I : VECTOR BUNDLES

Definition : Let B be a topological space. An n-dimensional vector bundle over B is a topological space E together with a map $p : E \to B$ such that

i) $E_b := p^{-1}(b)$ has an structure of n-dim v.s. $\forall b \in B$

ii) Every point $b \in B$ has an open neighbourhood $\mathcal{U}$ s.t. there exists an homeomorphism $h : p^{-1}(\mathcal{U}) \xrightarrow{\sim} \mathcal{U} \times \mathbb{R}^n$ taking $E_b = p^{-1}(b)$ to $\{b\} \times \mathbb{R}^n \simeq \mathbb{R}^n$ by a linear isomorphism, so in particular the following diagram commutes.

$$\begin{tikzcd} p^{-1}(\mathcal{U}) \arrow[rr, "h", "\sim"'] \arrow[dr, "p"'] & & \mathcal{U} \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ & \mathcal{U} & \end{tikzcd}$$

Examples : 1) $E := B \times \mathbb{R}^n \xrightarrow{\pi_1} B$ the trivial vector bundle, which we will denote as $\underline{n}$.

2) $E = \dfrac{[0,1] \times \mathbb{R}}{(0,t) \sim (1,-t)} \xrightarrow{\bar{p}} \dfrac{[0,1]}{0 \sim 1} \simeq S_1$, the Mobius bundle

3) $TS_n = \{ (x,v) \in S_n \times \mathbb{R}^{n+1} : x \perp v \} \longrightarrow S_n$

4) $NS_n = \{ (x, \lambda x) \in S_n \times \mathbb{R}^n : \lambda \in \mathbb{R} \} \longrightarrow S_n$

5) $\gamma := \{ ([v], v) \in \mathbb{P}_n \times \mathbb{R}^{n+1} \} \longrightarrow \mathbb{P}_n$, the tautological line bundle

Definition : A line bundle is a 1-dim v. bundle.

In general, for a v.b. $E \xrightarrow{p} B$, E is called the total space, B the base space, E_b the fibers and h the local trivialization.

__Definition__ : Let $p: E \to B$, $p': E' \to B$ be vector bundles over B. A __morphism of vector bundles__ is a continuous map $F: E \to E'$ which sends E_x to E'_x by a linear map $F_x := F|_{E_x}: E_x \to E'_x$, so in particular the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

An __isomorphism__ of vector bundles is a morphism of v.b. s.t. F is homeomorphism and F_x are linear isomorphisms.

- We will denote $Vect^n(B) := \{$ isomorphism classes of rank n vector bundles $\}$.

__Definition__: Let $p: E \to B$, $p': E' \to B'$ be vector bundles, and let $f: B \to B'$ be a continuous map. A __bundle map covering f__ is a continuous map $F: E \to E'$ which sends E_x to $E'_{f(x)}$ by a linear map $F_x := F|_{E_x}: E_x \to E'_{f(x)}$, so in particular the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

__Lemma__: Let $F: E \to E'$ a vector bundle morphism over B. If F_x are linear isomorphisms $\forall x \in B \Rightarrow$ $\Rightarrow F$ is a v.b. isomorphism.

__Definition__: A __section__ of $p: E \to B$ is a continuous map $s: B \to E$ s.t. $p \circ s = Id_B$.

__Example__: Given a v.b. $E \to B$, the __zero-section__ $0: B \to E$ takes $x \in B$ to $0 \in E_b$.

Definition: Let $p: E \to B$ be a v.b. A frame of E is a collection $s = (s_1, \dots s_n)$ of sections s.t. $\{s_1(x), \dots, s_n(x)\}$ is a basis of E_x $\forall x \in B$. A local frame is a frame for $E_{|U}$ $(U \subset B)$

· The local trivialty condition ensures the existence of local frames around every point. In particular if U is a trivializing open set, any section will be $s = f_1 s_1 + \dots + f_n s_n$ for some $f_i \in \mathcal{C}(U)$.

Lemma: Let $p: E \to B$ be a v.b. of dimension n.

$$E \text{ is isomorphic to } \underline{n} \iff \exists\, s_1 \dots s_n \text{ sections } : \{s_1(x), \dots, s_n(x)\} \text{ basis of } E_x \ \forall x$$

· (Cocycles): If $p: E \to B$ is a v.b. and $\{U_\alpha\}$ is a cover with local trivializations $h_\alpha: p^{-1}(U_\alpha) \to U$ in the intersections $(h_\beta h_\alpha^{-1})_{|U_\alpha \cap U_\beta} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \xrightarrow{\sim} (U_\alpha \cap U_\beta) \times \mathbb{R}^n$ for some maps

$$(x, v) \longmapsto (x, g_{\beta\alpha}(x)(v))$$

$g_{\beta\alpha} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$, called cocycles or gluing functions, that satisfy

1) $g_{\gamma\beta}\, g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$

2) $g_{\alpha\alpha} = Id$.

Conversely, given a set $\{g_{\beta\alpha}\}$ as before associated to a cover $\{U_\alpha\}$ of B, we can construct a vector bundle as

$$E := \frac{\coprod_\alpha U_\alpha \times \mathbb{R}^n}{(x, v) \sim (x, g_{\beta\alpha}(x)(v))} \quad \text{for } x \in U_\alpha \cap U_\beta.$$

CONSTRUCTIONS ON VECTOR BUNDLES

<u>Lemma</u>: Let $p: E \to B$ be a v.b.

1) If U is a trivializing open neighborhood of $x \in B$, then any other open nbhd $V \subset U$ is also trivializi

2) If $A \subset B$, then $p: p^{-1}(A) \to A$ is a vector bundle

3) If $p': E' \to B'$ is another v.b., then $p \times p': E \times E' \to B \times B'$ is a vector bundle.

<u>Definition</u>: Let $p: E \to B$, $p': E' \to B$ be vector bundles over B. The <u>direct sum</u> of E and E' is

$$E \oplus E' := \{ (v, v') \in E \times E' : p(v) = p'(v') \} \quad (= E \times_B E')$$

endowed with the projection $r: E \oplus E' \to B$, $(v, v') \mapsto p(v) = p'(v')$.

<u>Proposition</u>: $E \oplus E' \to B$ is a vector bundle

- In terms of cocycles, the ones for $E \oplus E'$ are $g_{\beta\alpha} \oplus g'_{\beta\alpha}$.

<u>Example</u>: $TS_n \oplus \underline{1} \simeq TS_n \oplus NS_n \simeq \underline{n+1}$.

<u>Definition</u>: A topological space B is <u>paracompact</u> if it is Hausdorff and for every open cover $\{U_\alpha\}$ there exists a <u>partition of unity</u> subordinated to the cover, ie, a collection $\{\phi_\alpha\} \subset \mathcal{C}(B)$ such that

i) $\phi_\alpha \geq 0$ and $\operatorname{supp} \phi_\alpha \subseteq U_\alpha$

ii) The family $\{\operatorname{supp} \phi_\alpha\}$ is <u>locally finite</u>: every point $x \in B$ has a nbhd in which only finitely many intersect.

iii) $\sum_\alpha \phi_\alpha \equiv 1$.

<u>Proposition</u>: Hausdorff + compact $\Rightarrow$ paracompact.

Proposition: Every CW-complex is paracompact.

Theorem: Every Hausdorff, locally compact, second countable space is paracompact.

Corollary: Every smooth manifold is paracompact.

Definition: An inner product on a vector bundle $p: E \to B$ is a continuous map

$$\langle \cdot, \cdot \rangle : E \oplus E \longrightarrow \mathbb{R}$$

which restricts fiberwise to a positive definite symmetric bilinear form (= inner product).

Proposition: If the base B is paracompact, every vector bundle $E \to B$ admits an inner product.

Definition: A vector subbundle of a vector bundle $p: E \to B$ is a subspace $E_0 \subset E$ s.t.

i) $E_0 \cap E_b$ is a vector subspace of E_b, $\forall b \in B$

ii) The restriction $p|_{E_0} : E_0 \longrightarrow B$ is a vector bundle.

