I : LIE GROUP STRUCTURE

__Definition__: A __Lie group__ G is a smooth manifold endowed with a group structure, such that the maps

$$G \times G \longrightarrow G \qquad ; \qquad G \longrightarrow G$$
$$(x, y) \longmapsto xy \qquad\qquad\qquad x \longmapsto x^{-1}$$

are $\mathcal{C}^\infty$.

__Examples__: 1) $(\mathbb{R}^n, +)$, 2) $(\mathbb{C}^n = \mathbb{R}^{2n}, +)$, 3) $(\mathbb{R}^* = \mathbb{R} - 0, \cdot)$

Actually, the condition $G \to G$, $x \mapsto x^{-1}$ is redundant; it follows from the 1^{st} one and the implicit function thm.

__Lemma__: Let G be a Lie group, and $H \leq G$ a submanifold which is also a subgroup. Then H is a Lie group in a natural way.

__Lemma__: G, G' Lie groups $\Rightarrow$ $G \times G'$ is a Lie group with the product $(g, g') \cdot (h, h') := (gh, g'h')$

__Definition__. A __Lie group homomorphism__ is a map $\varphi : G \to H$ between Lie groups which is both group homomorphism and smooth. We say it is an __isomorphism__ of Lie group if φ is diffeomorphism and group isomorphism.

__Lemma__: $GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$ is a Lie group (wrt the product).

__Definition__: The __left__ and __right multiplication__ are the maps

$$L_g : G \longrightarrow G \qquad , \qquad R_g : G \longrightarrow G$$
$$x \longmapsto gx \qquad\qquad\qquad x \longmapsto xg$$

which are diffeomorphisms.

Lemma: Let G be a Lie group and $H \leq G$ a subgroup. Then

$$H \text{ is a submanifold (thus Lie group)} \iff H \text{ is a submanifold at } e.$$

Here we are applying the following characterization of smooth submanifolds: if M is a smooth mnfd and $N \subset M$, then N is submanifold $\iff$ every point $x \in N$ has a chart (U, φ), $U \xrightarrow[\sim]{\varphi} \tilde{U} \subset \mathbb{R}^m$ such that $\varphi(U \cap N) = \tilde{U} \cap \mathbb{R}^d$ for some $d \leq m$. Basically this is the fact that smooth mnfd are where some coordinates vanish.

Lemma: $\det_{*, I} = \mathrm{tr} : T_I GL(n, \mathbb{R}) = M_n(\mathbb{R}) \to \mathbb{R}$.

Corollary. $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A = 1\}$ is a Lie subgroup.

- General machinery to give examples of Lie subgroups?

Theorem (closed subgroup). Let G be a Lie group and $H \leq G$ a subgroup of G.

$$H \text{ closed} \iff H \text{ is a submanifold (thus Lie group).}$$

Corollary. $O(n) = \{A \in GL(n, \mathbb{R}) : AA^t = I\}$ is a Lie subgroup.

In general, $O(p,q) = \{A \in GL(n, \mathbb{R}) : A \text{ isometry of a } (p,q) \text{ signature metric}\}$ is too.

- We can do a free basis version of $GL(n, \mathbb{R})$ and its Lie subgroups: consider E a vector space and let $GL(E) = \mathrm{Aut}(E)$ (E $\mathbb{R}$-vs or $\mathbb{C}$-vs). $\mathrm{End}(E) \simeq \mathbb{R}^{n^2}$ and $GL(E)$ is open, so Lie group. Given an euclidean metric g on E, we have

$$O(E) = \{\text{isometries } \sigma : E \to E\}, \quad SO(E) = \{\text{rotations} : \text{isometries } \sigma : E \to E \text{ with } \det = 1\}$$

are also Lie groups.

LIE ALGEBRA OF A LIE GROUP

- Denote by $\mathcal{D}(M) = \{$ vector fields on $M\}$. Call $l_g = (L_g)_*$ and $r_g = (R_g)_*$.

<u>Definition</u>: A vector field $D \in \mathcal{D}(M)$ is called <u>left invariant</u> if

$$D_{gx} = l_g(D_x).$$

Denote $\mathcal{D}^L(M) = \{$ left-inv. v.f $\}$.

Given $X \in T_eG$, we can define a left-inv. vf $(D_X)_g = l_g X$.

<u>Proposition</u>: The map $\varepsilon : \mathcal{D}^L(M) \xrightarrow{\sim} T_eG$ is a v.s. isomorphism

$$D \longmapsto D_e$$
$$D_X \longleftarrow X$$

<u>Definition</u>: Let M be a mfd and $D \in \mathcal{D}(M)$. An <u>integral curve</u> of D is a curve $\sigma : I \to M$ st $\sigma'(t) = D_{\sigma(t)}$ $\forall t \in I$, where $\sigma'(t_0) = \sigma_*\left((\partial_t)_{t_0}\right)$. We say that σ is the <u>maximal integral curve</u> when it has a maximal interval of definition.

In what follows we will denote as α_X the maximal integral curve of D_X with initial point e.

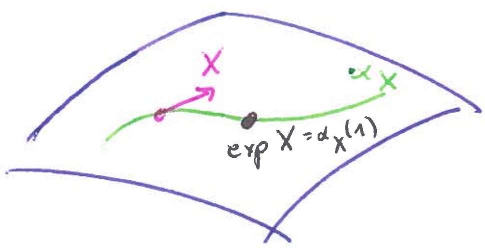
<u>Proposition</u>: 1) Any left-invariant vf is complete, ie, $\alpha_X : \mathbb{R} \to M$

2) $\alpha_X(t+s) = \alpha_X(t) \cdot \alpha_X(s)$, $\forall t, s \in \mathbb{R}$.

3) The map $\mathbb{R} \times T_eG \to G$, $(t, X) \longmapsto \alpha_X(t)$ is smooth.

<u>Definition</u>: Let G be a Lie group. The <u>exponential map</u> is defined as

$$\exp : T_eG \to G$$
$$X \longmapsto \alpha_X(1).$$



Example: 1) $\exp : M_n(\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$, $A \longmapsto e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

2) $\exp : \mathrm{End}(E) \longrightarrow GL(E)$, $T \longmapsto e^T = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$.

Proposition: 1) $\exp : T_eG \longrightarrow G$ is smooth

2) $\exp(tX) = \alpha_X(t)$

3) $\exp_{*,0} = \mathrm{Id} : T_eG \simeq T_0(T_eG) \longrightarrow T_eG$, ie, exp is a local diffeomorphism at 0.

4) $\exp((t+s)X) = \exp tX \cdot \exp sX$.

5) $\exp -X = (\exp X)^{-1}$.

Definition: A one-parameter subgroup of G is a Lie group homomorphism

$$\alpha : (\mathbb{R}, +) \longrightarrow G$$

For instance, α_X is a one-par. subg. More? No!

Lemma: $\mathrm{Hom}_{\text{Lie gp}}(\mathbb{R}, G) \simeq T_eG$, ie, there is a one-to-one correspondence

$$\begin{array}{ccc} T_eG & = & \{\text{one-parameter subgroups of } G\} \\ X & \longmapsto & \alpha_X \\ \alpha'(0) & \longleftarrow & \alpha \end{array}$$

Proposition: Let $\varphi : G \longrightarrow H$ be a Lie group hom. Then the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ T_eG \arrow[u, "\exp^G"] \arrow[r, "\varphi_*"] & T_eH \arrow[u, "\exp^H"'] \end{tikzcd}$$

• For a Lie group G, set $C_x : G \to G$, $y \mapsto xyx^{-1}$ the conjugation map, which is a Lie group hom (smooth + group hom)

Definition: The adjoint action of x is $Ad_x := (C_x)_{*,e} : T_eG \to T_eG$; and the adjoint representation of G on T_eG is

$$\begin{aligned} Ad : G &\longrightarrow GL(T_eG) \\ x &\longmapsto Ad(x) \end{aligned}.$$

Lemma. This map Ad is indeed a Lie group homomorphism.

Remark: "Ad_x linearizes C_x": since exp is local diff at 0, we have the diagram

$$\begin{tikzcd} G & \supset U_0 \arrow[r, "C_x"] & U \\ T_eG \arrow[u, "\exp"] & \supset V_0 \arrow[u, "\wr"'] \arrow[r, "Ad(x)"] & V \arrow[u, "\wr"'] \end{tikzcd}$$

the vertical arrows provide charts so that C_x becomes Ad_x in that particular charts. Since Ad_x is linear, we can find loc. coord. s.t. C_x becomes a linear map (which is Ad_x).

Definition: The adjoint action of T_eG on itself is

$$(Ad)_{*,e} = ad : T_eG \longrightarrow End(T_eG)$$

since $T_{Id}(GL(T_eG)) = End(T_eG)$).

