined (and it exists) if $\gamma_\varepsilon(0) = \text{const}$ and $b(\varepsilon, 0) = 0$ $\forall \varepsilon$.

- The condition $b(\varepsilon,0) = \text{cst}$ means $\frac{du}{d\varepsilon}(\varepsilon, 0) = 0$. We'll see later.

Definition: A Poisson path homotopy is any Poisson homotopy (γ, a, b) such that

i) γ is a path homotopy, i.e., $\gamma(\varepsilon, 0) = \text{const}$, $\gamma(\varepsilon, 1) = \text{const}$

ii) $b(\varepsilon, 0) = 0 = b(\varepsilon, 1)$

We will say that two Poisson paths (γ_0, a_0), (γ_1, a_1) are Poisson path-homotopic if there exists a Poisson path homotopy with $a_0(t) = a(0,t)$ and $a_1(t) = a(1,t)$.

- By the prop, for a Poisson homotopy $u_t(\varepsilon)$ is a lift of (γ_t, b_t), then $i_{\frac{du}{d\varepsilon}(\varepsilon,1)}\omega = \mu^* b(\varepsilon, 1)$ so the condition $b(\varepsilon,1) = 0$ (and analogously $b(\varepsilon,0) = 0$) means $\frac{du}{d\varepsilon}(\varepsilon, 0) = 0 = \frac{du}{d\varepsilon}(\varepsilon, 1)$, and then $u(1) = \tilde{u}(1)$ for $u, \tilde{u}$ the lift of two Poisson homotopic paths.

- Denoting by

$$\Sigma_\pi(M, x_0, x_1) := \frac{P_\pi(M, x_0, x_1)}{\text{Poisson path-homotopy}}$$

we see that the "action" $\Sigma_\pi(M, x_0, x_1) \times \mu^{-1}(x_0) \longrightarrow \mu^{-1}(x_1)$ is well defined.

- Putting all together,

Definition The Poisson homotopy groupoid of a Poisson manifold (M, π) is

$$\Sigma_\pi(M) := \frac{\text{Poisson paths}}{\text{Poisson path-homotopy}}$$

• $\Sigma_\pi(M)$ comes with two maps.

$$\begin{tikzcd} \Sigma_\pi(M) \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ M \end{tikzcd} \qquad \begin{array}{ll} s([\gamma,a]) := \gamma(0) & \underline{\text{source map}} \\ t([\gamma,a]) := \gamma(1) & \underline{\text{target map}} \end{array}$$

and with a partial multiplication: if $g,h \in \Sigma_\pi(M)$: $s(g) = t(h)$, then $g\cdot h$ is the concatenation

$$\begin{tikzcd} t(g) & s(g) = t(h) \arrow[l, bend right, "g"'] & s(h) \arrow[l, bend right, "h"'] \end{tikzcd}$$

• This says that the previous "action" can be expressed as an action of $\Sigma_\pi(M)$ on any complete SR

$$\Sigma_\pi(M) \times_M S \longrightarrow S$$

$$([\gamma,a], p) \longmapsto u(1) \quad , \quad \begin{array}{l} u = \text{lift of } (\gamma,a) \text{ starting at } p. \\ \gamma(0) = \mu(p) \end{array}$$

where $\Sigma_\pi(M) \times_M S$ is the fibered product of $(\Sigma_\pi(M), s)$ with (S,μ), ie,

$$\Sigma_\pi(M) \times_M S = \{ (g,p) \in \Sigma_\pi(M) \times S : s(g) = \mu(p) \}.$$

VIII. POISSON HOMOTOPY GROUP

Definition: The Poisson homotopy group is

$$G_{\pi,p} := \{ g \in \Sigma_\pi(M) \ : \ s(g) = t(g) = p \}$$

which is a group with the concatenation.

Example: If $p \in M$ is a point on (M,π) with $\pi_p = 0$, then Poisson paths = paths on $T_p^*M = \mathfrak{g} = \ker \pi_p^\#$ the isotropy Lie algebra, and Poisson homotopy = $\mathfrak{g}$-homotopy, thus

$$G_{\pi,p} = \frac{\text{Poisson paths}}{\text{Poisson path-homotopy}} = \frac{\text{Paths on } \mathfrak{g}}{\mathfrak{g}\text{-homotopy}} \simeq G$$

1-connected Lie group integrating $\mathfrak{g}$.

Proposition: Let (M,π) a regular Poisson manifold (ie, rank π = const), and let (L,ω) the symplectic leaf passing through $p \in M$. There is an exact sequence of groups

$$\pi_2(L,p) \xrightarrow{\partial} \nu_p^*(L) \xrightarrow{f} G_{\pi,p} \xrightarrow{p} \pi_1(L,p) \longrightarrow 0$$

where $\nu_p^*(L) = (T_pL)^0 = \ker \pi_p^\# = \mathfrak{g}_p(\pi)$ is the conormal space at p, and

- $p([\gamma, a]) = \gamma$, $-\ f(\xi) = [\xi, p]$ constant Poisson path

- For $[\sigma] \in \pi_2(L,p)$, represented by $\sigma : S_2 \to M$, $\sigma(n) = p$ (n north pole), take $v \in \nu_p(L) = \frac{T_pM}{T_pL}$ and $\sigma_\varepsilon : S_2 \to M$: $\sigma_0 = \sigma$ and $\sigma_\varepsilon(S_2) \subset L_\varepsilon$, $(L_\varepsilon, \omega_\varepsilon)$ sympl. leaf through $p_\varepsilon = \sigma_\varepsilon(n)$ *. Then if we define $A_\varepsilon := \int_{S_2} \sigma_\varepsilon^* \omega_\varepsilon$, then $\partial[\sigma]$ is the 1-form s.t.

$$\partial[\sigma](v) := \frac{dA_\varepsilon}{d\varepsilon}(0).$$

The map ∂ is called the monodromy map.