Proposition: If $E \to B$ has an inner product, and $E_0 \subset E$ is a vector subbundle, there exists a vector subbundle $E_0^{\perp} \subset E$ s.t. $E_0 \oplus E_0^{\perp} \simeq E$.

Theorem: Let $E \to B$ a v.b. over a compact, Hausdorff base. There exists a vector bundle $E' \to B$ such that $E \oplus E'$ is isomorphic to the trivial bundle.

Definition: Let $E, E' \to B$ be two vector bundles over B. The tensor product $E \otimes E'$ of E and E' is another v.b. which, as a set, it is

$$E \otimes E' = \coprod_{b \in B} E_b \otimes E'_b$$

together with the obvious projection $P: E \otimes E' \to B$, which is endowed with the unique topology making the bijective maps $h \otimes h' : P^{-1}(U) \to U \times \mathbb{R}^{n \cdot m}$ homeomorphisms (where h, h' are local trivializations).

Lemma: The previous topology is well-defined.

Proposition: Let $E, E' \to B$ be two v.s. over B. Given $E \oplus E' \to \mathbb{R}$ a continuous fiberwise bilinear map, there exists a unique continuous fiberwise linear map $E \otimes E' \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{tikzcd} E \oplus E' \arrow[r] \arrow[d, "\pi"'] & \mathbb{R} \\ E \otimes E' \arrow[ur, dashed] & \end{tikzcd}$$

PULLBACK BUNDLE

Given a cont. map $f: A \to B$, we want to construct $f^*: Vect^n(B) \to Vect^n(A)$.

Theorem: Given a continuous map $f: A \to B$ and a vector bundle $E \xrightarrow{p} B$, there exists a unique (up to iso) vector bundle $f^*(E) \to A$ with a vector bundle map $F: f^*(E) \to E$ covering f which takes $f^*(E)_a$ isomorphically to $E_{f(a)}$. Explicitly,

$$f^*(E) = \{(a,v) \in A \times E : f(a) = p(v)\} \quad (= A \times_B E)$$

This vector bundle $f^*(E) \to A$ is called the pullback of E by f.

Therefore we obtain a well-defined map $f^*: Vect^n(B) \to Vect^n(A)$.

Definition: The restriction of a vector bundle $p: E \to B$ to a subspace A is the pullback v.s. by the inclusion.

Lemma: Let A be a paracompact space, and take a vector bundle $E \to A \times [0,1]$. The restrictions to $A \times \{0\}$ and $A \times \{1\}$ are isomorphic.

Theorem: Let A be a paracompact space and $p: E \to B$ a v.s. Homotopic maps $f_0 \simeq f_1 : A \to B$ induce pullbacks $f_0^*(E) \simeq f_1^*(E)$ isomorphic.

Corollary: A homotopy equivalence $f: A \to B$ between paracompact spaces induces a bijection $f^*: Vect^n(B) \xrightarrow{\sim} Vect^n(A)$

II CLASSIFICATON OF VECTOR BUNDLES

- Let's construct vector bundles (of dim n) over $S_k = S_{k-1} \cup_{\partial \mathbb{D}_k^+ \sqcup \partial \mathbb{D}_k^-} \mathbb{D}_k^+ \sqcup \mathbb{D}_k^-$ with a map $f : S_{k-1} \to GL_n($

Consider

$$E_f := \frac{\mathbb{D}_k^+ \times \mathbb{R}^n \sqcup \mathbb{D}_k^- \times \mathbb{R}^n}{(x,v) \sim (x, f(x)(v))}, \text{ pushout of the diagram } \begin{array}{ccc} \partial \mathbb{D}^k \times \mathbb{R}^n & \xrightarrow{i \times f} & D^k \times \mathbb{R}^n \\ \downarrow i \times Id & & \downarrow \\ D^k \times \mathbb{R}^n & \longrightarrow & E_f \end{array}.$$

with $(x,v) \in \partial \mathbb{D}_k^-$ and $(x, f(x)(v)) \in \partial \mathbb{D}_k^+$. This construction is a particular case of the general construction using gluing functions.

<u>Definition</u>: We will call <u>clutching function</u> to the map $f : S_{k-1} \to GL_n(\mathbb{R}) / GL_n(\mathbb{C})$.

<u>Lemma</u>: Homotopic clutching functions $f \simeq g : S_{k-1} \to GL_n(\mathbb{R})$ define isomorphic vector bundles $E_f \simeq E_g$.

- This defines a map

$$\Phi : [S_{k-1}, GL_n(\mathbb{R})] \longrightarrow Vect^n(S_k)$$
$$f \longmapsto E_f$$

<u>Proposition</u>: The map $\Phi_{\mathbb{C}} : [S_{k-1}, GL_n(\mathbb{C})] \to Vect^n_{\mathbb{C}}(S_k)$ is a bijection.

<u>Lemma</u>: $GL_n(\mathbb{R})$ has two path components,

$$GL_n^+(\mathbb{R}) = \{\det > 0\}, \quad GL_n^-(\mathbb{R}) = \{\det < 0\};$$

and $GL_n(\mathbb{C})$ is path-connected.

UNIVERSAL BUNDLE

Definition: The Grassmann manifold $G_n(\mathbb{R}^k)$ $(n \leq k)$ is the collection of all n-dim linear subspaces of $\mathbb{R}^k$.

Eg: $$G_1(\mathbb{R}^k) = \mathbb{P}(\mathbb{R}^k) = \mathbb{P}_{k-1} \simeq G_{k-1}(\mathbb{R}^k)$$
$$(V^{k-1})^\perp \longleftarrow V^{k-1}$$

- How to define a topology on $G_n(\mathbb{R}^k)$?

Definition: The Stiefel manifold $V_n(\mathbb{R}^k)$ is the set of all possible orthonormal n-frames in $\mathbb{R}^k$, ie, collections of n l.i & orthonormal vectors in $\mathbb{R}^k$.

- Since vectors in these n-tuples have module 1, we can see $V_n(\mathbb{R}^k) \subset S_{k-1} \times \overset{n}{\cdots} \times S_{k-1}$ (orthogonal n-tuples). In particular, it's compact (closed in a compact). Therefore, we can endow $G_n(\mathbb{R}^k)$ with the quotient topology given by the surjective map

$$V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k)$$
$$\{v_1, \dots v_n\} \longmapsto \langle v_1, \dots v_n \rangle$$

so $G_n(\mathbb{R}^k)$ is compact as well

- Recall that the $\mathbb{R}$-linearization of $\mathbb{N}$ gives $\mathbb{R}^\infty := \mathbb{R}[\mathbb{N}]$, and it is topologized with the final topology (coherent topology) given by the inclusions $\mathbb{R}^n = \mathbb{R}[\{1,..,n\}] \to \mathbb{R}[\mathbb{N}] = \mathbb{R}^\infty$ (induced by the injection $\{1, \dots n\} \to \mathbb{N}$).

We now let $G_n(\mathbb{R}^\infty) := \overset{\varinjlim_k G_n(\mathbb{R}^k)}{\bigcup_k G_n(\mathbb{R}^k)}$ again topologized by the coherent/final topology $G_n(\mathbb{R}^k) \to G_n(\mathbb{R}^\infty)$. There are canonical n-dim bundles over $G_n(\mathbb{R}^k)$ and $G_n(\mathbb{R}^\infty)$:

$$E_n(\mathbb{R}^k) := \{(\ell, v) \in G_n(\mathbb{R}^k) \times \mathbb{R}^k : v \in \ell\} \longrightarrow G_n(\mathbb{R}^k), \quad (\ell, v) \mapsto \ell$$

The inclusions $\mathbb{R}^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^{k+2}$ give $E_n(\mathbb{R}^k) \subset E_n(\mathbb{R}^{k+1}) \subset \dots$ and we set

$$E_n(\mathbb{R}^\infty) := \bigcup_k E_n(\mathbb{R}^k)$$ again endowed with the final topology.