In particular we have a bilinear map:

Definition. The Lie bracket of $X, Y \in T_eG$ is

$$\begin{aligned} [\cdot,\cdot] : T_eG \times T_eG &\longrightarrow T_eG \\ (X, Y) &\longmapsto [X,Y] := ad_X Y \left(= ad(X)Y\right). \end{aligned}$$

Proposition: $[X,Y] = \frac{d}{dt}\Big|_{t=0} Ad_{\exp tX}\, Y$.

Example: $[A,B] = AB - BA$ the commutator of matrices for $GL(n,\mathbb{R})$.

Remark: The comm. square of Lie group hom relates Ad and ad as

$$\begin{array}{ccc} G & \xrightarrow{Ad} & GL(T_eG) \\ {\scriptstyle \exp}\uparrow & & \uparrow e^{\cdot} \\ T_eG & \xrightarrow{ad} & End(T_eG) \end{array} \qquad \boxed{e^{ad X} = Ad(\exp X)}$$

Lemma (Properties of $[\cdot,\cdot]$):

1) $[\cdot,\cdot]$ is bilinear
2) $[\cdot,\cdot]$ is skew-symmetric
3) If $\varphi : G \to H$ is a Lie group hom, then $\varphi_* : T_eG \to T_eH$ "preserves brackets", ie,

$$\varphi_*[X,Y]_G = [\varphi_* X, \varphi_* Y]_H \qquad \forall X,Y \in T_eG.$$

4) (Jacobi Identity):

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0.$$

With more generality,

Definition. A Lie algebra $\mathfrak{g}$ is a vector space endowed with a map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that

1) $[\cdot,\cdot]$ bilinear
2) $[\cdot,\cdot]$ skew-sym
3) Jacobi

Corollary: If G is a Lie group, $\mathfrak{g} := T_eG$ is a Lie algebra endowed with $[X,Y] := \mathrm{ad}_X Y$.

Definition: A Lie algebra homomorphism is a linear map $T : \mathfrak{g}_1 \to \mathfrak{g}_2$ st $T[e,v] = [Te, Tv]$.

Remark: The chain rule says that

$$\begin{array}{ccc} \mathrm{LieGps} & \longrightarrow & \mathrm{LieAlg} \\ G & \longmapsto & \mathfrak{g} \end{array}$$

is a functor (the Lie functor)

Lemma. If $X, Y \in \mathfrak{g} = T_eG$ commute, ie, $[X,Y] = 0 \Rightarrow \exp X$ and $\exp Y$ commute, $\exp X \cdot \exp Y = \exp Y \cdot \exp X$; and in particular

$$\exp X \cdot \exp Y = \exp(X+Y).$$

- If $(\mathfrak{g}, [\cdot,\cdot])$ is commutative, $\exp X$ and $\exp Y$ commute. Does it mean that G is abelian? NO !!!

Definition. Denote $G_e := \langle \exp X : X \in \mathfrak{g} \rangle$ the smallest subgroup containing $\exp X$'s. Explicitly,

$$G_e = \{ \exp X_1 \cdots \exp X_k \ : X_j \in \mathfrak{g} \}.$$ the component of the Id of G

Lemma: G_e is open in G

Lemma: In a topological group, if a subgroup is open, it is also closed.

Corollary. G_e is open and closed.

Lemma. G_e is the path-component (= connected component) of G containing e.

Corollary: G is connected $\Longleftrightarrow G = G_e$.

Lemma. $\mathfrak{g}$ abelian $\Leftrightarrow G_e$ abelian, and if G is connected, $\mathfrak{g}$ abelian $\Leftrightarrow G$ abelian.

Theorem (Classification of connected, abelian Lie groups): Every connected, abelian Lie group is diffeomorphic to $\mathbb{T}^p \times \mathbb{R}^q$ for some $p, q \in \mathbb{N}_0$.

Corollary. Every abelian, compact, connected Lie group is diffeomorphic to $\mathbb{T}^p$.

Corollary. Every abelian Lie group is diffeomorphic to $\mathbb{T}^p \times \mathbb{R}^q \times Z$, with Z discrete.

LIE SUBGROUPS

Definition. Let G be a Lie group. A *Lie subgroup* of G is a subgroup $H < G$ which is a Lie group in itself and such that the inclusion $i: H \longrightarrow G$ is a Lie group homomorphism.

This is a more general notion of the previous statements, in which H had to be closed.

Lemma. Let $X \in \mathfrak{g}$. Then $\alpha_X(\mathbb{R})$ is a Lie subgroup.

Proposition. If $\varphi: H \longrightarrow G$ is a injective Lie group hom, $\Rightarrow \varphi_*: \mathfrak{h} \longrightarrow \mathfrak{g}$ is injective too, and by homogeneity φ is an immersion.

Corollary. If $H < G$ is a Lie subgroup, then $i_*: \mathfrak{h} \longrightarrow \mathfrak{g}$ is injective, and the Lie algebra of H $\mathfrak{h}$ can be identified with the Lie subalgebra $i_*(\mathfrak{h})$ of $\mathfrak{g}$, $\mathfrak{h} \simeq i_*(\mathfrak{h}) \subseteq \mathfrak{g}$.

Theorem. Under the previous identification, the Lie algebra of a Lie subgroup $H < G$ is given by

$$\mathfrak{h} = \{ X \in \mathfrak{g} : \exp tX \in H \quad \forall t \in \mathbb{R} \}.$$

Examples:

1) $GL(n, \mathbb{R}) = \{ A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \det A \neq 0 \} \rightsquigarrow \mathfrak{gl}(n, \mathbb{R}) = \mathcal{M}_{n\times n}(\mathbb{R})$

2) $SL(n, \mathbb{R}) = \{ A \in GL(n, \mathbb{R}) : \det A = 1 \} \longrightarrow \mathfrak{sl}(n, \mathbb{R}) = \{ X \in \mathcal{M}_{n\times n}(\mathbb{R}) : \operatorname{tr} X = 0 \}$

3) $O(n, \mathbb{R}) = \{ \text{———} : AA^t = I \} \rightsquigarrow \mathfrak{o}(n, \mathbb{R}) = \{ X \in \mathcal{M}_{n\times n}(\mathbb{R}) : X^t = -X \}$

4) $SO(n, \mathbb{R}) = \{ \text{———} : AA^t = I \text{ and } \det A = 1 \} \rightsquigarrow \mathfrak{so}(n, \mathbb{R}) = \mathfrak{o}(n)$

5) $U(n) = \{ A \in GL(n, \mathbb{C}) : AA^* = I \} \longrightarrow \mathfrak{u}(n) = \{ X \in \mathcal{M}_{n\times n}(\mathbb{C}) : X^* = -X \}$ $\quad [(A^*)_{ij} = \overline{A_{ji}}$

6) $SU(n) = \{ \text{———} : AA^* = I \text{ and } \det A = 1 \} \rightsquigarrow \mathfrak{su}(n) = \{ X \in \mathcal{M}_{n\times n}(\mathbb{C}) : X^* = -X \text{ and } \operatorname{tr} X = 0 \}$

Theorem: Let G be a Lie group, and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra of $\mathfrak{g}$. Then

$$H := \langle \exp tX \rangle_{X \in \mathfrak{h}}$$

has a unique structure of Lie subgroup with Lie algebra $\mathfrak{h}$. In particular, there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{connected Lie subgroups} \\ \text{of } G \end{array} \right\} \;=\; \left\{ \begin{array}{c} \text{Lie subalgebras} \\ \text{of } \mathfrak{g} \end{array} \right\}$$

$$H \longmapsto \mathfrak{h}$$

$$\langle \exp \mathfrak{h} \rangle \longleftarrow\!\shortmid\; \mathfrak{h}$$

- Now we have enough tools to perform the proof of the "closed subgroup" thm.

Corollary: Let G, H be Lie groups, and let $\varphi : G \to H$ be a group homomorphism. If φ is continuous then φ is also $\mathcal{C}^\infty$.

Proposition: Consider the $\mathbb{R}$-basis $\left\{ \tau_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\}$ of $\mathfrak{su}(2)$.

1) The composite $\mathfrak{su}(2) \xrightarrow{\mathrm{ad}} \mathrm{End}(\mathfrak{su}(2)) \xrightarrow{\mathrm{mat}} M_3(\mathbb{R})$ gives a Lie algebra isomorphism

$$\mathfrak{su}(2) \xrightarrow{\sim} \mathfrak{so}(3).$$

2) The composite $SU(2) \xrightarrow{\mathrm{Ad}} GL(\mathfrak{su}(2)) \xrightarrow{\mathrm{mat}} M_3(\mathbb{R})$ takes values in $SO(3)$, $\varphi : SU(2) \to SO(3)$, and φ is a 2-sheeted covering map. In particular $SU(2)$ is the universal covering of $SO(3)$ and

$$\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}.$$

II : LIE GROUP ACTIONS

<u>Definition</u>: Let M be a set and H be a group. A (right) <u>action</u> of H on M is a map

$$\begin{aligned} M \times H &\longrightarrow M \\ (m, h) &\longmapsto m * h \overset{\text{not}}{=} mh \end{aligned}$$

such that

i) $m * e = m \quad \forall m \in M$

ii) $(m * h_1) * h_2 = m * (h_1 h_2)$.