*: such that $\left[\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} p_\varepsilon\right] = v$. The existence of this family can be proven using "Reeb stability".

EXISTENCE OF SYMPLECTIC REALIZATIONS

__Definition__: Let (M,π) be Poisson. A __Poisson spray__ is a vector field $V \in \mathfrak{X}(T^*M)$ satisfying

i) $pr_{*,\xi}(V_\xi) = \pi^\# \xi$, where $pr : T^*M \to M$

ii) $m_{s\,*,\xi}(V_\xi) = \frac{1}{s} V_{m_s(\xi)}$, where $m_s : T^*M \to T^*M$ is to multiply by $s \in \mathbb{R}$.

__Theorem__ (Existence of symplectic realizations): Every Poisson manifold admits symplectic realizations.

More precisely, let $V \in \mathfrak{X}(T^*M)$ be a Poisson spray. Then there is an open neighbourhood $U \subset T^*M$ of the zero-section on which the 2-form

$$\omega := \int_0^1 (\varphi_V^t)^* \omega_{can}\, dt$$

is symplectic and $pr : (U, \omega) \to (M, \pi)$ is a symplectic realization.

• For that,

__Lemma__: Identify M with the 0-section, $M = \{ 0_p \in T_p^*M : p \in M \} \subseteq T^*M$. Then

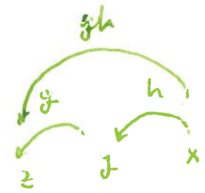
$$T(T^*M)|_M = TM \oplus T^*M .$$

IX : LIE GROUPOIDS

Definition: A groupoid $\mathcal{G} \rightrightarrows M$ consists of a set M ("points"), a set $\mathcal{G}$ ("arrows"), together with the following maps:

i) source / target : two maps $s,t : \mathcal{G} \to M$. For $g \in M$, $x,y \in M$, we write $g : x \to y$ or $y \xleftarrow{g} x$ to indicate that $s(g) = x$, $t(g) = y$.

ii) multiplication : a map $m : \mathcal{G}_2 \to \mathcal{G}$, $(g,h) \longmapsto m(g,h) := gh$, defined on the set of composable arrows $\mathcal{G}_2 = \{(g,h) \in \mathcal{G} \times \mathcal{G} : t(h) = s(g)\} = \{(g,h) : \xleftarrow{g}\xleftarrow{h}\}$, and such that



- $s(gh) = s(h)$, $t(gh) = t(g)$
- $(gh)k = g(hk)$ whenever $\xleftarrow{g}\xleftarrow{h}\xleftarrow{k}$

iii) unit : a map $u : M \to \mathcal{G}$, $u(x) := 1_x$ such that

- $s(1_x) = t(1_x) = x$
- $g \cdot 1_{s(g)} = 1_{t(g)} \cdot g$

iv) inverse : a map $\tau : \mathcal{G} \to \mathcal{G}$, $\tau(g) := g^{-1}$ such that

- $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$
- $g^{-1} g = 1_{s(g)}$, $g g^{-1} = 1_{t(g)}$

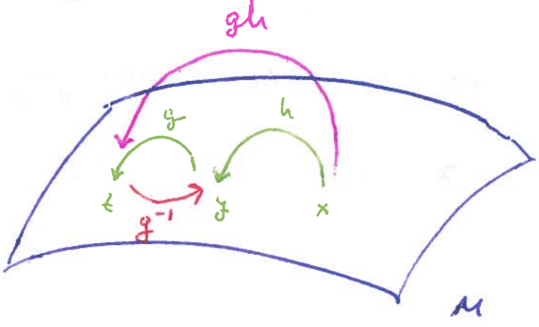
We also say that $\mathcal{G}$ is a groupoid over M, $\mathcal{G} \rightrightarrows M$.

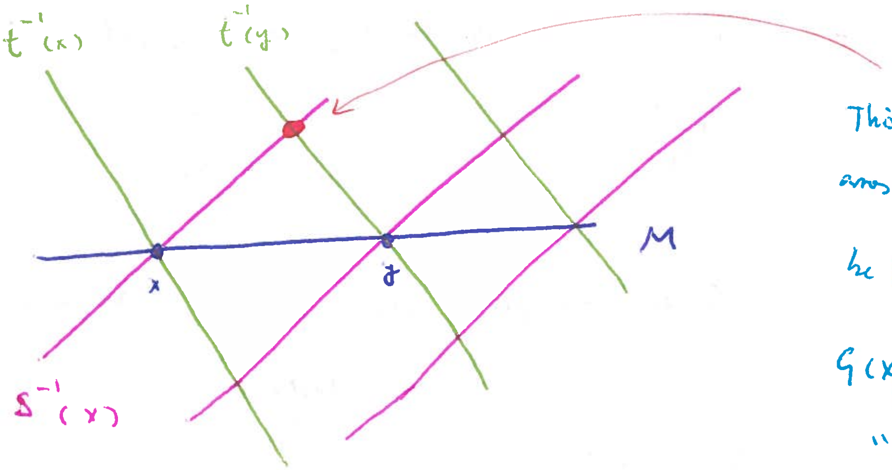
Definition: A Lie groupoid over a manifold M is a groupoid $\mathcal{G} \rightrightarrows M$ together with a smooth structure on $\mathcal{G}$ such that all the maps are smooth and