Lemma: The projection $p : E_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$, $(\ell, v) \mapsto \ell$ is a vector bundle for $k \in \mathbb{N}$ and $k = \infty$.

- Let us write $G_n \overset{not}{=} G_n(\mathbb{R}^\infty)$, $E_n \overset{not}{=} E_n(\mathbb{R}^\infty)$.

Theorem. If X is paracompact, we have a bijection

$$\begin{aligned} [X, G_n] &\xrightarrow{\sim} \mathrm{Vect}^n(X) \\ [f] &\longmapsto f^*(E_n) \end{aligned}$$

- Therefore, vector bundles over a fixed base are classified by homotopy classes of maps to G_n.

Definition. The space G_n is called the <u>classifying space</u> for n-dim v.b, and $E_n \to G_n$ is called the <u>universal bundle</u>.

III. K-THEORY

THE GROTHENDIECK GROUP

Definition. A *monoid* $(M, \cdot, 1)$ is a set M with a binary operation which is associative and 1 is a neutral element (i.e., a group but without inverses). It is a commutative monoid when $\cdot$ is commutative.

Definition: A *(unital) semiring* is a set $(R, +, 0, \cdot, 1)$ where $(R, +, 0)$ is a commutative monoid and $(R, \cdot, 1)$ is a monoid, the operation is distributive and $0 \cdot a = 0 = a \cdot 0$. I.e., a ring but without inverses for $+$ or $\cdot$.

Eg. 1) $(\mathbb{N}, +, \cdot)$

2) Denote $\mathrm{Vect}^*(X)$ the isomorphy classes of v.b. of all possible dimensions (here we admit that a vector bundle hasn't constant rank, although they have to have on the connected components of X). Then

$$\left(\mathrm{Vect}^*(X), \oplus, \underline{0}, \otimes, \underline{1}\right)$$

(where $\underline{0}$ is the vector bundle $X \times 0 \simeq X \to X$) is a commutative semiring.

Definition: Let $(M, +, 0)$ be a commutative monoid. Define on $M \times M$

$$(m_1, m_2) \sim (m_1', m_2') \iff \exists\, k \in M : m_1 + m_2' + k = m_1' + m_2 + k,$$

the *Grothendieck group* of M is the quotient

$$K(M) := \frac{M \times M}{\sim}.$$

Alternative construction:

$$K(M) = \frac{\bigoplus_M \mathbb{Z}}{\langle m + m' - (m \oplus m') \rangle}$$

We denote $[(m_1, m_2)]$ by $m_1 - m_2$, and the operation $(m_1 - m_2) + (m_1' - m_2') = (m_1 + m_1') - (m_2 - m_2')$ endows $K(M)$ with a abelian group structure, where the inverse of $m_1 = m_1 - 0$ is $-m_1 = 0 - m_1$ (there's a canonical injection $M \to K(M)$, $m \mapsto [(m, 0)]$). If M is a commutative semiring then $K(M)$ is a ring.

This has universal property

$\mathrm{Hom}(M, G) = \mathrm{Hom}(K(M), G)$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & G \\ K(M) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Proposition: If M is a commutative monoid (resp. comm. semiring), then $K(M)$ is an abelian group (resp. comm. ring). In particular, the Grothendieck construction defines a functor

$$K : \mathrm{Mon} \longrightarrow \mathrm{AbGp} \qquad (K : \mathrm{SemiRing} \longrightarrow \mathrm{Rings}).$$

THE K-THEORY OF A SPACE

In the following all spaces will be considered compact and Hausdorff, thus paracompact.

Definition. Let X be a compact Hausdorff top space. The <u>(complex) K-theory</u> of X is

$$K(X) := K(\mathrm{Vect}^*_{\mathbb{C}}(X))$$

and its <u>real K-theory</u>

$$KO(X) := K(\mathrm{Vect}^*_{\mathbb{R}}(X)).$$

Example: $\mathrm{Vect}^*_{\mathbb{K}}(*) = \mathbb{N}$, thus $K(*) = \mathbb{Z} = KO(*)$.

- Note that we can consider the class of a vb E, namely $[E] - [0] \overset{\text{not}}{=} E$.

Definition. Two v.b. $E, F \to X$ are <u>stably isomorphic</u> if $\exists\, n \geq 0$ st $E \oplus \underline{n} \simeq F \oplus \underline{n}$. If E is stably isomorphic to a trivial vb, then we say that E is <u>stably trivial</u>.

Lemma: "K-theory only distinguish up to stable isomorphism". Ie,

$$E = F \in \begin{matrix} K(X) \\ KO(X) \end{matrix} \iff E, F \text{ are stably isomorphic.}$$

Corollary: If E is stably trivial, then $E = \underline{n} \in K(X)$ $(KO(X))$.

Eg: TS^n is stably trivial, since $TS^n \oplus \underline{1} \simeq \underline{n+1}$.

- Observe that we have obvious injection $\mathbb{Z} \hookrightarrow K(X)$ $(KO(X))$, $n \longmapsto \underline{n}$. In particular, this endows $K(X)$ and $KO(X)$ with structures of $\mathbb{Z}$-algebras.

- Every element in $K(X)$ $(KO(X))$ is represented by a difference $E - \underline{n}$.

Definition: The reduced K-theory of X is

$$\widetilde{K}(X) := K(X)/\mathbb{Z} \qquad \left(\widetilde{KO}(X) := KO(X)/\mathbb{Z} \right)$$

Proposition: In reduced K-theory,

$$E = F \in \widetilde{K}(X) \quad (\widetilde{KO}) \iff \exists\, n, m \geq 0 : E \oplus \underline{n} \simeq F \oplus \underline{m}.$$

Lemma: Homotopic maps induce the same maps in K-theory:

$$f \simeq g : X \to Y \quad \to \quad f^* = g^* : K(Y) \to K(X) \quad (KO)$$

In particular, K, KO, $\widetilde{K}$, $\widetilde{KO}$ define functors $CH \to Ring / Ab$ (depending whether we consider reduced or unreduced K-theory). Why isn't $\widetilde{K}$, $\widetilde{KO}$ rings? Well, $\mathbb{Z} \hookrightarrow K(X)$ doesn't have to be an ideal.

But: if we pick a preferred basepoint x_0, then the inclusion $x_0 \hookrightarrow X$ induces a map

$$K(X) \xrightarrow{\dim} K(*) = \mathbb{Z}, \quad E - F \longmapsto \dim E_{x_0} - \dim F_{x_0}$$

which is a retraction to $\mathbb{Z} \hookrightarrow K(X)$, $n \mapsto \underline{n}$. In particular, this gives a split ses

$$0 \to \mathbb{Z} \hookrightarrow K(X) \to \widetilde{K}(X) \to 0$$

so:

$$\boxed{K(X) \simeq \widetilde{K}(X) \oplus \mathbb{Z}}, \qquad \boxed{KO(X) \simeq \widetilde{KO}(X) \oplus \mathbb{Z}}$$

(splits are not natural, depend on the choice of a basepoint, more exactly, of its connected component). In particular, this says that $\operatorname{Ker} r \overset{\pi}{\simeq} \widetilde{K}(X)$, so since r is a ring hom, $\operatorname{Ker} r$ is an ideal, so $\widetilde{K}(X)$ inherits a multiplication. The moral is:

Proposition: If (X, x_0) is a pointed space, then $\widetilde{K}(X)$ and $\widetilde{KO}(X)$ are rings and define functors

$$\widetilde{K}, \widetilde{KO} : CH_*^{op} \to Ring.$$

LONG EXACT SEQUENCE

Theorem: let X be a compact Hausdorff space and $A \subseteq X$ a closed subspace (thus compact Hausdorff). Then the sequence $A \xhookrightarrow{i} X \xrightarrow{\pi} X/A$ induces an exact sequence

$$\widetilde{K}(X/A) \xrightarrow{\pi^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A)$$

; note that X/A is also compact Hausdorff).