Analogously, one can define a left action by $h * m$, and both are the same since for a right action $*$, setting $h \bullet m := m * h^{-1}$ is a left action.

If H acts on M, we will write $M \circlearrowleft H$.

<u>Definition</u>: Let $M \circlearrowleft H$. There is an equivalence relation on M

$$m_1 \equiv m_2 \iff \exists h : m_2 = m_1 * h,$$

and the quotient space is denoted by M/H.

<u>Definition</u>: When M is a top space and H a top group, we say that the action is continuous when the map $M \times H \to M$ is too, and when M is a manifold and H a Lie group, we say that the action is smooth when $M \times H \to M$ is smooth.

<u>Lemma</u>: Let M be a top space and H a top gp.

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, dashed, "\pi^* f"] & \\ M/H \arrow[r, "f"] & N \end{tikzcd}$$

1) $\pi : M \to M/H$ is open

2) $\pi^* : \mathcal{C}(M/H, N) \xrightarrow{\sim} \mathcal{C}(M,N)^H$, $f \longmapsto f \circ \pi$, is bijective,

where $\mathcal{C}(M,N)^H \subset \mathcal{C}(M,N)$ is the subset of <u>H-invariant</u> maps, i.e., $g : M \to N$ s.t. $g(m*h) = g(m)\ \forall h$

Q: Set M a manifold and H a Lie group, $M \circlearrowleft H$. Is it possible to endow M/H with a smooth structure? In general, the answer is no! But in some cases we can!

· Denote as M^H the set of H-invariant elements, ie, elements s.t. $m * h = m$ $\forall h \in H$.

Ex: If $M = V \times H$, $V \subset \mathbb{R}^n$ open, there is a natural action $(x,h) * h' := (x, hh')$; and $M/H \simeq V$, so the quotient inherits a structure of manifold.

Definition: A right action $M \circlearrowleft H$ is of principal fiber bundle type (PFB type) if

i) $K \subset M$ compact $\Rightarrow$ KH closed

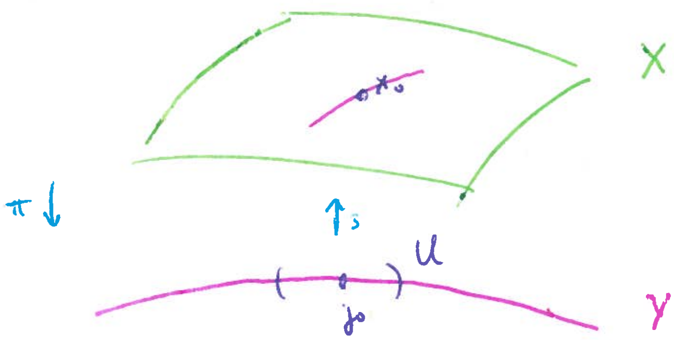
ii) Every point $m \in M$ has a H-invariant open nbhd U (ie, $Uh \subset U$ $\forall h \in H$) st there is a diffeomorphism

$$\varphi : U \xrightarrow{\sim} V \times H \quad , \quad V \subset \mathbb{R}^n \text{ open}$$

intertwining the H-actions, ie, st φ is an H-set morphism, ie, $\varphi(m' * h) = \varphi(m') * h$ $\forall m' \in U$, $h \in H$.

Lemma: i) $\Longleftrightarrow$ M/H is Hausdorff.

Lemma: Let $\pi : X \to Y$ be a surjective submersion of manifolds. Then π allows local sections, ie for every point $x_0 \in X$, $\pi(x_0) = y_0$, $\exists$ U open nbhd of y_0 and a map $s : U \to X$ smooth st $s(y_0) = x_0$ and $\pi \circ s = \mathrm{Id}_U$.



Corollary (Smoothness principle): Let $\pi : X \to Y$ be a surjective submersion and $f : Y \to Z$ a map to a manifold Z. Then

$$\boxed{\pi^* f \text{ is } \mathcal{C}^\infty \iff f \text{ is } \mathcal{C}^\infty}$$

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\pi^* f"] & \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

Corollary: Let X be a manifold; let Y be a set, and let $\pi : X \to Y$ be a surjective map. If there is a structure of $\mathcal{C}^\infty$ manifold on Y s.t. π is submersion, then it is unique.

The previous corollary guarantees the uniqueness in the following

Theorem: Suppose $M \circlearrowleft H$ is a smooth action of PFB type. Then M/H has a unique structure of smooth manifold such that $\pi : M \to M/H$ is a submersion.

With more generality,

Definition: A principal fiber bundle (PFB) with base B and group structure H, or a principal H-bundle over B, is a manifold P endowed with a H-action together with a map $\pi : P \to B$ such that

i) π is H-invariant, $\pi(p * h) = \pi(p)$.

ii) Every point $b \in B$ has a nbhd V and a diffeomorphism $\phi : \pi^{-1}(V) \to V \times H$ which is a morphism of H-sets ($V \times H$ endowed with the trivial action) making the following diagram commute,

$$\begin{tikzcd} \pi^{-1}(V) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & V \times H \arrow[dl, "pr_1"] \\ & B & \end{tikzcd}$$

Example: If $M \circlearrowleft H$ is of PFB type, then $\pi : M \to M/H$ is a PFB.

• We need easy to handle conditions which guarantee that an action is of PFB type and so that M/H is a manifold.

Definition: Let $M \circlearrowleft H$. We say that the action is

a) **free** if $I_x = \{h \in H : xh = x\} = \{e\}$ $\forall x \in M$, ie, if $xh = x \Rightarrow h = e$.

b) **proper** if the map $M \times H \longrightarrow M \times M$, $(m,h) \longmapsto (m, mh)$ is proper.

Definition. An **infinitesimal action** of $\mathfrak{h}$ on M is a Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{X}(M)$; and if $M \circlearrowleft H$ then there is a canonical infinitesimal action given by assigning to every $X \in \mathfrak{h}$ the vector field $D_X \in \mathfrak{X}(M)$ given

$$(D_X)_m := \frac{d}{dt}\Big|_{t=0} m * \exp tX$$

Lemma: $I_m = \{h \in H : m * h = m\}$ is a closed subgroup of H, thus Lie group on its own, with Lie algebra

$$\mathfrak{h}_m := \mathrm{Lie}(I_m) = \{X \in \mathfrak{h} : (D_X)_m = 0\}$$

**Theorem*: Let $M \circlearrowleft H$ be a smooth action. Then

The action is of PFB type $\Longleftrightarrow$ the action is proper and free

Proposition: Let G be a Lie group and $H < G$ a closed subgroup, and consider the action of H on G by right multiplication. Then the action is free and proper, so there exists a unique smooth structure on G/H s.t. $G \longrightarrow G/H$ is a submersion. Moreover, if H is normal (so G/H is a group on itself) then G/H both as a quotient group and as the quotient by an action are compatible: the canonical map $\pi : G \to G/H$ is a Lie group homomorphism.

Proposition. Let H be a normal, closed subgroup of a Lie group G. Then the Lie algebra of the Lie group G/H is isomorphic (via π_*) to $\mathfrak{g}/\mathfrak{h}$.

Definition: An action $M \circlearrowleft H$ is said to be transitive if $m * H = M$ $\forall m \in M$, i.e., if the orbit of any point is the entire space.

Proposition: Set $\alpha_m : H \to M$, $h \mapsto mh$. The induced map $\overline{\alpha_m} : G/I_m \to M$ is an immersion (one to one to $m*H$), and if the action is transitive, then it is a diffeomorphism.

$$\overline{\alpha_m} : G/I_m \xrightarrow{\sim} M .$$

Example: $SO(n+1) \circlearrowleft S^n$ smooth & transitively, and the isotropy group at e_1 is isomorphic to $SO(n)$. Moreover:

$$S^n \simeq SO(n+1)/SO(n) .$$

INTEGRATION OVER A LIE GROUP

We want to make sense of $\int_G f(x)\,dx$ for a Lie group, at least for $f \in \mathcal{C}_c^\infty(G)$ compactly supported.

Theorem: There exists an integral $I : \mathcal{C}_c^\infty(G) \to \mathbb{C}$, $f \longmapsto \int_G f\,dx$, unique up to multiplication by a positive scalar, such that

1) I is $\mathbb{C}$-linear
2) $f \geq 0 \Rightarrow I(f) \geq 0$, and $f = 0 \Leftrightarrow I(f) = 0$
3) (left invariant) $I(L_g^* f) = I(f)$, i.e., $\int f(gx)\,dx = \int f(x)\,dx$ $\forall g \in G$.

