I : VECTOR BUNDLES

<u>Definition</u>: A (real) <u>vector bundle</u> of rank k over a manifold M is a manifold E together with a smooth map $\pi : E \to M$ such that

i) $E_x := \pi^{-1}(x)$ is a k-dim vector space, $\forall x \in M$

ii) Every $x \in M$ has a nbhd U and a diffeomorphism $\phi : E_U \xrightarrow{\sim} U \times \mathbb{R}^n$, $E_U := \pi^{-1}(U)$ s.t.

$$\begin{tikzcd} E_U \arrow[rr, "\phi", "\sim"'] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

commutes and $\phi_x : E_x \longrightarrow \{x\} \times \mathbb{R}^n \simeq \mathbb{R}^n$ is a linear map (necessarily iso).

We say that E_x is the <u>fiber</u> over x, and ϕ is a <u>trivialization</u> of E over U. Complex vector bundles are the same by substituting $\mathbb{R}$ by $\mathbb{C}$.

<u>Examples</u>: 1) The trivial vector bundle $M \times \mathbb{R}^n \to M$.

2) TM and T^*M tangent and cotangent bundles. For a chart $U \subset M$, $TU \xrightarrow{\sim} U \times \mathbb{R}^n$ gives an atlas for TM, but also a trivialization of TM (resp. T^*M) as vector bundle. In particular this says that

to give local trivializations of E $\Rightarrow$ to give an atlas on E.

3) $\tau := \{(v, [v]) \in \mathbb{R}^{n+1} \times \mathbb{RP}^n\} \longrightarrow \mathbb{RP}^n$, the <u>tautological line bundle</u>.

Definition: Let $E, F \to M$ be vb over M. A <u>vector bundle morphism</u> $f: E \to F$ is a smooth map st E_x maps to F_x linearly, ie, that the diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] & & F \arrow[dl] \\ & M & \end{tikzcd}$$

commutes and f_x is linear. We say that f is an <u>isomorphism</u> when it has an inverse, ie, when f is diffeomorphism

Lemma: Let $H: E \to F$ be a vb map, and let $\phi: E_U \xrightarrow{\sim} U \times \mathbb{R}^n$, $\psi: F_U \xrightarrow{\sim} U \times \mathbb{R}^m$ be local trivializations over U. Then there exist a unique smooth function $h: U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ st

$$\begin{tikzcd} E_U \arrow[r, "H"] \arrow[d, "\phi"', "\wr"] & F_U \arrow[d, "\wr"', "\psi"] \\ U \times \mathbb{R}^n \arrow[r, dashed] & U \times \mathbb{R}^m \\ (x, v) \arrow[r, mapsto] & (x, h(x) v) \end{tikzcd}$$

Corollary: A map $H: E \to F$ is a vb isomorphism $\Leftrightarrow$ H is a vb morphism which is fiberwise isomorphism.

Definition: A <u>section</u> of $\pi: E \to M$ is a smooth map $s: M \to E$ st $\pi \circ s = Id_M$. Ie, $s(x) \in E_x$.

Lemma. Let $\phi: E_U \xrightarrow{\sim} U \times \mathbb{R}^n$ be a local trivialization of $\pi: E \to M$, & $s: U \to E_U$ is a section, there are unique $f_1 \dots f_n \in \mathcal{C}^\infty(M)$ st

$$\begin{tikzcd} E_U \arrow[r, "\phi", "\sim"'] \arrow[dr, "\pi"'] & U \times \mathbb{R}^n \arrow[d, "pr_1"] \\ & U \arrow[ul, bend left, "s"] \end{tikzcd}$$

$$\phi \circ s = (Id, f_1, \dots, f_n)$$

- $\Gamma(E) = \{\text{sections of } E\}$

Examples: 1) The <u>zero section</u> $0: M \to E$, $x \mapsto 0 \in E_x$.

2) $\Gamma(TM) = \mathfrak{X}(M)$

3) $\Gamma(T^*M) = \Omega^1(M)$

Definition: A *frame* of a vb $E \to M$ is a collection of sections $\{s_1, \dots, s_m\}$ st $\{s_1(x), \dots, s_m(x)\}$ is a basis of E_x $\forall x \in M$. A *local frame* is a frame of E_U, $U \subset M$.

Lemma: The local triviality condition is equivalent to the existence of local frames around every point.

- The space $\Gamma(E)$ has a structure of $\mathcal{C}^\infty(M)$-module, and the previous lemma says that $\Gamma(E_U)$ is a free $\mathcal{C}^\infty(U)$-module of rank m.