Lemma: If $A \subset X$ is a contractible closed subspace, then the projection $\pi : X \to X/A$ induces a bijection $\pi^* : Vect^*(X/A) \xrightarrow{\sim} Vect^*(X)$, hence isomorphisms $\pi^* : K(X/A) \xrightarrow{\sim} K(X)$, in reduced and unreduced, complex and real).

Definition: let X be a space. The cone of X is $CX := \dfrac{X \times I}{X \times \{1\}}$, the suspension of X is $SX := \dfrac{CX}{X \times \{0\}}$; and the reduced suspension of a pointed space (X, x_0) is $\Sigma X = \dfrac{SX}{\{x_0\} \times I}$.

- If $(X, \{x_0\})$ has the HEP, then $SX \simeq \Sigma X$ (because $\{x_0\} \times I$ is contractible).

Definition: let (X, x_0), (Y, y_0) be pointed spaces. The wedge sum is $X \vee Y = \dfrac{X \sqcup Y}{x_0 \sim y_0}$, and the smash product $X \wedge Y := \dfrac{X \times Y}{X \vee Y}$.

Lemma: 1) $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$

2) $SS^n = S^{n+1}$, $\Sigma S^n = S^{n+1}$, $S^n \wedge S^m = S^{n+m}$.

3) $S^n \wedge X = \Sigma^n X$

Theorem: let X be a compact Hausdorff space and $A \subset X$ closed. There is a l.e.s in reduced K-theory

$$\widetilde{K}(S^2(X/A)) \to \widetilde{K}(S^2X) \to \widetilde{K}(S^2A) \to$$
$$\to \widetilde{K}(S(X/A)) \to \widetilde{K}(SX) \to \widetilde{K}(SA) \to$$
$$\to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A)$$

Corollary: $\widetilde{K}(X \vee Y) \simeq \widetilde{K}(X) \oplus \widetilde{K}(Y)$.

Corollary: There is a split ses

$$0 \to \widetilde{K}(S^n(X \wedge Y)) \to \widetilde{K}(S^n(X \times Y)) \to \widetilde{K}(S^n(X \vee Y)) \to 0.$$

Example: $K(X \amalg Y) = K(X) \oplus K(Y)$.

PRODUCT THEOREM

Lemma: let A, B, C be R-algebras. Then there is an R-algebra isomorphism

$$\begin{aligned} \mathrm{Hom}_{R\text{-alg}}(A \otimes_R B, C) &= \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C) \\ \phi &\longmapsto (\phi_1(a) := \phi(a \otimes 1),\ \phi_2(b) := \phi(1 \otimes b)) \\ \phi(a \otimes b) := \phi_1(a)\phi_2(b) &\longleftarrow (\phi_1, \phi_2) \end{aligned}$$

Definition: let X, Y be compact, Hausdorff spaces and let $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$. The **external product** is the unique $\mathbb{Z}$-algebra homomorphism

$$\mu : K(X) \otimes K(Y) \to K(X \times Y)$$

corresponding with the pair (π_1^*, π_2^*), that is, $\mu(a \otimes b) = \pi_1^*(a)\, \pi_2^*(b)$.

Example: $\pi_2^*(\underline{1} = Y \times \mathbb{C}) = \underline{1} = (X \times Y) \times \mathbb{C}$; thus $\mu(E \otimes \underline{1}) = \pi_1^*(E)$, for any vs $E \to X$.

Lemma. let E_g, E_f be vs over S^k comy from clutching factors $f, g : S^{k-1} \to GL_n(\mathbb{C})$. Then

$$E_{fg} \oplus \underline{n} \simeq E_f \oplus E_g.$$

Proposition: let H the tautological line bundle over $\mathbb{CP}^1 \simeq S^2$. Then the equation $(H-1)^2 = 0$ holds in K-theory, thus there is a well defined $\mathbb{Z}$-alg homomorphism

$$\frac{\mathbb{Z}[H]}{(H-1)^2} \xrightarrow{\ i\ } K(S^2)$$

* **Theorem (Fundamental Product)**: Let X be a compact Hausdorff space. Then the composite

$$K(X) \otimes \frac{\mathbb{Z}[H]}{(H-1)^2} \xrightarrow{\mathrm{Id} \otimes i} K(X) \otimes K(S^2) \xrightarrow{\mu} K(X \times S^2)$$

is a $\mathbb{Z}$-alg. isomorphism.

- The proof is boring and requires a lot of work, I will skip everything.

Corollary: $K(S^2) \simeq \frac{\mathbb{Z}[H]}{(H-1)^2}$

Corollary: The external product $\mu : K(X) \otimes K(S^2) \simeq K(X \times S^2)$ is a $\mathbb{Z}$-alg isomorphism.

Corollary: $\widetilde{K}(S^2) \simeq \mathbb{Z}$

Corollary (Bott periodicity): $\boxed{\widetilde{K}(S^n) = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}}$

- Do we have an external product in reduced K-theory? Yes! : There is a split ses (seen before)

$$0 \to \widetilde{K}(X \wedge Y) \xrightarrow{q^*} \widetilde{K}(X \times Y) \xrightarrow{i_X^* \oplus i_Y^*} \widetilde{K}(X) \oplus \widetilde{K}(Y) \to 0 .$$

This, together with the fact that $K(X) \simeq \widetilde{K}(X) \oplus \mathbb{Z}$, produces the diagram

$$\begin{array}{ccccccc}
K(X) \otimes K(Y) \simeq & \left(\widetilde{K}(X) \otimes \widetilde{K}(Y)\right) & \oplus & \widetilde{K}(X) & \oplus & \widetilde{K}(Y) & \oplus \ \mathbb{Z} \\
\mu \downarrow & \tilde{\mu} \downarrow & & \| & & \| & \| \\
K(X \times Y) \simeq & \widetilde{K}(X \wedge Y) & \oplus & \widetilde{K}(X) & \oplus & \widetilde{K}(Y) & \oplus \ \mathbb{Z}
\end{array}$$

Proposition: $\tilde{\mu} := \mu|_{\widetilde{K}(X) \otimes \widetilde{K}(Y)}$ is a ring hom and lands in $\widetilde{K}(X \wedge Y)$,

$$\tilde{\mu} : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$$ <u>reduced external product</u>

• We will denote by $*$ both μ and $\tilde{\mu}$.

• Since $S^n \wedge X = \Sigma^n X = S^n X / \mathbb{I}^n$, then the quotient map $S^n X \to S^n X/\mathbb{I}^n = S^n \wedge X$ induces iso in $\widetilde{K}$, since $\mathbb{I}^n$ is contractible. Let β be the composite.

$$
\begin{tikzcd}
\widetilde{K}(X) \arrow[r] \arrow[rrr, bend right, "\beta"'] & \widetilde{K}(S^2) \otimes \widetilde{K}(X) \arrow[r] & \widetilde{K}(S^2 \wedge X) = \widetilde{K}(S^2 X)
\end{tikzcd}
$$

where the first map is $a \mapsto (H-1) * a$.