Lemma: Every Lie group is orientable.

Definition: A Lie group G is unimodular if $|\det Ad(x)| = 1$ $\forall x \in G$.

Lemma: G unimodular $\Rightarrow$ I (left and) right invariant

Lemma: G compact $\Rightarrow$ G unimodular.

III : REPRESENTATION THEORY

• In the following G will be a Lie group and V a locally convex vector space over $\mathbb{C}$.

<u>Definition</u>: A <u>continuous representation</u> of G on V is a continuous left action

$$G \times V \to V$$

such that $\pi(x) : V \to V$ is a linear isomorphism for all $x \in G$. This is equivalent to give a Lie group homomorphism

$$\pi : G \to GL(V).$$

The representation is <u>finite dimensional</u> if $\dim V < \infty$.

<u>Example</u>: Let X be a set endowed with a left action $G \curvearrowright X$. It induces a representation of G on $\mathcal{F}(X) = \{$ maps $X \to \mathbb{C}\}$:

the <u>left regular representation</u> on G

$$L : G \to GL(\mathcal{F}(X))$$
$$g \longmapsto (L(g)\varphi)(x) := \varphi(g^{-1}x).$$

(similarly $(R(g)\varphi)(x) = \varphi(xg)$, for a right action).

<u>* The example</u>: $SU(2) = \{ A \in M_2(\mathbb{C}) : AA^* = I, \det A = 1 \} = \left\{ g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in M_2(\mathbb{C}) : |\alpha|^2 + |\beta|^2 = 1 \right\}$

For a map $\varphi : \mathbb{C}^2 \to \mathbb{C}$, the previous representation gives

$$(\pi(g)\varphi)(z_1, z_2) = \varphi\left(\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) = \varphi(\bar{\alpha} z_1 - \bar{\beta} z_2, -\beta z_1 + \alpha z_2)$$

What says that $\pi(g)$ preserves P_n = homogeneous polynomials of degree n, i.e., $\pi(g) P_n \subset P_n$.

Therefore, the restriction $\pi_n := \pi|_{P_n}$ defines a representation of $SU(2)$ on P_n,

$$\pi_n : SU(2) \times P_n \to P_n.$$

Definition: An *unitary representation* $(\pi, \mathcal{H})$ is a continuous representation on a Hilbert space $\mathcal{H}$ such that $\pi(x)$ is *unitary* $\forall x \in G$, ie, st $\pi(x) \in \mathcal{U}(\mathcal{H})$.

Theorem: Let (π, V) a finite-dimensional cont. representation of G. If G is *compact*, then π is *unitarizable*, ie, there exists an inner product on V st π is unitary.

Definition: Let (π, V) be a cont. representation of G.

a) A linear subspace $W \subseteq V$ is *invariant* if $\pi(g) W \subseteq W$ $\forall g \in G$

b) The representation is *irreducible* if the only closed invariant subspaces are the trivial ones: 0 and W.

Definition: Let (π_1, V_1), (π_2, V_2) be cont representations of G. An *intertwining map* or *intertwiner* from π_1 to π_2 is a cont. linear map $T : V_1 \to V_2$ such that

$$\begin{tikzcd} V_1 \arrow[r, "\pi_1(g)"] \arrow[d, "T"'] & V_1 \arrow[d, "T"] \\ V_2 \arrow[r, "\pi_2(g)"] & V_2 \end{tikzcd}$$

commutes for all $g \in G$.

We will denote as $\mathrm{Hom}_G(V_1, V_2)$ the set of intertwiners, and as $\mathrm{End}_G(V)$ when $V = V_1 = V_2$.

Definition: Two representations π_1, π_2 of G on V are said to be *equivalent* if there is an intertwiner T between them which is a topological linear isomorphism, ie, if $\exists T^{-1} \in \mathrm{Hom}_G(V_2, V_1)$.

We say that T is an *equivalence*, and we will write $\pi_1 \sim \pi_2$.

Lemma: Let $A, B \in End(V)$, with V a fin. dim $\mathbb{C}$-vs. If $[A,B]=0$, then

1) A preserves $Ker\, B$, ie, $A(Ker\, B) \subseteq Ker\, B$.

2) A preserves all eigenspaces of B.

Lemma (Schur): Let (π, V) be a cont. fin. dim. representation of G.

1) π irreducible $\Rightarrow$ $End_G(V) = \langle Id \rangle = \mathbb{C} \cdot Id$.

2) If π is unitarizable (eg if G is compact), then "$\Leftarrow$" also holds.

Corollary: The representations π_n of $SU(2)$ on P_n are irreducible.

Proposition: Every unitarizable f.d. cont. representation decomposes as direct sum of irreducible representations.

Concretely, if (π, V) is a unitarizable f.d. cont rep. of G, then there exist invariant subspaces $V_1, \dots, V_N$ of V st

1) $V = V_1 \oplus \cdots \oplus V_N$

2) $\pi_j := \pi|_{V_j}$ (well-def bc V_j is invariant) is a irreducible representation of G on V_j, and

$$\pi = \pi_1 \oplus \cdots \oplus \pi_N .$$

CHARACTER THEORY

Let (π, V) be a f.d cts rep. Given a onb $\{e_1 \dots e_n\}$ of V, we can consider the matrix of $\pi(x): V \to V$

which is $\pi(x) \equiv (\langle \pi(x) e_i, e_j \rangle)_{ij}$. In general

Definition. Let $v, w \in W$. A *matrix coefficient* of (π, V) is a function

$$m_{v,w}: G \longrightarrow \mathbb{C}$$
$$x \longmapsto m_{v,w}(x) := \langle \pi(x) v, w \rangle$$

Warning: From now on we will assume that G is compact.

Definition: Let (π, V) be a f.d.c.r. of G. The space of continuous functions of type π is,

$$\mathscr{C}(G)_\pi := \operatorname{span}\{ m_{v,w} : v, w \in V \} \subseteq \mathscr{C}(G).$$

and it doesn't depend on the choice of the inner product.

Lemma: $\pi \sim \pi' \Rightarrow \mathscr{C}(G)_\pi = \mathscr{C}(G)_{\pi'}$, and we'll denote it as $\mathscr{C}(G)_{[\pi]}$.

- We will write $\hat{G}$ for the collection of equivalence classes of irreducible f.d.c.r. of G. We will write $\pi \in \hat{G}$ meaning $[\pi] \in \hat{G}$.
- Write $\dim \pi := \dim V$.
- $\dim \mathscr{C}(G)_\pi \leq (\dim \pi)^2$.

Theorem (Schur orthogonality relations). Let G be a compact Lie gp and let $\pi, \pi' \in \hat{G}$ equipped with the unitarizing inner products.

1) $\pi \not\sim \pi' \Rightarrow \mathscr{C}(G)_\pi \perp \mathscr{C}(G)_{\pi'}$ in $L^2(G)$

2) If $v, v', w, w' \in (\pi, V)$ then $\langle m_{v,w}, m_{v',w'} \rangle = \dfrac{1}{\dim \pi} \cdot \langle v, v' \rangle \cdot \overline{\langle w, w' \rangle}$.

Lemma: 1) $\pi \not\sim \pi' \Leftrightarrow \operatorname{Hom}_G(V_\pi, V_{\pi'}) = 0$.

2) Given $w \in V_\pi$, $v \in V_{\pi'}$, set $L_{v,w} := \langle -, w \rangle \cdot v : V_\pi \to V_{\pi'}$.

If $\pi = \pi' \Rightarrow \operatorname{tr} L_{v,w} = \langle v, w \rangle$.

Definition: Let (π, V) be a f.d.c.r. of G. The character of π is

$$\begin{aligned} \chi_\pi : G &\to \mathbb{C} \\ x &\mapsto \chi_\pi(x) := \operatorname{tr} \pi(x). \end{aligned}$$

Proposition (Properties):

1) χ_π is smooth

2) π irreducible $\Rightarrow$ $\chi_\pi \in \mathcal{E}(G)_\pi$.

3) $\chi_\pi(y x y^{-1}) = \chi_\pi(x)$ (conjugation invariant)

4) $\pi, \pi' \in \hat{G}$, G compact, and $\pi \not\sim \pi'$ $\Rightarrow$ $\chi_\pi \perp \chi_{\pi'}$.

5) $\pi \sim \pi' \Rightarrow \chi_\pi = \chi_{\pi'}$.

6) If π irreducible, $\langle \chi_\pi, \chi_\pi \rangle_{L^2(G)} = 1$.