i) s,t are submersions

ii) u embedding

• Some pictures of Lie groupoids





This "fat dot" is the set of arrows $\{g : y \xleftarrow{g} x\}$. It can be huge! We will call it

$$\mathcal{G}(x,y) := \{g : y \xleftarrow{g} x\}$$

"set of arrows from x to y".

Definition: Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.

a) The s-fiber above $x \in M$ is $\mathcal{G}(x,-) := s^{-1}(x) = \{g \in \mathcal{G} : s(g) = x\} = \{g : \xleftarrow{g} x\}$

(and similarly the t-fibers $\mathcal{G}(-,y) = t^{-1}(y)$)

b) the right translation R_g by an arrow $g : x \to y$ is $R_g : s^{-1}(y) \to s^{-1}(x)$, $h \mapsto hg$

$\xleftarrow{h} y \qquad \xleftarrow{h} y \xleftarrow{g} x$

(which is diff. with inverse $R_{g^{-1}}$).

c) The isotropy group of $\mathcal{G}$ at $x \in M$ is

$$\mathcal{G}_x = \mathcal{G}(x,x) = s^{-1}(x) \cap t^{-1}(x) = \{g \in \mathcal{G} : s(g) = t(g) = x\} = \{g : x \xleftarrow{g} x\}$$

d) The orbit of $\mathcal{G}$ through $x \in M$ is

$$\mathcal{O}_x := \{y \in M : \exists\, g : y \xleftarrow{g} x\} = t(\mathcal{G}(x,-)) = \{\text{points connected to } x \text{ by an arrow}\}$$

Lemma: There is a natural action $\mathcal{G}(x,-) \times \mathcal{G}_x \to \mathcal{G}(x,-)$, which is free and proper, so $\mathcal{G}(x,-)/\mathcal{G}_x \simeq \mathcal{O}_x$ has a unique smooth struct. s.t. $t : \mathcal{G}(x,-) \to \mathcal{O}_x$ is submersion.

Examples : 1) Lie group = Lie groupoid over a point.

2) The pair groupoid $G := M \times M \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} M$, which has exactly one arrow joining two points. Thus the arrows are pairs $y \xleftarrow{(x,y)} x$. Moreover, $(y,z)\cdot(x,y) = (x,z)$, $1_x = (x,x)$, $(x,y)^{-1} = (y,x)$,

$s^{-1}(x) = M$, $t^{-1}(y) = M$.

3) The fundamental groupoid $\Pi(M)$:= path homotopy classes of paths $\gamma : [0,1] \to M$

$$\Pi(M) \overset{s}{\underset{t}{\rightrightarrows}} M, \qquad s([\gamma]) = \gamma(0), \quad t([\gamma]) = \gamma(1)$$

4) The action groupoid : if G is a Lie group acting on M, then we consider $G \ltimes M$ which is $G \times M \overset{s}{\underset{t}{\rightrightarrows}} M$,

$$s(g,x) = x, \quad t(g,x) = g\cdot x \qquad g x \xleftarrow{(g,x)} x$$

5) Flows of vector fields : for $X \in \mathfrak{X}(M)$, consider $\mathcal{D}(X) \subseteq \mathbb{R} \times M$ maximal domain where φ_X^t is defined. It defines a Lie groupoid $\mathcal{D}(X) \overset{s}{\underset{t}{\rightrightarrows}} M$, with $s(t,x) = x$, $t(t,x) = \varphi_X^t(x)$ and multiplication $(t,x)\cdot(s,y) = (t+s, y)$

$$\varphi_X^t(x) \xleftarrow{(t,x)} x$$

6) Gauge groupoid : If P is a principal G-bundle over M, $P \circlearrowleft G$, $P \to M$, G Lie group, then the gauge groupoid is the quotient of the pair groupoid modulo the G-action (acting diagonally).

$P \otimes_G P := P \times P / G = \{[p,q] : p,q \in G\}$, together with $P \otimes_G P \rightrightarrows M$, $s([p,q]) = pr(p)$, $t([p,q]) = pr(q)$.