***Theorem** (Bott Periodicity): $\beta : \widetilde{K}(X) \xrightarrow{\sim} \widetilde{K}(S^2 X)$ is an isomorphism.

Corollary: If $n \in \mathbb{N}$, $\widetilde{K}(S^{2n}) \simeq \mathbb{Z}$ is generated by the n-fold reduced external product $(H-1) * \overset{n}{\cdots} * (H-1)$

K-THEORY AS A COHOMOLOGY THEORY

Definition: A reduced generalized cohomology theory on pointed CW-complexes is a sequence of contravariant functors $\{\tilde{h}^n : CW_* \to AbGps : n \geq 0\}$ satisfying the Eilenberg - Steenrod axioms:

i) (Homotopy invariance): $f \simeq g : X \to Y \Rightarrow f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)$.

ii) (Long exact sequence): For a CW pair (X, A), there are natural group homomorphisms, called coboundary maps $\delta : \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A)$, fitting in a les

$$
\cdots \to \tilde{h}^n(X/A) \to \tilde{h}^n(X) \to \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A) \to \cdots
$$

iii) (Sums) For a wedge sum $X = \bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow X$ induce iso

$$
\prod_\alpha i_\alpha^* : \tilde{h}^n(X) \to \prod \tilde{h}^n(X_\alpha)
$$

• How to come up with a cohomology theory out of K-theory? Inspired by the les in reduced K-theory and the ES axioms,

Definition: Set $\tilde{K}^{-n}(X) := \tilde{K}(S^n X)$ and the relative K theory as $\tilde{K}^{-n}(X,A) := \tilde{K}(S^n(X/A))$ with $n \geq 0$.

• With this notation we get

$$
\begin{tikzcd}
 & \tilde{K}(X) \arrow[d, "\beta \,\wr\, \text{Bott}"] & \tilde{K}^0(A) \arrow[d, "\beta \,\wr\, \text{Bott}"] \\
\cdots \arrow[r] & \tilde{K}^{-2}(X) \arrow[r] & \tilde{K}^{-2}(A) \arrow[dll, rounded corners] \\
\tilde{K}^{-1}(X,A) \arrow[r] & \tilde{K}^{-1}(X) \arrow[r] & \tilde{K}^{-1}(A) \arrow[dll, rounded corners] \\
\tilde{K}^{0}(X,A) \arrow[r] & \tilde{K}^{0}(X) \arrow[r] & \tilde{K}^{0}(A).
\end{tikzcd}
$$

The negative coefficients were introduced to have a cohomology rising the degree, but Bott periodicity allows to do

Definition. Let $n \geq 0$. Then set $\tilde{K}^{2n}(X) := \tilde{K}(X) = \tilde{K}^0(X)$ and $\tilde{K}^{2n+1}(X) := \tilde{K}(SX) = \tilde{K}^{-1}(X)$

• Therefore, the les of the $\tilde{K}^n$ groups boils down to a 6-terms exact sequence:

$$
\begin{tikzcd}
\tilde{K}^0(X,A) \arrow[r] & \tilde{K}^0(X) \arrow[r] & \tilde{K}^0(A) \arrow[dll, rounded corners] \\
\tilde{K}^1(X,A) \arrow[r] & \tilde{K}^1(X) \arrow[r] & \tilde{K}^1(A) \arrow[ull, rounded corners]
\end{tikzcd}
$$

• The reduced external product induces a product between K-theory groups,

$$\tilde{K}^n(X) \otimes \tilde{K}^m(Y) \to \tilde{K}^{n+m}(X \wedge Y)$$

Definition. The (reduced) K-theory ring is $\tilde{K}^*(X) := \tilde{K}^0(X) \oplus \tilde{K}^1(X)$.

• This defines a product $\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \to \tilde{K}^*(X \wedge Y)$. If $Y = X$, composing with the map $\tilde{K}^*(X \wedge X) \to \tilde{K}^*(X)$ induced by the diagonal map $X \to X \wedge X$, $x \mapsto [(x,x)]$, we get a multiplication $\tilde{K}^*(X) \otimes \tilde{K}^*(X) \to \tilde{K}^*(X)$ making $\tilde{K}^*(X)$ into a (graded) ring, which generalizes the multiplication on $\widetilde{K}(X)$.

- We also have products in the relative version of K-theory; namely

$$\widetilde{K}^0(X,A) \otimes \widetilde{K}(Y,B) \longrightarrow \widetilde{K}(X \times Y, X \times B \cup A \times Y),$$

using the identification $X/A \wedge Y/B = (X \times Y) / X \times B \cup A \times Y$.

- In the case $Y = X$, composing with the corresponding diagonal map, we get a product

$$\widetilde{K}^0(X,A) \otimes \widetilde{K}(X,B) \longrightarrow \widetilde{K}^0(X, A \cup B).$$

<u>Proposition</u>: If $X = A \cup B$ is union of two compact contractible subspaces, then the product of $\widetilde{K}^0(X)$ is trivial.

<u>Corollary</u>: The product of $\widetilde{K}^0(S^n) \simeq \mathbb{Z}$ is trivial.

* <u>Theorem</u>: The functors $\widetilde{K}^n(-)$ define a reduced generalized cohomology theory.

- But caution! K-theory does not satisfy the dim axiom! So $\widetilde{K}^n \neq \widetilde{H}^n$!

<u>Proposition</u>: 1) $\widetilde{K}^p(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}^n, & p = 0 \\ 0, & p = 1. \end{cases}$

2) $\widetilde{K}^p(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & p = 0 \\ 0, & p = 1. \end{cases}$

- Observe that for a pair (X,A), $\widetilde{K}^0(A)$ can be viewed as a $\widetilde{K}^0(X)$-module, by setting $a \cdot b := i^*(a) \cdot b$ and: the diagonal $X \to X \wedge X$ induces a quotient map $X/A \to X \wedge (X/A)$. Similarly, $\widetilde{K}(X,A)$ is also a $\widetilde{K}^0(X)$ mod and this a product $\widetilde{K}^0(X) \otimes \widetilde{K}^0(X,A) \to \widetilde{K}(X,A)$.

<u>Theorem</u>: The following is a exact triangle of $\widetilde{K}^0(X)$-modules:

$$\begin{tikzcd} \widetilde{K}^0(X,A) \arrow[rr] & & \widetilde{K}^0(X) \arrow[dl] \\ & \widetilde{K}^0(A) \arrow[ul] & \end{tikzcd}$$

• Let's describe an unreduced version of the groups $\widetilde{K}^n$:

<u>Definition</u>: The <u>higher unreduced K-theory groups of X</u> are

$$K^n(X) := \widetilde{K}^n(X_*)$$

where $X_* = X \amalg *$. For a pair of spaces (X,A), set

$$K^n(X,A) := \widetilde{K}^n(X,A)$$

<u>Lemma</u>: $K^0(X) = K(X)$, $K^1(X) = \widetilde{K}^1(X)$, and the six-term les is still valid for unreduced groups.

THE SPLITTING PRINCIPLE

Definition: A finite cell complex is a top space X together with a filtration

$$X_0 \subset X_1 \subset \dots \subset X_k = X$$

st X_0 is a finite discrete space and X_n arises from X_{n-1} by attaching m_i-cells (of any possible dimension)

- Any finite cell complex is homotopy equivalent to a CW-complex.

Proposition: If X is a finite cell complex, then $\tilde{K}^0(X)$ is a finitely generated group with at most as many generators as the number of cells of X.

If all cells have even dimension, then

$$K^0(X) = \bigoplus_{\# \text{cells}} \mathbb{Z} \qquad , \qquad K^1(X) = 0 .$$

Theorem: $K(\mathbb{C}P^n) = \dfrac{\mathbb{Z}[x]}{(x^{n+1})}$, where $x = L - 1$, and L is the canonical line bundle over $\mathbb{C}P^n$

* **Theorem** (Leray-Hirsch for K-theory): Let $E \to B$ be a fiber bundle with typical fiber F, with E and B Hausdorff compact, such that $K^0(F)$ is a free abelian group. Suppose that there exist classes $c_1, \dots, c_k \in K^0(E)$ which restrict to a basis for $K^0(F)$ in each fiber. If either

(a) B is a finite cell complex

(b) F is a finite cell complex having all cells of even dimension

then $K^0(E)$ is a free $K^0(B)$-module with basis $\{c_1, \dots, c_k\}$.

Example: Let $E \to X$ be a v.b of constant rank n, thus fiber bundle with typical fiber $\mathbb{C}^n$.