Corollary: Let π be a fdcr of a compact Lie group G. If $\pi \sim \bigoplus_{i=1}^{N} \pi_i$, then

$$\boxed{\chi_\pi = \sum_{i=1}^{N} \chi_{\pi_i}}$$

Definition: Let $\delta \in \hat{G}$ and $\pi \sim \bigoplus_{i=1}^{N} \pi_i$. The multiplicity of δ and π is

$$m(\delta, \pi) := \# \{ i : \delta \sim \pi_i \}.$$

eme: The multiplicity is a well-defined natural number, i.e., it does not depend on the decomposition.
n particular, $m(\delta, \pi) = \langle \chi_\pi, \chi_\delta \rangle_{L^2(G)}$, and any π is $\pi \sim \bigoplus_{\delta \in \hat{G}} m(\delta, \pi) \cdot \delta$

Theorem (Classification of f.d. representations): Let π, π' be fdcr of a compact Lie group G.

$$\pi \sim \pi' \iff \chi_\pi = \chi_{\pi'}.$$

Definitions: Denote V_π to the v.s. V of a rep. (π, V).

a) The contragradient or dual representation of π on G in V_π^* is

$$\check{\pi} : G \longrightarrow GL(V_\pi^*)$$
$$x \longmapsto \check{\pi}(x) := \pi(x^{-1})^*.$$

b) The <u>tensor product</u> representation of π and ρ rep. of G is

$$\pi \otimes \rho : G \longrightarrow GL(V_\pi \otimes V_\rho)$$
$$x \longmapsto \pi(x) \otimes \rho(x).$$

c) The <u>exterior tensor product</u> representation of (π, V_π) of G and (ρ, V_ρ) on H is

$$\pi \hat{\otimes} \rho : G \times H \longrightarrow GL(V_\pi \otimes V_\rho)$$
$$(x, y) \longmapsto \pi(x) \otimes \rho(y).$$

<u>Lemma</u>:

1) $\chi_{\check{\pi}}(x) = \chi_\pi(x^{-1})$

2) $\chi_{\pi \otimes \rho} = \chi_\pi \cdot \chi_\rho$

3) $\chi_{\pi \hat{\otimes} \rho}(x, y) = \chi_\pi(x) \cdot \chi_\rho(y)$.

<u>Lemma</u>: Let (δ, V_δ) be a representation with an unitarizing inner product.

1) $\mathcal{C}(G)_\delta$ is invariant under the left & right regular representations L and R, i.e., $L(g)\mathcal{C}(G)_\delta \subseteq \mathcal{C}(G)_\delta$

2) $V_\delta \longrightarrow \mathcal{C}(G)_\delta$, $v \longmapsto m_{v,w}$ intertwines δ and R for all w.

3) $V_\delta \longrightarrow \mathcal{C}(G)_\delta$, $w \longmapsto m_{v,w}$ intertwines δ and L, for all v. (this is an anti-linear map

<u>Corollary</u>: $m : V_\delta \otimes \overline{V_\delta} \xrightarrow{\sim} \mathcal{C}(G)_\delta$ is an unitary isomorphism, (where $\overline{V_\delta}$ is the conjugate complex v.s.), and it intertwines the representations $\delta \hat{\otimes} \bar{\delta}$ and $R \times L$ (where $\bar{\delta}$ is δ viewed in $\overline{V_\delta}$, an anti-linear map on $\overline{V_\delta}$).

<u>Lemma</u>. The polarity $\overline{V_\delta} \longrightarrow V_\delta^*$, $w \longmapsto \langle\ , w\rangle$, intertwines $\bar{\delta}$ and $\check{\delta}$.

Theorem (Peter–Weyl): Let G be a compact Lie group.

1) $$\overline{\bigoplus_{\delta \in \hat{G}}^{\perp} \mathcal{C}(G)_\delta} = L^2(G)$$

2) $$R \times L\big|_{\mathcal{C}(G)_\delta} \simeq \delta \hat{\otimes} \delta^* \simeq \delta \otimes \bar{\delta}$$

3) $R \times L$ decomposes as sum of irreducible representations on $L^2(G)$ as
$$R \times L = \bigoplus_{\delta \in \hat{G}} \delta \hat{\otimes} \delta^*.$$

Lemma: Via the iso $V_\delta \otimes \overline{V_\delta} \simeq \mathrm{End}(V_\delta)$, the map $m: V_\delta \otimes \overline{V_\delta} \to \mathcal{C}(G)_\delta$, $v \otimes w \mapsto m_{v,w}$, becomes $T_\delta : \mathrm{End}(V_\delta) \to \mathcal{C}(G)$, $A \mapsto \mathrm{tr}(\delta(x) \circ A)$.

Definition: The class functions are
$$L^2(G, \text{class}) := \{\varphi \in L^2(G) \quad \varphi(xgx^{-1}) = \varphi(g) \ \forall x, g\} = L^2(G)^{\Delta_G}$$

Corollary: $\{\chi_\delta : \delta \in \hat{G}\}$ form a complete orthonormal basis for $L^2(G, \text{class})$.

Corollary: Suppose $\{\delta_j : j \in J\} \subseteq \hat{G}$ with $\delta_i \neq \delta_j$ for $i \neq j$. If span $\{\chi_{\delta_j} : j \in J\}$ is dense in $\mathcal{C}(G, \text{class})$, then $\hat{G} = \{\delta_j : j \in J\}$.

Corollary: $\widehat{SU(2)} = \{(\pi_n, P_n) : n \in \mathbb{N}\}$.

REPRESENTATIONS OF LIE ALGEBRAS

Definition: Let $\mathfrak{g}$ be a Lie algebra. A representation of $\mathfrak{g}$ in a v.s. V is a Lie algebra homomorphism
$$\rho : \mathfrak{g} \to \mathrm{End}(V),$$
or, a bilinear map $\mathfrak{g} \times V \to V$ s.t. $\rho[X,Y] = \rho(X)\rho(Y) - \rho(Y)\rho(X)$. In general we will write $\rho(X)v \overset{\text{not}}{=} Xv$. One says that V is a $\mathfrak{g}$-module.

- In general we will assume f.d. representations.

• For a representation of a Lie algebra, we have the analogous notions of an invariant subspace, irreducible representation, intertwiner, ... Also:

Lemma (Schur): Let (ρ, V) be a representation of $\mathfrak{g}$. If ρ is irreducible $\Rightarrow \operatorname{End}_{\mathfrak{g}}(V) = \langle \mathrm{Id} \rangle$.

Proposition: If $\pi: G \to GL(V)$ is a f.d.c.r. of G, then $\pi_*: \mathfrak{g} \to \operatorname{End}(V)$ is a rep. of $\mathfrak{g}$, called the infinitesimal representation, and

$$\begin{tikzcd} G \arrow[r, "\pi"] & GL(V) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\pi_*"] & \operatorname{End}(V) \arrow[u, "e^{\circ}"'] \end{tikzcd} \qquad , \qquad \boxed{\pi(\exp X) = e^{\pi_* X}}$$

Lemma: Let (π, V) be a f.d.c.r. of G.

1) $W \subseteq V$ G-invariant $\Rightarrow W$ is $\mathfrak{g}$-invariant
2) π_* irreducible $\Rightarrow \pi$ irreducible
3) $T: V \to V'$ G-equivariant $\Rightarrow T$ $\mathfrak{g}$-equivariant.
4) If G is connected, then "$\Leftarrow$" holds for 1) - 3).

COMPLEXIFICATION OF A LIE ALGEBRA

Definition: Let $\mathfrak{g}$ be a real Lie algebra. Its complexification is $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, which is a complex Lie algebra, endowed with the natural extension of the Lie bracket.

• As real vector space, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i \cdot \mathfrak{g}$.

Example: $\underline{su(2)_{\mathbb{C}}} = su(2) \oplus i \cdot su(2) = \underline{sl(2;\mathbb{C})}$.

If V is a $\mathfrak{g}$-module, (V $\mathbb{C}$-v.s.), then V is a $\mathfrak{g}_{\mathbb{C}}$-module as well, via $(X+iY) \cdot v := Xv + Y(iv)$.

Lemma: Let V be a $\mathfrak{g}$-module, and let $W \leq V$.

1) W $\mathfrak{g}$-invariant $\Rightarrow$ W $\mathfrak{g}_{\mathbb{C}}$-invariant

2) V is $\mathfrak{g}$-irreducible $\Rightarrow$ V is $\mathfrak{g}_{\mathbb{C}}$-irreducible.