LIE ALGEBROIDS

Definition: A Lie algebroid over M is a triple $(A, [\cdot,\cdot]_A, \#)$ consisting of:

i) A vector bundle $A \to M$

ii) A Lie bracket on sections, $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$

iii) A morphism of vector bundles $\# : A \to TM$, called the anchor

satisfying the following Leibniz rule:

$$[\alpha, f\beta]_A = (\#\alpha)f \cdot \beta + f\cdot[\alpha,\beta]_A, \qquad f \in C^\infty(M), \quad \alpha,\beta \in \Gamma(A),$$

where $\# : \Gamma(A) \longrightarrow \Gamma(TM) = \mathfrak{X}(M)$ is the induced map of sections $(M \xrightarrow{\alpha} A) \longmapsto (M \xrightarrow{\alpha} A \xrightarrow{\#} TM)$

- Ie, A is "like" the tangent bundle, it is something that behaves as TM (i) & ii)) and can be related to TM (iii))

__Lemma__: The induced map $\# : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism,

$$\# [\alpha, \beta]_A = [\# \alpha, \# \beta]$$

__Definition__. Let $(A, [\cdot,\cdot]_A, \#)$ a Lie algebroid. The __isotropy Lie algebra__ of A at $x \in M$ is

$$\mathfrak{g}_x = \mathfrak{g}_{A,x} := Ker \left(\#_x : A_x \longrightarrow T_x M \right)$$

__Lemma__. $\mathfrak{g}_x$ inherits a Lie algebra structure from $[\cdot,\cdot]_A$: there exists a unique Lie bracket $[\cdot,\cdot]_{\mathfrak{g}_x}$ on $\mathfrak{g}_x$ such that $[\alpha, \beta]_A (x) = [\alpha(x), \beta(x)]_{\mathfrak{g}_x}$, for $\alpha, \beta \in \Gamma(A)$: $\alpha(x), \beta(x) \in \mathfrak{g}_x$.

__Examples__. 1) Lie algebras = Lie algebroids over a point.

2) Poisson manifolds : $A := T^*M$, $\# := \pi^{\#} : T^*M \to TM$, $[\cdot,\cdot]_A := [\cdot,\cdot]_{\pi}$ bracket of 1-forms.

THE LIE ALGEBROID OF A LIE GROUPOID

- In Lie groups, we can associate a Lie algebra to every Lie group; and for every Lie algebra there exists a unique 1-connected Lie group integrating the Lie alg. Do we have an analogous situation here?

- Let $\mathcal{G} \rightrightarrows M$ a Lie groupoid. We associate it a Lie algebroid:

  - The vector bundle A is $A_x := T_{1_x} (\mathcal{G}(x,-)) = T_{1_x} (s^{-1}(x))$, $x \in M$.

    As vector bundle it also can be seen as $A = u^*(Ker\, s_*)$, $s_* : T\mathcal{G} \to TM$.

  - The anchor map is $\#_x := \left(t|_{s^{-1}(x)} \right)_{*, 1_x} : T_{1_x}(s^{-1}(x)) \to T_x M$

  - Lie bracket $[\cdot,\cdot]_A$? In Lie groups, we took left-inv. v.f. Now? Similar.

Definition: A vector field $X \in \mathfrak{X}(G)$ is called <u>right-invariant</u> if

i) X is tangent to the fibers of s, i.e., $X_h \in T_h(s^{-1}(y))$ $\qquad (s(h) = y)$

ii) $(R_g)_{*,h} : T_h(s^{-1}(y)) \longrightarrow T_{hg}(s^{-1}(x))$ satisfies $(R_g)_{*,h}(X_h) = X_{hg}$ $\quad \forall\, h,g \in G$ composables.

Note that i) is necessary for ii) to make sense.

- I claim that every $\alpha \in \Gamma(A)$ gives rise to a right invariant v.f. $\alpha^R \in \mathfrak{X}^{inv}(G)$: for $y \overset{g}{\curvearrowleft} x$ consider $(R_g)_{*,1_y} : A_y = T_{1_y}(s^{-1}(y)) \longrightarrow T_g(s^{-1}(x))$ and define

$$\alpha^R_g := (R_g)_{*,1_y}(\alpha_y).$$

In the other way, any right inv. v.f. X arises as α^R for some $\alpha \in \Gamma(A)$, namely the one given by $\alpha_y = X_{1_y}$. The moral is

Theorem: The previous discussion gives a bijection

$$\Gamma(A) \xrightarrow{\ \sim\ } \mathfrak{X}^{inv}(G).$$

Moreover, if $X, Y \in \mathfrak{X}^{inv}(G) \Rightarrow [X,Y] \in \mathfrak{X}^{inv}(G)$, so there exists a unique bracket of sections

$$[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$$

such that $([\alpha,\beta]_A)^R = [\alpha^R, \beta^R]$.

Theorem: $(A, \#, [\cdot,\cdot]_A)$ becomes a Lie algebroid

Examples: 1) G Lie group ($=$ Lie groupoid over a point) $\rightsquigarrow$ $\mathfrak{g}$ its Lie algebroid

2) $M \times M \rightrightarrows M$ pair groupoid $\rightsquigarrow$ TM its Lie algebroid

- For the question, is every Lie algebroid integrable? <u>NO</u>! There is obstructions of a more topological nature.

X : SYMPLECTIC GROUPOIDS

Definition: Let $\Sigma \overset{s}{\underset{t}{\rightrightarrows}} M$ be a Lie groupoid. A k-form $\omega \in \Omega^k(\Sigma)$ is said to be multiplicative if pulling back to $\Sigma_2 \subseteq \Sigma \times \Sigma$ (composable arrows) via $pr_1, pr_2, m : \Sigma_2 \to \Sigma$ one has

$$m^*\omega = pr_1^*\omega + pr_2^*\omega$$

Definition: A symplectic groupoid (Σ, ω) over M is a Lie groupoid $\Sigma \rightrightarrows M$ together with a symplectic form $\omega \in \Omega^2(\Sigma)$ which is multiplicative. (in particular it is a sympl. mnfd).