Let $E^* = E$ − zero-section and define the projective bundle $P(E) \to X$ as

$$P(E) = E^*/\sim, \qquad v \sim \lambda v, \quad v \in E_x, \ \lambda \in \mathbb{C}^*,$$

so that $P(E) \to X$ is a fiber bundle with typical fiber $\mathbb{CP}^{n-1}$.

*Theorem (Splitting Principle): Given a vb $E \to X$ over a compact Hausdorff space X, there exists a compact Hausdorff space $F(E)$ and a map $p: F(E) \to X$ st

1) The induced map $p^*: K^0(X) \to K^0(F(E))$ is injective
2) $p^*(E)$ splits as a direct sum of line bundles.

ADAMS OPERATIONS

*Theorem: Let X be a compact Hausdorff space. There exist ring homomorphisms (wrt * external product)

$$\psi^k: K(X) \to K(X), \quad k \geq 0$$

satisfying

1) $\psi^k \circ f^* = f^* \circ \psi^k$, $\quad f: X \to Y$, ie, ψ^k are natural
2) $\psi^k(L) = L^k$, for L a line bundle
3) $\psi^k \circ \psi^l = \psi^{kl}$
4) If p prime, then $\psi^p(\alpha) = \alpha^p + p\,\beta$ for some $\beta \in K(X)$.

These are called the Adams operations.

Proposition: ψ^k restricts to operations $\psi^k: \widetilde{K}(X) \to \widetilde{K}(X)$ satisfying the same properties.

Example: If X is contractible, $K(X) = \mathbb{Z}$, then $\psi^k = \mathrm{Id}$ $\forall k$, since there's a unique ring hom. $\mathbb{Z} \to \mathbb{Z}$.

Proposition: $\psi^k : K(S^{2n}) = \mathbb{Z} \longrightarrow \mathbb{Z} = K(S^{2n})$ is multiplication by k^n.

APPLICATION: DIVISION ALGEBRAS AND PARALLELIZABLE SPHERES

Definition: A division ring R is a non-zero ring with no trivial zero-divisors and where every non-zero element is invertible. If R is also a K-algebra, then we say that R is a division algebra (over K).

If R is associative, then the condition of not having non-trivial zero-divisors follows from the second one.

Q: For what $n \geq 0$ is $\mathbb{R}^n$ a division algebra?

Examples: 1) $\mathbb{R}$ and $\mathbb{C} \simeq \mathbb{R}^2$ are fields, thus division algebras.

2) Quaternions $\mathbb{H}$, (Hamilton 1843) $\mathbb{R}^4$ with basis $\{1, i, j, k\}$ and identities

$$i^2 = j^2 = k^2 = -1, \qquad ijk = -1.$$

3) Octonions $\mathbb{O}$ (Graves, Cayley): $\mathbb{R}^8$.

Definition: An H-space (H after Hopf) is a pointed topological space (X, e) together with a continuous map $\mu : X \times X \to X$, $\mu(x,y) \overset{\text{not}}{=} xy$ such that e acts as a unit: $xe = x = ex$, $\forall x \in X$.

Examples: 1) Every topological group is an H-space: $(\mathbb{R}, +)$, $(\mathbb{C}, +)$, $(\mathbb{R}^*, \cdot)$, $GL_n(\mathbb{R})$, ...

2) $S^0 \subset \mathbb{R}$, $S^1 \subset \mathbb{C}$, $S^3 \subset \mathbb{H}$, $S^7 \subset \mathbb{O}$ are H-spaces with the multiplication restricted to the sphere.

Lemma: If $\mathbb{R}^n$ is a division algebra, then S^{n-1} is an H-space.

• So we have just translated our original algebraic problem to a topological one!

Proposition. The even spheres S^{2k} cannot be H-spaces, $k>0$.

• (The Hopf invariant): Let $f: S^{2n-1} \to S^n$ (and $n \geq 2$ even) be a map between spheres that we think as an attaching map for $X = S^n \cup_f \mathbb{D}^{2n}$, which is a CW-complex with 1 0-cell, 1-n-cell and 1-2n-cell. The les of the pair (X, S^n) provides a ses

$$0 \to \tilde{K}(S^{2n}) = \mathbb{Z} \xrightarrow{\pi^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(S^n) = \mathbb{Z} \to 0$$

for $S^n \xrightarrow{i} X \xrightarrow{\pi} X/S^n = S^{2n}$. Let $x \in \tilde{K}(S^{2n})$ be the gen n-fold of the gen of $\tilde{K}(S^2)$, and $y \in \tilde{K}(S^n)$ the $(n/2)$-fold, generator too. Since i^* surj, let $a \in \tilde{K}(X)$ be a lift of y, $i^*a = y$; and let $b := \pi^* x$. Since $\tilde{K}(S^n) = \mathbb{Z}$ is a free $\mathbb{Z}$-module, the previous sequence splits, so a and b generate $\tilde{K}(X)$. ~~Also~~ Now, since the ring structure of $\tilde{K}(S^{n})$ is trivial (pag. 9.2), in particular $y^2=0$, thus $a^2 \in \operatorname{Ker} i^* = \operatorname{Im} \pi^*$, so $\exists\, K \in \mathbb{Z}$ st $a^2 = \pi(Kx) = Kb$.

Lemma: The previous integer K is well-defined.

Definition. The previous integer k is called the Hopf invariant of $f: S^{2n-1} \to S^n$, and it is denoted as $h(f)$.

Example. Consider the Hopf map $\eta: \partial D^4 = S^3 \to S^2$, attaching map of $\mathbb{CP}^2$ ~~from~~ arising from $\mathbb{CP}^1$. Then $h(\eta) = 1$.

Proposition: Let $n \geq 2$ be even. If S^{n-1} has an H-space structure, then there exists a map $f: S^{2n-1} \to S^n$ with Hopf invariant ± 1.

*Theorem (Adams-Atiyah, 1964) : Let $n \geq 2$ be even. If there exists a map $f: S^{2n-1} \to S^n$ with Hopf invariant ± 1, then $n = 2, 4, 8$.

To conclude we need

Lemma : If 2^m divides $3^m - 1$, then $m = 1, 2, 4$.

Definition : A smooth manifold M is parallelizable if there exists a global basis of vector fields, ie, if $\mathfrak{X}(M)$ is a free $\mathcal{C}^\infty(M)$-module of rank $m = \dim M$.

Obviously, being parallelizable is equivalent to say that TM is trivializable.

Q : What spheres are parallelizable ?

Lemma : If S^{n-1} is parallelizable, then it is an H-space.

Corollary : The following statements occur only for $n = 1, 2, 4, 8$:

1) $\mathbb{R}^n$ is a division algebra
2) S^{n-1} is an H-space
3) S^{n-1} is parallelizable
4) $\pi_{2n-1}(S^n)$ contains an elmt with Hopf invariant (not for $n=1$ with the def we gave).

IV: CHARACTERISTIC CLASSES

- In this chapter all spaces will be considered to be paracompact.
- In general, characteristic classes of vector bundles will measure triviality or the number of l.i. sections that a vb has over the various skeleta of a CW-complex.

<u>Definition</u>: A <u>characteristic class</u> is a function associating to each vector bundle $E \to B$ of dim n a class $x(E) \in H^k(B;A)$ for some $n,k \in \mathbb{N}$ and ab grp A, with the naturality property that $f^*(x(E)) = x(f^*(E))$.

- Recall that $Vect^n_{\mathbb{R}}(X) \cong [X, BO(n)]$ and $Vect^n_{\mathbb{C}}(X) \cong [X, BU(n)]$ (also called G_n).

For the universal bundle $E_n \to G_n$ there is a char class $x = x(E_n) \in H^k(G_n;A)$, which also determine any char. class over any $E \to B$ since $x(E) = x(f^*(E_n)) = f^*(x)$, since any E is f^*E_n for some $f: B \to G_n$.