* The example: (π_n, P_n) irred. rep. of $SU(2)$ $\Rightarrow$ (π_{n*}, P_n) irred. rep of $su(2)$ $\Rightarrow$

$\Rightarrow (\pi_{n*})_{\mathbb{C}}$ irre. rep. of $sl(2,\mathbb{C}) = su(2)_{\mathbb{C}}$. Let us compute the latter, in the basis

$$sl(2,\mathbb{C}) = \mathbb{C}\cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{H} \oplus \mathbb{C}\cdot \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{X} \oplus \mathbb{C}\cdot \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{Y}.$$

(the rep. is determined by the linear maps $H\cdot, X\cdot, Y\cdot : P_n \to P_n$). The matrices of this linear maps in the basis $\{P_k = z_1^{n-k} \cdot z_2^{k}\}$ are

$$H\cdot \equiv \begin{pmatrix} n & & & \\ & n-2 & & \\ & & \ddots & \\ & & & -n \end{pmatrix}, \qquad X\cdot \equiv \begin{pmatrix} 0 & -1 & & & \\ & \ddots & -2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -n \\ & & & & 0 \end{pmatrix}$$

$$Y\cdot \equiv \begin{pmatrix} 0 & & & & \\ -n & \ddots & & & \\ & 1-n & \ddots & & \\ & & \ddots & \ddots & \\ & & & -1 & 0 \end{pmatrix}$$

- Our next goal is to classify all $sl(2,\mathbb{C})$-modules, i.e., all representations of $sl(2,\mathbb{C})$ up to equivalence.

As before, if V is a representation of $sl(2,\mathbb{C})$, $H \in sl(2,\mathbb{C})$ induces an endomorphism $H: V \to V$.

Definition: We say that $\lambda \in \mathbb{C}$ is an <u>H-weight</u> if λ is an eigenvalue of $H\cdot$, i.e., if

$$V_\lambda := \{v \in V : Hv = \lambda v\} = \operatorname{Ker}(H - \lambda I) \neq 0.$$

· The basis of $\mathfrak{sl}(2,\mathbb{C})$ $\{H, X, Y\}$ satisfies the relations

$$[H,X] = 2X, \qquad [H,Y] = -2Y, \qquad [X,Y] = H.$$

Lemma:

1) $X \cdot V_\lambda \subseteq V_{\lambda+2}$

2) $Y \cdot V_\lambda \subseteq V_{\lambda-2}$

Definition: Let V be a fd $\mathfrak{sl}(2,\mathbb{C})$-module. An element $v \in V$ is primitive if $Xv = 0$.

Lemma: Let V be a fd $\mathfrak{sl}(2,\mathbb{C})$-module. There is a primitive vector $v \in V$ which is also a weight vector.

Proposition: ______________ , and let $v \in V$ be a primitive vector which is also a weight vector, of weight $\lambda \in \mathbb{C}$. Then:

1) $\exists\, n \in \mathbb{N} : \{v, Yv, Y^2v, \dots, Y^n\}$ forms a basis for V.

2) $X Y^k v = c_k \cdot Y^{n-1} v$, where $c_k = k \cdot (\lambda - (k-1))$.

3) $\lambda = n$.

Finally we get

Theorem: The $\mathfrak{sl}(2,\mathbb{C})$-modules P_n, $n \in \mathbb{N}$, exhaust all irreducible fd $\mathfrak{sl}(2,\mathbb{C})$-modules, up to equivalence. Ie, $\widehat{\mathfrak{sl}(2,\mathbb{C})} = \{((\pi_{n_+})_{\mathbb{C}}, P_n) : n \geq 1\}$.

ALGEBRAIC ANALYSIS OF A COMPACT LIE ALGEBRA

Definition: A Lie algebra $\mathfrak{g}$ is compact if it is (isomorphic to) the Lie algebra of a compact Lie group.

Lemma: Let $T: V \to V$ be a semisimple (= diagonalizable) endomorphism. If $W \subset V$ is T-invariant, then $T_{|W}: W \to W$ is semisimple too.

Definition: A torus in $\mathfrak{g}$ is a abelian subalgebra $\mathfrak{t} \subset \mathfrak{g}$, ie, $[\mathfrak{t},\mathfrak{t}] = 0$. We say that a torus is maximal if it is not contained in a bigger torus.

Definition: Let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal torus, and denote $\mathfrak{t}^*_{\mathbb{C}} := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, \mathbb{R})$.

We say that $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ is a weight if

$$V_\lambda := \{ v \in V : H \cdot v = \lambda(H) \cdot v \quad \forall H \in \mathfrak{t} \} \neq 0 .$$

We will denote the set of weights as $\Lambda(V, \mathfrak{t}) \subset \mathfrak{t}^*_{\mathbb{C}}$.

Definition: We say that a $\mathfrak{t}$-module V is semisimple when $H \cdot : V \to V$ is semisimple $\forall H \in \mathfrak{t}$.

Theorem: If V is a f.d. semisimple $\mathfrak{t}$-module, then it decomposes as direct sum of weight spaces,

$$\boxed{V = \bigoplus_{\lambda \in \Lambda(V, \mathfrak{t})} V_\lambda}$$

Lemma: Let (π, V) be a f.d. rep of a compact Lie gp G, and let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal torus. Then all weights are "imaginary", ie, $\Lambda(V, \mathfrak{t}) \subset i\mathfrak{t}^* \subset \mathfrak{t}^*_{\mathbb{C}}$.

- Let $\mathfrak{g}$ be a compact Lie algebra. There is a natural structure of $\mathfrak{g}_{\mathbb{C}}$-module on $\mathfrak{g}_{\mathbb{C}}$, given by the complexification of ad, $\quad \mathrm{ad} : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$, ie, $X \in \mathfrak{g}_{\mathbb{C}}$ and $v \in \mathfrak{g}_{\mathbb{C}}$, $X \cdot v = \mathrm{ad}(X) v = [X, v]$. As before for $\alpha \in \mathfrak{t}^*_{\mathbb{C}}$ we can consider $(\mathfrak{g}_{\mathbb{C}})_\alpha$, and in particular for the 0-weight, $(\mathfrak{g}_{\mathbb{C}})_0 = \mathfrak{t}$.

Definition: The roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t})$ are the non-zero weights, $R = R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}) := \Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}) - 0$.

Theorem (Root space decomposition): For any compact Lie algebra $\mathfrak{g}$ there is a decomposition

$$\boxed{\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t})} (\mathfrak{g}_{\mathbb{C}})_\alpha \right)}$$

Example: $su(2)_{\mathbb{C}} = sl(2,\mathbb{C}) = \underbrace{\mathbb{C}\cdot H}_{t_{\mathbb{C}}} \oplus \underbrace{\mathbb{C}\cdot X}_{(su(2)_{\mathbb{C}})_{\alpha}} \oplus \underbrace{\mathbb{C}\cdot Y}_{(su(2)_{\mathbb{C}})_{-\alpha}}$; $t = \mathbb{R}\cdot iH$ maximal torus

where α is determined by the condition $\alpha(H) = 2$.

Proposition: Let V be a $g_{\mathbb{C}}$-module. Then $(g_{\mathbb{C}})_\alpha V_\lambda \subset V_{\lambda+\alpha}$ $\forall \lambda \in t_{\mathbb{C}}^*$, $\alpha \in R(g_{\mathbb{C}}, t)$

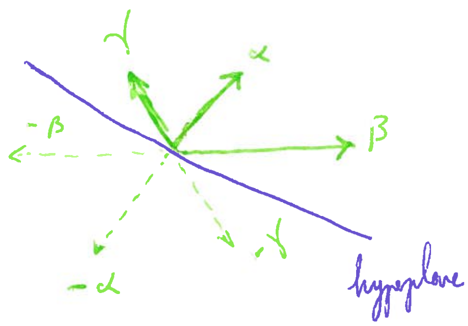
Lemma: Let V be an irreducible $g_{\mathbb{C}}$-module. Then it decomposes as

$$V = \bigoplus_{\lambda \in \Lambda(V)} V_\lambda = \bigoplus_{\lambda \in \Lambda(V) \cap (\lambda_0 + \mathbb{Z}R)} V_\lambda .$$

Corollary: For $\lambda, \mu \in \Lambda(V)$, $\lambda - \mu \in \mathbb{Z}R$

Proposition: Let g be a compact Lie algebra. If $\alpha \in R \Rightarrow -\alpha \in R$.

The roots R are a finite set in general. Any hyperplane which does not contain any roots separates "positive" roots and "negative". An hyperplane here is, since $R \subset it^*$, is determined by $H_0 \in it$ (thus $\{\lambda : \lambda(H_0) = 0\} \subset it^*$ is the hyperplane). Then define



$$R^+ := \{\alpha \in R : \alpha(H_0) > 0\}, \quad R^- = \{\alpha \in R : \alpha(H_0) < 0\}$$

In particular, $\mathbb{N}R^+ \cap (-\mathbb{N}R^+) = 0$.