Theorem I: Given a symplectic groupoid (Σ, ω) over M, there exists a unique Poisson structure π on M such that

$$t : (\Sigma, \omega) \longrightarrow (M, \pi)$$

is a Poisson map. Such π is called the induced Poisson structure on the base.

Moreover, the Lie algebroid A of Σ is canonically isomorphic to the algebroid $(T^*M, \pi^\#, [\cdot,\cdot]_\pi)$ associated to π. More precisely, the vector bundle morphism

$$\sigma_\omega : A \longrightarrow T^*M$$

which corresponds in sections with

$$\sigma_\omega : \Gamma(A) \longrightarrow \Omega^1(M)$$
$$\alpha \longmapsto (i_{\alpha^R}\omega)_{|M}$$

is an isomorphism of Lie algebroids.

Furthermore, $t : (\Sigma, \omega) \longrightarrow (M, \pi)$ is a complete sympl. realization of (M, π).

Here, M is identified with $\{1_x : x \in M\} \subseteq \Sigma$, and u is viewed as an inclusion, so $(i_{\alpha^R}\omega)_{|M} = u^*(i_{\alpha^R}\omega)$.

"For proving this,

Proposition: For any symplectic groupoid one has:

i) $\dim \Sigma = 2 \dim M$

ii) $(\mathrm{Ker}\, s_*)^{\perp} = \mathrm{Ker}\, t_*$

iii) M is ~~Lagrangian~~ lagrangian in (Σ, ω), i.e., $T_{1_x} M = (T_{1_x} M)^{\perp}$.

In particular, it holds

$$\boxed{i_{\alpha^R} \omega = t^*(\sigma_\omega(\alpha))}$$

Corollary: The following equalities hold:

$$(IM\,1) \quad \sigma_\omega(\rho\beta) = -\sigma_\omega(\rho\alpha)$$

$$(IM\,2) \quad \sigma_\omega[\alpha,\beta] = \mathcal{L}_{\rho\alpha}\sigma_\omega\beta - \mathcal{L}_{\rho\beta}\sigma_\omega\alpha - d(\sigma_\omega(\alpha)(\rho\beta))$$

where IM stands for "infinitesimally multiplicative".

Corollary: If Σ has connected s-fibers, $\omega, \omega' \in \Omega^2(\Sigma)$ are multiplicative and satisfy $i_{\alpha^R} d\omega = 0 = i_{\alpha^R} d\omega'$ $\forall \alpha \in \Gamma(A)$, then it holds

$$\omega = \omega' \iff \sigma_\omega = \sigma_{\omega'} .$$

Theorem II: Let (M, π) be Poisson whose Lie algebroid $(T^*M, [\cdot,\cdot]_\pi, \pi^\#)$ is integrable. Let $\Sigma \rightrightarrows M$ a Lie groupoid with 1-connected s-fibers and with a fixed isomorphism of Lie algebroids

$$\sigma : A \longrightarrow T^*M \qquad (A = \mathrm{Lie}(\Sigma))$$

Then Σ carries a unique closed and multiplicative 2-form $\omega \in \Omega^2(M)$ with $\sigma = \sigma_\omega$. Moreover, (Σ, ω) becomes a symplectic groupoid for which the induced Poisson structure on the base is π.

XI : POISSON COHOMOLOGY

- In every manifold we have the De Rham complex $(\Omega^\bullet(M), d)$ satisfying $d^2=0$, which is a cochain complex,

$$\mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \to \cdots \to \Omega^n(M) \to 0 \cdots$$

and the k-th De Rham cohomology group is

$$H^k_{dR}(M) := \frac{\mathrm{Ker}\,(d: \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{Im}\,(d: \Omega^{k-1}(M) \to \Omega^k(M))}$$

- In a Poisson manifold (M,π), we have a similar situation:

Definition: The Poisson complex is a cochain complex $(\mathfrak{X}^\bullet(M), d_\pi)$, where $d_\pi := [\pi, -]$ and $[\,;\,]$ is the Schouten bracket of k-vector fields.

- Note that d_π is a degree $+1$ morphism, and the graded Jacobi ensures that $d_\pi^2=0$.

Definition: The Poisson cohomology groups are

$$H^k(M,\pi) := \frac{\mathrm{Ker}\,(d_\pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M))}{\mathrm{Im}\,(d_\pi : \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^k(M))}$$

Examples: 1) $H^0(M,\pi)$ = Casimir functions.

2) $H^1(M,\pi) = \dfrac{\text{Poisson v.f.}}{\text{Hamiltonian v.f.}}$

Lemma: Let (M,π) be a Poisson manifold and take $\omega_M \in \Omega^{top}(M)$ a volume form. Consider the map

$$\theta : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$$
$$f \longmapsto \theta(f) := \operatorname{div} X_f ,$$

ie, $\theta(f)$ is the unique function such that $\mathcal{L}_{X_f}\omega_M = \theta(f)\cdot\omega_M$. Then

1) θ is a derivation (thus a vector field)

2) θ is a Poisson vector field (ie, $\mathcal{L}_\theta \pi = 0$)

3) If ω_M' is another volume form, $\omega_M' = \pm e^u \omega_M$, $u \in \mathcal{C}^\infty(M)$ (assuming M connected) then if θ' is the v.f associated to ω_M', then it holds $\theta' = \theta - X_u$.