STIEFEL-WHITNEY CLASSES AND CHERN CLASSES

<u>Theorem</u>: There is a unique sequence of functions $w_1, w_2, \dots$ assigning to each real vector bundle $E \to B$ a class $w_i(E) \in H^i(B;\mathbb{Z}/2)$, depending only on the isomorphism class of E, such that

i) $w_i(f^*(E)) = f^*(w_i(E))$,

ii) $w(E \oplus E') = w(E) \cup w(E')$, where $w(E) = 1 + w_1(E) + w_2(E) + \dots \in H^*(B;\mathbb{Z}/2)$.

iii) $w_i(E) = 0$ if $i > \dim E$

iv) if $\gamma \to \mathbb{R}P^\infty$ is the taut. line bundle, then $w_1(\gamma)$ is a generator of $H^1(\mathbb{R}P^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

Each of the classes $w_i(E)$ is the <u>i-th Stiefel-Whitney class</u> of E, and $w(E) := 1 + w_1(E) + \dots$ (the sum is finite by iii)) is the <u>total Stiefel-Whitney class</u> of E

- The analogues for complex vb are the Chern classes:

Theorem: There is a unique sequence of functions $c_1, c_2, \dots$ assigning to each complex vb $E \to B$ a class $c_i(E) \in H^{2i}(B;\mathbb{Z})$, depending only on the isomorphism class of E, such that

i) $c_i(f^*(E)) = f^*(c_i(E))$

ii) $c(E \oplus E') = c(E) \cup c(E')$, $c(E) = 1 + \sum c_i(E)$

iii) $c_i(E) = 0$ if $i > \dim E$

iv) If $\gamma \to \mathbb{C}P^\infty$ is the tautological line bundle, $c_1(\gamma)$ is a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}/2$ specified in advance.

We call $c_i(E)$ the <u>i-th Chern class</u>, and $c(E) := 1 + c_1(E) + c_2(E) \dots \in H^*(B;\mathbb{Z})$ is the <u>total Chern class</u>.

<u>Corollary</u>: $w_i(\underline{n}) = 0$, $c_i(\underline{n}) = 0$, $\underline{n}$ = trivial vb of dim n.

<u>Corollary</u>: If E is stably trivial, then $w_i(E) = 0$.

<u>Corollary</u>: If $E \oplus E'$, then the SW (C) class of E determine the SW (C) class of E'. Concretely, $w(E') = \overline{w}(E)$, where $\overline{w} = 1 + \overline{w}_1 + \dots$ can be computed inductively from a relation

<u>Corollary</u> (Whitney Duality): Let M be a Riemannian manifold, TM the tangent bundle and NM the normal bundle. Then $w_i(NM) = \overline{w}_i(TM)$.

<u>Corollary</u>: If $\gamma \to \mathbb{R}P^n$ is the tautol. line bundle over $\mathbb{R}P^n$, then $w(\gamma) = 1 + a$, where $a = w_1(\gamma)$ is the gen of $H^1(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

• If M is a smooth manifold, we let $w_i(M) := w_i(TM)$.

<u>Proposition</u>: $T\mathbb{R}P^n \oplus \underline{1} \cong \gamma \oplus \dots \oplus \gamma$, so $w(\mathbb{R}P^n) = (1+a)^{n+1} = 1 + \sum_{i=1}^{n} \binom{n+1}{i} a^i$.

<u>Corollary</u> (Stiefel): $w(\mathbb{R}P^n) = 1 \iff n+1$ is a power of 2. So the only candidates of $\mathbb{R}P^n$ to be parallelizable are $\mathbb{R}P^1$, $\mathbb{R}P^3$, $\mathbb{R}P^7$, $\mathbb{R}P^{15}$, ...

<u>Corollary</u>: If $\mathbb{R}P^{2^r}$ can be immersed in $\mathbb{R}^{2^r + k}$, then k must be at least $2^r - 1$.

• Let us briefly explain how SW and C classes are defined. They are based in the theorem below.

• Observe that if $p: E \to B$ is a fibre bundle, and R is a ring, then we can make $H^*(E;R)$ into a $H^*(B;R)$-module by setting $b \cdot e := p^*b \cup e$, where $b \in H^*(B;R)$ and $e \in H^*(E;R)$.

<u>Theorem</u> (Leray-Hirsch): Let $p: E \to B$ be a fibre bundle with typical fibre F. If the following holds,

a) The map $i^*: H^n(E;R) \to H^n(F;R)$ induced by the inclusion $i: F \hookrightarrow E$ is surjective $\forall n$,

b) $H^n(F;R)$ is a free R-module of finite rank $\forall n$,

then $H^*(E;R)$ is a free $H^*(B;R)$-module, where a basis is given by lifts of the basic elements of $H^*(F;R)$. In other words, the map

$$\begin{aligned} H^*(B;R) \otimes_R H^*(F;R) &\xrightarrow{\cong} H^*(E;R) \\ b \otimes i^*(x) &\longmapsto p^*b \cup x \end{aligned}$$

is an isomorphism.

• Given an n-dim vector bundle $E \to B$, get the projective fibre bundle $P(E) \to B$, with typical fibre $\mathbb{RP}^{n-1}$. There are classes $x_i \in H^i(P(E); \mathbb{Z}/2)$ restricting to a gen. of $H^i(\mathbb{RP}^{n-1}; \mathbb{Z}/2)$ in each fibre for $i = 0, \dots, n-1$. If $x = x_1$, then $x_i = x^i$. By the Leray-Hirsch theorem, $H^*(P(E);\mathbb{Z}/2)$ is a free $H^*(B;\mathbb{Z}/2)$-module with basis $\{1, x, \dots, x^{n-1}\}$, so x^n can be expressed as a linear combination

$$x^n + w_1(E)\, x^{n-1} + \dots + w_n(E) \cdot 1 = 0 .$$

• There is an obvious similarity between SW and C classes. It is even better:

<u>Theorem</u>: View an n-dim complex vector bundle $E \to B$ as a $2n$-dim real v.b. Then $w_{2i+1}(E) = 0$ and if $\pi: H^{2i}(B;\mathbb{Z}) \to H^{2i}(B;\mathbb{Z}/2)$ is the morphism induced by $\mathbb{Z} \to \mathbb{Z}/2$, then $w_{2i}(E) = \pi(c_i(E))$.

• As in the introduction, set $w_i = w_i(E_n)$, where $E_n \to G_n = BO(n)$ is the universal bundle.
Similarly set $c_i = c_i(E_n)$, here $E_n \to G_n = BU(n)$.

<u>Theorem</u>: 1) $H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \dots, w_n]$, and

2) $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$.

<u>Lemma</u>: The set $Vect^1(B)$ of line bundles over B is an ab. gp wrt $\otimes$ (and any element is its inverse)

<u>Theorem</u>: Suppose X has the homotopy type of a CW-complex. Then the first SW class

$$w_1 : Vect^1_{\mathbb{R}}(X) \xrightarrow{\cong} H^1(X; \mathbb{Z}/2)$$

and the first Chern class

$$c_1 : Vect^1_{\mathbb{C}}(X) \xrightarrow{\cong} H^2(X; \mathbb{Z})$$

are group isomorphisms.

<u>Proposition</u>: Suppose that X has the hty type of a CW-cx. Then $E \to X$ is orientable $\iff w_1(E) = 0$.

<u>THOM ISOMORPHISM AND CHERN CHARACTER</u>

<u>Definition</u>: An n-dim real vector bundle $E \to B$ is <u>cohomologically R-orientable</u>, R ring, if there is a cohomology class $\mu \in H^n(D(E), S(E); R)$ such that $\mu_x := i^*\mu \in H^n(D(E_x), S(E_x); R)$, $x \in B$, is a generator.

<u>Proposition</u>: $E \to B$ is orientable $\iff$ E is cohom. $\mathbb{Z}$-orientable.