Lemma: Let $E, F \to M$ be vb. There is a bijection

$$\begin{aligned} \mathrm{Hom}_{vb}(E,F) &\longleftrightarrow \mathrm{Hom}_{\mathcal{C}^\infty(M)\text{-mod}}(\Gamma(E), \Gamma(F)) \\ u &\longmapsto (s \mapsto u \circ s) \end{aligned}$$

Lemma (Construction): Let $\{E_x : x \in M\}$ be a family of vector spaces of the same dim; and let

$$\{ s^U = (s_1^U \dots s_m^U) \ : U \in \mathcal{O} \}$$

be a family of discrete (ie, without any smoothness condition) local frames, $\mathcal{O}$ open cover of M. If the "change of coordinate functions" from s^U to s^V in the overlapping $U \cap V$ are smooth, then there exists a unique smooth structure of E which makes it into a vector bundle over M st $\{s^U\}$ are smooth local frames.

: How do different trivializations compare?

Definition: Let $\phi, \psi : E_U \xrightarrow{\sim} U \times \mathbb{R}^m$ be two trivializations over U. We call *transition function* between ϕ and ψ to the unique $g : U \to GL(m, \mathbb{R}) \subset \mathcal{L}(\mathbb{R}, \mathbb{R})$ such that

$$\begin{aligned} U \times \mathbb{R}^m \xrightarrow{\phi^{-1}} \ &E_U \xrightarrow{\psi} U \times \mathbb{R}^m \\ (x, v) &\longmapsto (x, g(x) v) \end{aligned}$$

- If $\mathcal{U} = \{U_\alpha\}$ is an open cover of trivializable open sets with trivializations $\phi_\alpha : E_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^m$, then for every intersection we can consider the transition functions $g^\alpha_\beta : U_{\alpha\beta} := U_\alpha \cap U_\beta \to GL(n,\mathbb{R})$, i.e.,

$$\phi_\beta \circ \phi_\alpha^{-1}(x,v) = (x, g^\alpha_\beta(x)\cdot v).$$

Of course, these $\{g^\alpha_\beta\}$ satisfies relations, namely

$$\left.\begin{array}{ll} g^\alpha_\alpha = \mathrm{Id} & (\underline{\text{skew-symmetry}}) \\ g^\gamma_\alpha\, g^\beta_\gamma\, g^\alpha_\beta = \mathrm{Id} & (\underline{\text{cocycle condition}}) \end{array}\right\} (*)$$

"and they are the only ones!"

- Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of a manifold M, and consider $g^\alpha_\beta : U_{\alpha\beta} \to GL(n,\mathbb{R})$ be a smooth map, $\alpha,\beta\in A$.

Consider families $\check{g} = \{g^\alpha_\beta : \alpha,\beta \in A\}$. Not any family comes from a VB, needs to satisfy $(*)$. But if $\check{g}$ satisfies $(*)$, does it produce a VB? Yes!

<u>Theorem</u>: Let M be a manifold and $\mathcal{U} = \{U_\alpha : \alpha\in A\}$ be an open cover

1) If $\pi : E \to M$ is a VB with trivializations ϕ_α over $\mathcal{U}$, then the transition functions satisfy $(*)$.

2) If $\check{g} = \{g^\alpha_\beta\}$ is a family satisfying $(*)$, then there is a vector bundle $E \to M$ which admits trivializations over $\mathcal{U}$ and whose transition functions are g^α_β.

so,

$$\left\{\begin{array}{c} \text{vector bundles} \\ \pi : E \to M \text{ together with a} \\ \text{choice of trivializations over } \mathcal{U} \end{array}\right\} = \left\{\begin{array}{c} \text{families } \check{g} \\ \text{satisfying } (*) \end{array}\right\}.$$

- Observe that in this result we take a VB and trivializations. But how to get rid of the trivialization?

We could have chosen trivializations $\{\phi_\alpha\}$ and $\{\psi_\alpha\}$ over $\mathcal{U}$. This gives rise to functions

$h_\alpha : U_\alpha \to GL(n,\mathbb{R})$ coming out of $(\psi_\alpha \circ \phi_\alpha^{-1})(x,v) = (x, h_\alpha(x)v)$

If $\{g^\alpha_\beta\}$ are the transition factors for $\{\phi_\alpha\}$ and $\{\tilde{g}^\alpha_\beta\}$ are for $\{\psi_\alpha\}$, then they relate as

$$\tilde{g}^\alpha_\beta = h_\beta \, g^\alpha_\beta \, h_\alpha^{-1} .$$

Therefore the theorem can be improved as

<u>Corollary</u>: There is a 1-1 correspondence

$$\left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{vector bundles } E \to M \\ \text{of rank } n \text{ that can be} \\ \text{trivialized over } \mathcal{U} \end{array} \right\} = \left\{ \begin{array}{l} \text{families } \check{g} \\ \text{satisfying } (*) \end{array} \right\} / \sim$$

where $\check{g}_1 \sim \check{g}_2 \Leftrightarrow \exists \, \check{h} = \{ h_\alpha : U_\alpha \to GL(n,\mathbb{R}) \} \; : \; g_{2\,\beta}^{\;\alpha} = h_\beta \, g_{1\,\beta}^{\;\alpha} \, h_\alpha^{-1}$.