Corollary: The modular class $\mathrm{mod}_\pi := [\theta] \in H^1(M,\pi)$ does not depend of the chosen volume form. Moreover, $\mathrm{mod}_\pi = 0 \iff M$ admits a volume form which is invariant under Ham. v.f

· Can we say something about $H^2(M,\pi)$?

Definition: A smooth deformation of π is a smooth family $\{\pi_t\}_{t\in[0,1]}$ of Poisson structures on M such that $\pi_0 = \pi$.

· Taking $\frac{d}{dt}$ in $[\pi_t,\pi_t]=0 \Rightarrow d_{\pi_t}(w = \dot{\pi}_t) = 0$.

Definition: An infinitesimal deformation of π is a bivector $w \in \mathfrak{X}^2(M)$, $d_\pi w = 0$.

Definition: A trivial deformation is a smooth deformation $\{\pi_t\}_{t\in[0,1]}$ s.t. there exists a family of Poisson diffeomorphisms $\varphi_t : (M,\pi_t) \xrightarrow{\sim} (M,\pi)$ with $\varphi_0 = \mathrm{Id}$.

· This says that $H^2(M,\pi) = \dfrac{\text{Infinitesimal deformations}}{\text{Trivial infinitesimal deformations}}$

- Remember that a linear map $f: E \to E'$ induces a map $f_\bullet : \Lambda^p E \to \Lambda^p E'$,

$(f_\bullet T^p)(\omega_1^i, \ldots, \omega_p^i) := T^p(f^*\omega_1^i, -, f^*\omega_p^i)$; or more concrete, $e_1 \wedge \cdots \wedge e_p \longmapsto f(e_1) \wedge \cdots \wedge f(e_p)$

Proposition : The map $\pi^\# : \Lambda^\bullet T^*M \to \Lambda^\bullet TM$, $\pi^\#(\alpha_1 \wedge \cdots \wedge \alpha_k) = \pi^\#\alpha_1 \wedge \cdots \wedge \pi^\#\alpha_k$ coming from $\pi^\# : T^*M \to TM$ induces a chain map on sections,

$$\pi^\# : (\Omega^\bullet(M), d) \longrightarrow (\mathcal{X}^\bullet(M), d_\pi),$$

and the latter induces a map on cohomology,

$$[\pi^\#] : H^\bullet_{dR}(M) \longrightarrow H^\bullet(M, \pi).$$

LINEARIZATION AROUND FIXED POINTS

Definition : We say that $p \in M$ is a fixed point of (M, π) if $\pi_p = 0$.

- If $\pi_p = 0$, then $\mathfrak{g}_p(\pi) = \mathrm{Ker}\, \pi_p^\# = T_p^*M$ is a Lie algebra with the bracket $[d_p f, d_p g]_{\mathfrak{g}_p} := d_p\{f, g\}$. Therefore, we can consider $(\underset{\overset{\shortparallel}{T_pM}}{\mathfrak{g}_p^*(\pi)}, \pi_{\mathfrak{g}_p})$ the Linear Poisson structure.

Definition : We say that $\pi_{\mathfrak{g}_p}$ is the linearization of π at p.

- In a coordinated chart $(U; u_1, \ldots, u_n)$ around p, if $\pi = \frac{1}{2}\sum_{i,j=1}^{n} \pi_{ij}\, \partial_{u_i} \wedge \partial_{u_j}$ $(\pi_{ij}(p) = 0)$, then

$$\pi_{\mathfrak{g}_p} = \frac{1}{2} \sum_{i,j,k} \frac{\partial \pi_{ij}}{\partial u_k}(0)\, u_k\, \partial_{u_i} \wedge \partial_{u_j}$$

Definition : A Poisson structure π is said to be linearizable around a fixed point $p \in M$ if there is a Poisson isomorphism

$$\phi : (U \subset M, \pi) \xrightarrow{\sim} (V \subset \mathfrak{g}_p^*, \pi_{\mathfrak{g}_p}) \quad , p \in U$$

such that $\phi(p) = 0$.

XII : LINEARIZATION OF POISSON STRUCTURES

Definition: A Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is called Poisson non-degenerated if for all Poisson manifold (M,π) and $p \in M$ such that $\pi_p = 0$ and $\mathfrak{g}_p \simeq \mathfrak{g}$ we have that π is linearizable around p.

- It seems a condition almost impossible to check, but..

Theorem (Conn): Any semisimple compact Lie algebra is Poisson non-degenerate.

- $\mathfrak{sl}(2)$ is not Poisson non-deg.

Lemma (Moser deformation). Let $\{\pi_t\}_{t\in[0,1]}$ be a smooth deformation of Poisson structures. If $X_t \in \mathfrak{X}(M)$ be a time-dependent vector field s.t. $\dot{\pi}_t = d_{\pi_t}(X_t)$, then its flow ϕ_t satisfies $\phi_t^* \pi_t = \pi_0$.