• <u>Theorem</u> (Thom): let $p: E \to B$ be a n-dim orientable vector bundle, over a compact base B. There exists a cohomology class $u \in H^n(E, E - B; \mathbb{Z})$, called the <u>Thom class</u>, which restricts to μ_x for any $x \in B$. Moreover,

$$\begin{array}{ccc} H^k(B; \mathbb{Z}) & \xrightarrow{\cong} & H^{n+k}(D(E), S(E); \mathbb{Z}) \cong H^{n+k}(T(E); \mathbb{Z}) \\ \alpha & \longmapsto & p^*\alpha \vee u \end{array}$$

is an isomorph $\forall i$, and $T(E) := D(E)/S(E)$ is the <u>Thom space</u> of E.

• Actually, the previous Thom isomorphism also holds in complex K-theory, and with cohomology with coeff in $\mathbb{Q}$. We would like to compare these two isos.

<u>Proposition</u>: There is a unique natural ring homomorphism, called the <u>Chern character</u>,

$$ch : K(X) \longrightarrow H^{*}(X;\mathbb{Q})$$

which satisfies $ch([L]) = e^{c_1(L)} = 1 + c_1(L) + \frac{1}{2!}c_1(L)^2 + \dots$ for a line bundle L.

• By naturality, we also get a map $ch: \widetilde{K}(X) \to \widetilde{H}^{*}(X;\mathbb{Q})$.

<u>Proposition</u>: The map $ch: \widetilde{K}(S^{2n}) \longrightarrow H^{2n}(S^{2n};\mathbb{Q})$ is injective with image $\widetilde{H}^{2n}(S^{2n};\mathbb{Z}) \subset \widetilde{H}^{2n}(S^{2n};\mathbb{Q})$.

<u>Proposition</u>: If X has the htpy type of a CW-complex, the map

$$K^{*}(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^{*}(X;\mathbb{Q})$$

induced by ch is an isomorphism. More precisely, $K(X)\otimes\mathbb{Q} \cong \bigoplus_{n\in\mathbb{N}} H^{2n}(X;\mathbb{Q})$ and $K^{1}(X)\otimes\mathbb{Q} \cong \bigoplus_{n\in\mathbb{N}} H^{2n+1}(X;\mathbb{Q})$.

• Let $p: E \to X$ be a complex VB. It can be shown that it always has a coh. orientation wrt K and $H(-;\mathbb{Q})$, so we have Thom isomorphisms. We would like to compare the composites

$$K(X) \xrightarrow[\cong]{\text{Thom}} K(T(E)) \xrightarrow{ch} H^{*}(T(E);\mathbb{Q}) \quad , \quad K(X) \xrightarrow{ch} H^{*}(X;\mathbb{Q}) \xrightarrow[\cong]{\text{Thom}} H(T(E);\mathbb{Q}).$$

It turns out that the failure of being equal is mesured by a charcteristic class, called the Todd class.

<u>Theorem</u>: There exists a unique function td assigning to every complex vector bundle $E \to X$ a class $td(E) \in H^{*}(X;\mathbb{Q})$, called the <u>Todd class</u>, satisfying

1) $f^{*}(td(E)) = td(f^{*}E)$,

2) $td(E\oplus E') = td(E) \smile td(E')$

3) $td(L) = \dfrac{c_1(L)}{1-e^{c_1(L)}}$, for a line bundle $L \to X$.

Proposition: The previous two composites are the same up to multiplication by $td(E)^{-1}$, ie, the following diagram commutes:

$$\begin{tikzcd} K(X) \arrow[r, "\cong"] \arrow[d, "ch"'] & K(T(E)) \arrow[r, "ch"] & H^{\bullet}(T(E);\mathbb{Q}) \\ H^{\bullet}(X;\mathbb{Q}) \arrow[r, "\cdot td(E)^{-1}"] & H^{\bullet}(X;\mathbb{Q}) \arrow[ur, "\cong"', "Thom"] & \end{tikzcd}$$

• We finish with an important theorem relating vb with algebraic geometry. If $\mathcal{F}$ is a coherent sheaf on X, its Euler char is $\chi(\mathcal{F}) := \sum_i (-1)^i \, rk(H^i(X;\mathcal{F}))$.

Theorem (Hirzebruch-Riemann-Roch): let $E \to M$ be a cx vb over a compact complex manifold M. If $\mathcal{E}$ is the sheaf of holomorphic sections of E, then

$$\chi(\mathcal{E}) = \int_M ch(E) \cup td(TM).$$

• The classical Riemann-Roch thm can be recovered from this.

Remark: The theorem above about c_1 and w_1 gives an alternative way to define them, given the bijections

$$\check{H}^1(X, \mathcal{C}(X,\mathbb{Z}/2); \mathcal{U}) \cong Vect^1_{\mathbb{R}}(X;\mathcal{U}) \quad , \quad \check{H}^2(X, \mathcal{C}(X;\mathbb{Z}); \mathcal{U}) \cong Vect^1_{\mathbb{C}}(X;\mathcal{U})$$

where $\mathcal{C}(X,\mathbb{Z}/2) \cong \mathbb{Z}/2$, $\mathcal{C}(X;\mathbb{Z}) \cong \mathbb{Z}$, $\mathcal{U}$ is a good cover and $Vect^1_{\mathbb{F}}(X;\mathcal{U})$ denotes the set of iso classes of line bundles that can be trivialized over $\mathcal{U}$.

EULER AND PONTRYAGIN CLASSES

• We will describe more refined char classes for real vector bundles, taking $\mathbb{Z}$ coeff rather than $\mathbb{Z}/2$.

Definition: let $E \to B$ be a orientable n-dimensional vb. The Euler class $e(E)$ is the image of the Thom class $u \in H^n(D(E), S(E); \mathbb{Z})$ under the composite

$$H^n(D(E), S(E); \mathbb{Z}) \longrightarrow H^n(D(E);\mathbb{Z}) \longrightarrow H^n(B;\mathbb{Z})$$

where the first arrow is the usual passage in cohom. and the second is induced by the inclusion $B \hookrightarrow D(E)$ as the 0-section.

<u>Lemma</u>: $e(E)$ only depends on the choice of orientation of E (and its iso class).

<u>Proposition</u> (Properties): Let $E \to B$ be a n-dim vb.

1) $e(f^*E) = f^*(e(E))$, where f^*E has the orientation induced by the pullback and E

2) $e(E \oplus E') = e(E) \cup e(E')$, where $E \oplus E'$ has the orientation induced by E and E'.

3) $w_n(E) = \pi(e(E))$, where $\pi : H^n(B;\mathbb{Z}) \to H^n(B;\mathbb{Z}/2)$ is induced by $\mathbb{Z} \to \mathbb{Z}/2$.

4) $c_n(E) = e(E)$, if E has a suitable choice of orientation (E viewed as $2n$-dim real vb)

5) $e(E) = -e(E)$, if $\dim E = \text{odd}$

6) $e(E) = 0$, if E has a nowhere vanishing section

• Why is it called Euler class?

<u>Theorem</u> (Chern - Gauss - Bonnet): Let M be a $2n$-dim oriented closed smooth manifold. Then

$$\chi(M) = e(TM) \cap [M].$$

In other words,

$$\chi(M) = \int_M e(TM).$$

<u>Definition</u>: Let $E \to B$ be an n-dim real vector bundle, and let $E_{\mathbb{C}} = E \otimes \mathbb{C}$ be its complexification (n-$\dim_{\mathbb{C}}$).

The <u>i-th Pontryagin class</u> of E is

$$p_i(E) := (-1)^i c_{2i}(E_{\mathbb{C}}) \in H^{4i}(B;\mathbb{Z}).$$

• How are these related with the previous ones?

<u>Proposition</u>: 1) $w_{2i}(E)^2 = \pi(p_i(E))$, where $\pi : H^{4i}(B;\mathbb{Z}) \to H^{4i}(B;\mathbb{Z}/2)$

2) $e(E)^2 = p_n(E)$, for an orientable $2n$-dim real vb $E \to B$.