Corollary: If $g_{\mathbb{C}}^+ := \bigoplus_{\alpha \in R^+} g_{\mathbb{C}\alpha}$ and $g_{\mathbb{C}}^- := \bigoplus_{\alpha \in R^-} g_{\mathbb{C}\alpha}$, then they are subalgebras and

$$\boxed{g_{\mathbb{C}} = g_{\mathbb{C}}^- \oplus t_{\mathbb{C}} \oplus g_{\mathbb{C}}^+}$$

Definition : let V be a f.d. $\mathfrak{g}_{\mathbb{C}}$-module. A highest weight vector of V (relative to a choice of R^+) is a weight vector $0 \neq v \in V$ s.t. $\mathfrak{g}_{\mathbb{C}}^+ v = 0$.

∘ We denote as $\mathfrak{g}_{\mathbb{C}}^+ W := \text{span}\{Xw : w \in W, X \in \mathfrak{g}_{\mathbb{C}}^+\}$.

Lemma : Every f.d. $\mathfrak{g}_{\mathbb{C}}$-module V has a highest weight vector.

Definition : let V be a $\mathfrak{g}_{\mathbb{C}}$-module. A cyclic vector $v \in V$ is a vector s.t. $V = \langle v \rangle_{\mathfrak{g}_{\mathbb{C}}}$ the $\mathfrak{g}_{\mathbb{C}}$-submodule generated by S.

Theorem : let V be a $\mathfrak{g}_{\mathbb{C}}$-module, and suppose V has a cyclic highest weight vector $v \in V$. Then

1) $\mathbb{C}v$ is a weight space V_{λ_0}.
2) $V = \text{span}\{Y_1 \cdots Y_n v : Y_1, \ldots, Y_n \in \mathfrak{g}_{\mathbb{C}}^-, n \geq 0\}$.
3) $\Lambda(V) = \lambda_0 - \mathbb{N}R^+$.
4) V has a unique maximal proper submodule.
5) A cyclic highest weight vector is unique up to a constant.

Corollary. If V is an irreducible f.d. $\mathfrak{g}_{\mathbb{C}}$-module, then V has a unique highest weight $\lambda_V \in \Lambda(\mathfrak{g}, \mathfrak{t})$

Theorem : let $\mathfrak{g}$ be a compact Lie algebra, and let V_1, V_2 be irreducible $\mathfrak{g}_{\mathbb{C}}$-modules. If they have the same highest weight (rel. to R^+), then V_1 and V_2 are isomorphic as $\mathfrak{g}_{\mathbb{C}}$-modules, i.e, the representations are equivalent.

THE KILLING FORM

. let $\text{Aut}(\mathfrak{g})$ be the set of all Lie algebra automorphisms $\mathfrak{g} \to \mathfrak{g}$, which is a closed subgp of $GL(\mathfrak{g})$ and therefore Lie gp on its own.

Definition. let $\mathfrak{g}$ be a Lie algebra. A derivation on $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ such that

$$D[X, Y] = [DX, Y] + [X, DY]. \quad \text{(Leibniz rule)}$$

We denote by $\text{Der}(\mathfrak{g})$ the set of derivations of $\mathfrak{g}$).

Proposition: $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Der}(\mathfrak{g})$.

Example: $\mathrm{ad}(X): \mathfrak{g} \to \mathfrak{g}$ is a derivation, by Jacobi, and therefore $\mathrm{ad}: \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$.

Definition: The inner automorphism of $\mathfrak{g}$ is the smallest group generated by elements $e^{\mathrm{ad}(X)}$.

$$\mathrm{Int}(\mathfrak{g}) := \langle e^{\mathrm{ad}(X)} : X \in \mathfrak{g} \rangle$$

which is a subgp of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$.

Definition: The Killing form of $\mathfrak{g}$ is the map

$$B : \mathfrak{g} \times \mathfrak{g} \to K \qquad (K = \mathbb{R} \text{ or } \mathbb{C})$$
$$(X, Y) \mapsto B(X,Y) := \mathrm{tr}\left(\mathrm{ad}(X) \circ \mathrm{ad}(Y)\right)$$

Proposition (Properties):

1) B is a symmetric metric on $\mathfrak{g}$.

2) $B(\varphi(X), \varphi(Y)) = B(X,Y) \quad \forall \varphi \in \mathrm{Aut}(\mathfrak{g})$.

3) $B(DX, Y) + B(X, DY) = 0 \quad \forall D \in \mathrm{Der}(\mathfrak{g})$

4) $B(\mathrm{ad}(Z)X, Y) + B(X, \mathrm{ad}(Z)Y) = 0$ (Invariance of the Killing form)

Definition: The center of a Lie algebra $\mathfrak{g}$ is

$$Z(\mathfrak{g}) := \{ X \in \mathfrak{g} : [X, \mathfrak{g}] = 0 \} = \mathrm{Ker}\ \mathrm{ad}.$$

Lemma. Let $\mathfrak{g}$ be a compact Lie algebra.

1) B is negative semidefinite, and $\mathfrak{g}^{\perp} = Z(\mathfrak{g})$.

2) $\exists\ \mathfrak{g}_1 \subset \mathfrak{g}$ ideal s.t. $B|_{\mathfrak{g}_1}$ is negative def and $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_1$.

Proposition: If the Killing form of $\mathfrak{g}$ is def negative, then

$$\mathrm{ad}: \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(\mathfrak{g})$$

is a Lie alg isomorphism, and therefore $\mathfrak{g}$ is compact.

Corollary $\mathfrak{g}$ is compact $\Leftrightarrow$ $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_1$, with $\mathfrak{g}_1$ an ideal and $B|_{\mathfrak{g}_1}$ def negative.

Definition: We say that a Lie algebra $\mathfrak{g}$ is simple if it is not abelian and has no proper ideals; and it is called semisimple if it is a direct sum of simple Lie algebras.

Theorem: The following are equivalent:

1) $\mathfrak{g}$ is compact and semisimple
2) The Killing form of $\mathfrak{g}$ is negative definite.
3) $\mathfrak{g}$ is compact and $Z(\mathfrak{g}) = 0$.

ROOTS IN A COMPACT, SEMISIMPLE LIE ALG

In what follows let $\mathfrak{g}$ be a compact, semisimple Lie algebra, and set $E := i\mathfrak{t}^*$, which is an euclidean space with $(B_{\mathbb{C}})|_{i\mathfrak{t}}$.

Lemma: $E = \mathrm{span}\, R$.

Lemma: Let $\lambda, \mu \in i\mathfrak{t}^*$, with $\lambda + \mu \neq 0$. Then $(\mathfrak{g}_{\mathbb{C}})_\lambda \perp (\mathfrak{g}_{\mathbb{C}})_\mu$ (wrt the Killing form).

We want to mimic the situation of $\mathfrak{sl}(2,\mathbb{C})$ in the general case: given $\alpha \in E$, a vector $H_\alpha \in i\mathfrak{t}$ is uniquely determined by the conditions $H_\alpha \perp \mathrm{Ker}\,\alpha$ and $\alpha(H_\alpha) = 2$. We want to find a $\mathfrak{sl}_2(\mathbb{C})$-triple.

Let $\tau: \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be the conjugation map: for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, $\tau(X+iY) = X - iY$.

Lemma. Let $\alpha \in R$. There is $0 \neq X_\alpha \in \mathfrak{g}_{\mathbb{C}}$ such that $\{H_\alpha, X_\alpha, Y_\alpha := -\tau(X_\alpha)\}$ is a standard $\mathfrak{sl}_2$-triple, ie,

$$[H_\alpha, X_\alpha] = 2X_\alpha, \qquad [H_\alpha, Y_\alpha] = -2Y_\alpha, \qquad [X_\alpha, Y_\alpha] = H_\alpha.$$

The previous result says that we have a copy of $sl(2,\mathbb{C})$ in any $\mathfrak{g}_{\mathbb{C}}$!!

Denote $\mathfrak{S}_{\alpha\,\mathbb{C}} := \mathrm{span}\{H_\alpha, X_\alpha, Y_\alpha\} \subseteq \mathfrak{g}_{\mathbb{C}}$ and

$$\mathfrak{S}_\alpha := \mathfrak{S}_{\alpha\,\mathbb{C}} \cap \mathfrak{g} = \mathrm{span}\{iH_\alpha, X_\alpha - Y_\alpha, i(X_\alpha + Y_\alpha)\} \simeq su(2).$$

Theorem: Let $\alpha \in R$. Then

1) $\dim_{\mathbb{C}} (\mathfrak{g}_{\mathbb{C}})_\alpha = 1$

2) $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$

3) $\mathfrak{S}_{\alpha\,\mathbb{C}} = \mathbb{C}H_\alpha \oplus (\mathfrak{g}_{\mathbb{C}})_\alpha \oplus (\mathfrak{g}_{\mathbb{C}})_{-\alpha}$

ROOT SYSTEMS

Definition: Let E be a real vector space, and let $0 \neq \alpha \in E$. A reflexion in α is a linear map $s: E \to E$ such that $s(\alpha) = -\alpha$ and $\mathrm{Ker}(s - \mathrm{Id}) \oplus \mathbb{R}\alpha = E$.