Theorem: Let $\pi \in \mathfrak{X}^2(U)$, $0 \in U \subset \mathbb{R}^n$ be a Poisson structure with $\pi(0) = 0$. Then π is linealizable at 0 if and only if there exists $X \in \mathfrak{X}(V)$, $0 \in V \subset U$ such that 1) $\pi = d_\pi X$ and 2) $X = \sum x_i \partial_{x_i} + h.o.t.(x_i)$ (h.o.t. = higher order terms)

LOG-SYMPLECTIC STRUCTURES

Definition: We say that a Poisson manifold (M_{2n}, π) is log-symplectic when $\pi^n = \pi \wedge \overset{n}{\cdots} \wedge \pi$ is transverse to the 0-section of $\Lambda^{2n} TM$.

- If M orientable, let $w \in \mathfrak{X}^{2n}(M)$ nowhere vanishing; $\omega_M = w^{-1}$, thus $\pi^n = f \cdot w$. Then π is log symplectic $\Leftrightarrow$ 0 is a regular value of f, and $N := f^{-1}(0) = \{p : \pi_p^n = 0\} = \{p : \operatorname{rank} \pi < 2n\}$ is a codim 1 submanifold called the singular locus of π.

Proposition: N is a Poisson submanifold with rank $\pi_{|N} = 2(n-1)$.

Moreover, for any $p \in N$ $\exists (U; x, y, p_1, q_1, \ldots, p_{n-1}, q_{n-1})$ such that

$$\pi_{|U} = x\, \partial_x \wedge \partial_y + \sum_{i=1}^{n-1} \partial_{p_i} \wedge \partial_{q_i}.$$

with $U \cap N \equiv \{x = 0\}$, and $\omega := \pi_{|U-N} = dy \wedge d(\log|x|) + \sum^{n-1} dq_i \wedge dp_i$, what gives the name

XIII : POISSON LIE GROUPS

Definition : A Poisson Lie group (PLG) is a Lie group G endowed with a Poisson structure π such that

$$m : (G,\pi) \times (G,\pi) \longrightarrow (G,\pi)$$
$$(g\ ,\ g') \longmapsto gg'$$

is a Poisson map ($G\times G$ is endowed with the product Poisson structure $\pi\times\pi$).

Definition : A morphism of PLG is a map $f : (G_1,\pi_1) \longrightarrow (G_2,\pi_2)$ which is a Poisson map and a Lie group homomorphism.

. V is a v.s., and π a Poisson structure, then π linear $\iff$ (V,π) is a PLG (with "+")

Notation : Since G acts on G by multiplying by the right / left, $L_g / R_g : G\to G$, their differentials induce an action of G on TG : $(L_g)_* : TG \to TG$, $(L_g)_* v = g\cdot v$. One can extend this map to a map on $\Lambda^\bullet TG$ (and therefore an action of G on $\Lambda^\bullet TG$,

$$(L_g)_\bullet,(R_g)_\bullet : \Lambda^\bullet TG \longrightarrow \Lambda^\bullet TG$$
$$\pi_h \longmapsto (L_g)_\bullet \pi_h \overset{not}{=} g\bullet\pi_h \quad (\text{or } \pi_h\bullet g)$$

Lemma : m is Poisson $\iff$ $\boxed{\pi_{gh} = \pi_g\cdot h + g\,\pi_h .}$

. $\pi_e = 0$ (For a PLG)

. $\tilde\pi : G \to \Lambda^2 \mathfrak{g}$, $\tilde\pi(g) = \tilde\pi_g := \pi_g\cdot g^{-1} \in \Lambda^2 T_eG$; and the boxed eq. from before can be rewritten as $\tilde\pi_{gh} = \tilde\pi_g + Ad_g(\tilde\pi_h)$ (Ad_g induced by $Ad_g : \mathfrak{g}\to\mathfrak{g}$).

. $(G,\pi) \to (G,\pi)$, $g\mapsto g^{-1}$ is anti-Poisson, ie, $(G,\pi)\to(G,-\pi)$, $g\mapsto g^{-1}$ is Poisson.

<u>Definition</u>. Let $\mathcal{V}$ be a representation of G, ie, $\Psi: G \to \mathrm{Aut}(\mathcal{V})$, $g \mapsto \Psi_g$; $\Psi_g(v) \overset{\text{not}}{=} g\cdot v$.

a) A <u>1-cocycle on G</u> with values on $\mathcal{V}$ is a smooth map $c: G \to \mathcal{V}$ satisfying

$$\boxed{c(g\cdot h) = c(g) + g\cdot c(h)}$$

b) If $\mathfrak{g}$ is the Lie algebra of G, then $\mathcal{V}$ is also a representation of $\mathcal{V}$, $a\cdot v := \frac{d}{d\varepsilon}\big|_{\varepsilon=0} \exp(\varepsilon a)\cdot v$.

A <u>1-cocycle on $\mathfrak{g}$</u> with values on $\mathcal{V}$ is a linear map $\gamma: \mathfrak{g} \to \mathcal{V}$ satisfying

$$\boxed{\gamma([a,b]) = a\gamma(b) - b\gamma(a)}$$

<u>Ex</u>: For PLG, we saw that $\tilde{\pi}: G \to \Lambda^2\mathfrak{g}$ is a 1-cocycle on the Lie group with values on $\Lambda^2\mathfrak{g}$.