Lemma: If $\alpha \in R$ and s_α is an orthonormal reflexion in α $(E = it^*)$, then explicitly it is

$$s_\alpha(\lambda) = \lambda - \frac{2\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha .$$

Theorem: $s_\alpha(R) = R$.

Theorem: Let $\alpha \in R$. There is $\varphi_\alpha \in \mathrm{Aut}(\mathfrak{g})$ st

1) $\varphi_\alpha(t_{\mathbb{C}}) \subset t_{\mathbb{C}}$

2) $\widetilde{\varphi_\alpha} = s_\alpha$, where $\widetilde{\varphi_\alpha}$ is the complex linear extension.

Proposition: The pair (E, R) has the following properties.

1) E is a real vector space of finite dim, and $R \subset E - 0$ is finite.

2) $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ $\forall \alpha \in R$

3) For all $\alpha \in R$, $\exists$ S_α reflexion in α st $S_\alpha(R) = R$, and such a reflexion is unique.

4) For all $\alpha, \beta \in R$ $S_\alpha(\beta) = \beta + k\alpha$ for some $k \in \mathbb{Z}$; and if one uses the Killing form

$$k = -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}.$$

Definition: The Weyl group of the root system (E, R) is

$$W(R) := \langle S_\alpha : \alpha \in R \rangle < GL(E), \text{ and finite.}$$

Definition: A pair (E, R) satisfying 1) – 4) is called an (abstract) root system.

Theorem: There is a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of compact, semisimple} \\ \text{Lie algebras} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of root systems} \end{array} \right\}$$

Definition: Let (E, R) be an abstract root system, and choose $\langle\cdot,\cdot\rangle$ $W(R)$-invariant. The Cartan numbers are the integer coefficients of the reflexions,

$$n_{\alpha\beta} := \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}.$$

For $\alpha, \beta \in R$, with the given inner product there is angles between them: $\varphi_{\alpha\beta}$ determined by

$\langle\alpha,\beta\rangle = |\alpha|\,|\beta| \cdot \cos\varphi_{\alpha\beta}$.

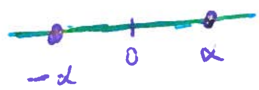
We have $n_{\alpha\beta}\, n_{\beta\alpha} = 4 \cdot \cos^2\varphi_{\alpha\beta} \in \mathbb{Z}$, and then the product must be $0, 1, 2, 3, 4$. 4 means $\varphi_{\alpha\beta} = 0$ or π $\Rightarrow$ $\alpha = \pm\beta$ (proportional roots). Else

Proposition: Let $\alpha, \beta \in R$ be non-proportional roots with $|\alpha| \leq |\beta|$. Then we have the following possibilities:

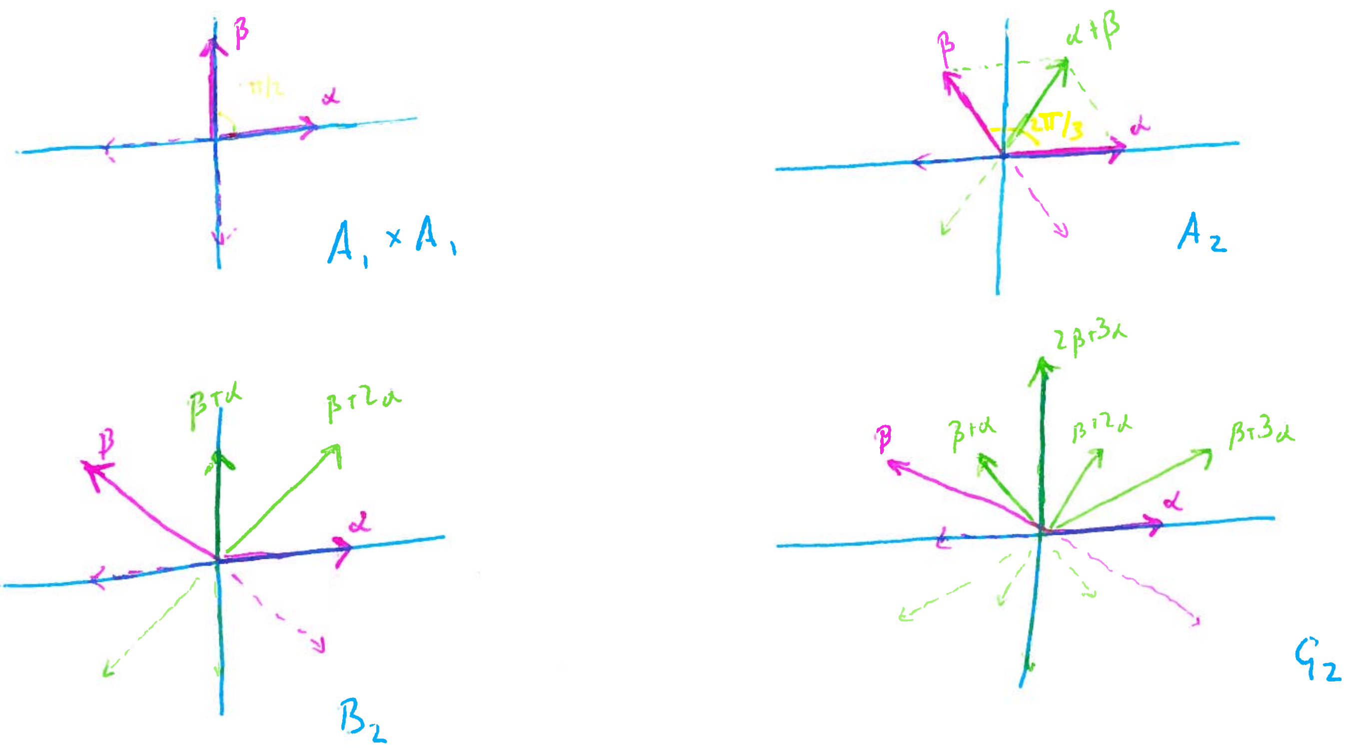
$n_{\alpha\beta} n_{\beta\alpha}$	$n_{\alpha\beta}$	$n_{\beta\alpha}$	$\gamma_{\alpha\beta}$	$\|\beta\|/\|\alpha\|$
0	0	0	$\pi/2$	indetermined
1	1	1	$\pi/3$	1
1	-1	-1	$2\pi/3$	1
2	2	1	$\pi/4$	$\sqrt{2}$
2	-2	-1	$3\pi/4$	$\sqrt{2}$
3	3	1	$\pi/6$	$\sqrt{3}$
3	-3	-1	$5\pi/6$	$\sqrt{3}$

Theorem (Classification of root systems): There are the following possibilities for root systems:

__dim E = 1__ :



__dim E = 2__ :



DYNKIN DIAGRAMS

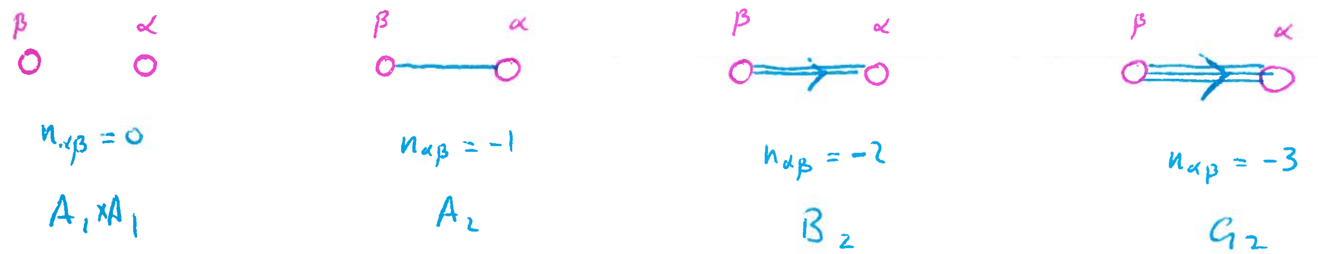
Definition: Let (E,R) be a root system. A fundamental system for R is a subset $S \subset R$ st

i) S is a basis for E (over $\mathbb{R}$)

ii) $R \subset \mathbb{N}S \cup (-\mathbb{N}S)$.

Definition. The Coxeter graph of (E,R) consists of a graph with as many nodes as $\alpha \in S$ and as many edges connecting $\alpha, \beta \in S$ as $n_{\alpha\beta} n_{\beta\alpha} = 0, \dots, 3$.

The Dynkin diagram of (E,R) is the Coxeter diagram together with arrows on edges, pointing towards the shorter root.



Theorem (Classification of simple compact Lie algebras): Up to isomorphism, a root system is completely determined by its Dynkin diagram.

Moreover, two simple compact Lie algebras are isomorphic $\iff$ they have the same Dynkin diagram.