<u>Proposition</u>: 1) If $c: G \to \mathcal{V}$ is a 1-cocycle on G $\Rightarrow$ $c_{*,e}: \mathfrak{g} \to \mathcal{V}$ is a 1-cocycle on $\mathfrak{g}$.

2) If G is a 1-connected Lie group, then there's a bijection

$$\{1\text{-cocycles on } G\} \Longleftrightarrow \{1\text{-cocycles on } \mathfrak{g}\}$$

$$c \longmapsto c_{*,e}.$$

• Let (G, π) a PLG. Then $\pi_e = 0 \Rightarrow \mathrm{Ker}\, \pi_e^\# = T_e^*G = \mathfrak{g}^*$ is a Lie algebra (the isotropy Lie algebra). Ie, $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie algebras. $\mathfrak{g}^*$ has a Lie bracket $[\cdot,\cdot]_{\mathfrak{g}^*}: \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ skew-sym, or equivalently, a bilinear map $[\cdot,\cdot]_{\mathfrak{g}^*}: \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$. Consider its dual linear map

$$\delta := ([\cdot,\cdot]_{\mathfrak{g}^*})^*: \mathfrak{g} \to \Lambda^2\mathfrak{g}$$

It turns out that $\delta = \tilde{\pi}_{*,e}$, thus δ is a $\mathfrak{g}$-1-cocycle. This has a name:

Definition: A Lie bialgebra (LBA) is a pair $(\mathfrak{g}, \delta)$ where

i) $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra

ii) $\delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ is a linear map such that $(\mathfrak{g}^*, \delta^*)$ is a Lie algebra, and δ is a $\mathfrak{g}$-1-cocycle (i.e., both Lie algebra structures are compatible),

$$\delta([a,b]) = ad_a(\delta(b)) - ad_b(\delta(a))$$

- Good to know / remember: $ad_a(u \wedge v) = [a,u] \wedge v + u \wedge [a,v]$

Corollary: (G, π) PLG $\Rightarrow$ $(\mathfrak{g}, \delta := \tilde{\pi}_{*,e})$ LBA.

Lemma: $(\mathfrak{g}, [\cdot,\cdot], \delta)$ LBA $\Rightarrow$ $(\mathfrak{g}^*, \delta^*, [\cdot,\cdot]^*)$ LBA.

Theorem (Lie): The Lie functor

$$Lie : \text{1-connected LG} \longrightarrow \text{LA}$$

is an equivalence of categories, i.e.

1) $Hom_{LG}(G, H) \xrightarrow{\sim} Hom_{LA}(\mathfrak{g}, \mathfrak{h})$ is a bijection

2) For all $\mathfrak{g} \in \text{LA}$ there exists G a 1-connected Lie group such that $Lie(G) \simeq \mathfrak{g}$.

Theorem (Drinfeld): The Lie functor

$$Lie : \text{1-connected PLG} \longrightarrow \text{LBA}$$

is an equivalence of categories

Definition: Let (G, π) be a 1-connected PLG with LBA $(\mathfrak{g}, \delta)$. The dual PLG (G^*, π^*) is the 1-connected PLG with LBA $(\mathfrak{g}^*, \delta^*)$.

Ex: $(G, 0)$ has as dual $(G^* = \mathfrak{g}^*, \pi^* = \pi_{linear})$.

• For an element $r \in \Lambda^2\mathfrak{g}$, it induces a map $r_{*,\mathfrak{g}} : \mathfrak{g} \to \Lambda^2\mathfrak{g}$, $r_{*,\mathfrak{g}}(a) := ad_a(r)$.

Definition: A coboundary LBA is a LBA $(\mathfrak{g}, \delta)$ s.t. $\delta = r_{*,\mathfrak{g}}$ for some $r \in \Lambda^2\mathfrak{g}$.

• In particular, δ is a 1-cocycle on $\mathfrak{g}$, $\delta([a,b]) = ad_{[a,b]} r = (ad_a \circ ad_b - ad_b \circ ad_a) r = ad_a \delta(b) - ad_b \delta(a)$

Definition. A coboundary PLG is a PLG (G, π) such that $\pi = r^L - r^R$ for some $r \in \Lambda^2\mathfrak{g}$ (and r^L, r^R denotes the induced bivector fields with the left & right translations), i.e.,

$\pi_g = g \cdot r - r \cdot g \iff \tilde{\pi}_g = Ad_g(r) - r$.

Proposition: Let (G, π) a 1-connected PLG with LBA $(\mathfrak{g}, \delta)$. Then

$$\underset{(\pi = r^L - r^R)}{(G, \pi) \text{ is a coboundary PLG}} \iff \underset{(\delta = r_{*,\mathfrak{g}})}{(\mathfrak{g}, \delta) \text{ is a coboundary LBA}}$$

Proposition: Let $r \in \Lambda^2\mathfrak{g}$.

1) $(G, \pi := r^L - r^R)$ is a PLG $\iff [r, r] \in (\Lambda^3\mathfrak{g})^G$

2) $(\mathfrak{g}, \delta := r_{*,\mathfrak{g}})$ is a LBA $\iff [r, r] \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$