<u>ČECH COHOMOLOGY</u>

- Let us fix an open cover $\mathcal{U} = \{U_\alpha\}$ of a connected manifold M.

<u>Definition</u>: Let A be an abelian group endowed with a smooth structure (i.e., a discrete group or a Lie gp). A <u>degree k Čech cochain with locally constant coefficients in A</u> for the cover $\mathcal{U}$ is a collection

$$\check{f} = \{ f_{\underline{\alpha}} : U_{\underline{\alpha}} \to A \text{ locally const} \; : \; \underline{\alpha} = (\alpha_0 \ldots \alpha_k) \}$$

satisfying $f_{\alpha_0 \ldots \alpha_i \alpha_{i+1} \ldots \alpha_k} = - f_{\alpha_0 \ldots \alpha_{i+1} \alpha_i \ldots \alpha_k}$. We denote $\check{\mathcal{C}}^k(M, \underline{A}; \mathcal{U})$ the abelian group of all Čech cochains of degree k. If $\mathcal{C}^\infty(X, \underline{A})$ denotes loc. cont. fnct. on X, then $\check{\mathcal{C}}^k(M; \underline{A}; \mathcal{U}) = \prod_{\underline{\alpha} = (\alpha_0 < \ldots < \alpha_k)} \check{\mathcal{C}}(U_{\underline{\alpha}}; \underline{A})$

- <u>Remark</u>: A loc. cont. function takes the same value in any connected component of its domain. If it is connected, then it's constant.

<u>Definition</u>: The <u>Čech differential</u> is the linear map $\delta^k : \check{\mathcal{C}}^k(M, \underline{A}, \mathcal{U}) \to \check{\mathcal{C}}^{k+1}(M, \underline{A}; \mathcal{U})$ given by

$$(\delta^k \check{f})_{\alpha_0 \ldots \alpha_{k+1}} := \sum_{i=1}^{k+1} (-1)^i \, f_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{k+1}} .$$

Lemma: $\delta^2 = 0$.

Definition: The *Čech complex* is the cochain complex

$$\check{C}^0 \xrightarrow{\delta^0} \check{C}^1 \xrightarrow{\delta^1} \check{C}^2 \xrightarrow{\delta^2} \dots$$

and the *Čech cohomology groups* of M with loc. const. coefficients in A wrt the cover $\mathcal{U}$ is

$$\check{H}^k(M, \underline{A}; \mathcal{U}) = \frac{\operatorname{Ker} \delta^k}{\operatorname{Im} \delta^{k-1}}.$$

Proposition: $\check{H}^0(M, \underline{A}; \mathcal{U}) = \{ f : M \to A \text{ loc. const.} \} \quad \left(= A^{|\pi_0(M)|} \right)$.

Proposition: Let $\check{g} \in \check{C}^1(M, \underline{\mathbb{R}^*}; \mathcal{U})$.

1) $\check{g}$ is a cocycle $\Longleftrightarrow$ $\check{g}$ satisfies the cocycle condition.

2) $[\check{g}] = [\check{g}'] \in \check{H}^1 \Longleftrightarrow \exists\, \check{h} \in \check{C}^0 : g'_{\alpha\beta} = h_\beta g_{\alpha\beta} h_\alpha^{-1}$.

~~Since $GL(1,\mathbb{R}) = \mathbb{R}^*$ and $GL(1,\mathbb{C}) = \mathbb{C}^*$~~, we get:

Theorem (Classification of line bundles, 1st version): Let $K = \mathbb{R}$ or $\mathbb{C}$. There's a 1-1 correspondence

$$\check{H}^1(M, \underline{K^*}; \mathcal{U}) = \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{line vector bundles that can} \\ \text{be trivialized over } \mathcal{U} \end{array} \right\}$$

What happens if we forget about the condition loc. const?

Definition: Let A be an ab. grp with a smooth str. A *degree k Čech cochain with coefficients in smooth functions with values in A* over $\mathcal{U}$ is a collection

$$\check{f} = \{ f_a : M \to A \text{ smooth} : a = (\alpha_0 \dots \alpha_k) \}$$

st $f_{\alpha_0 \dots \alpha_i \alpha_{i+1} \dots \alpha_k} = - f_{\alpha_0 \dots \alpha_{i+1} \alpha_i \dots \alpha_k}$. The set is denoted $\check{C}^k(M, \overbrace{C^\infty(M,A)}^{A\ (\text{o.jud})}; \mathcal{U}) = \prod_{a = (\alpha_0 < \dots < \alpha_k)} C^\infty(U_a, A$

• For this, still $\delta^2 = 0$ and one gets cohomology gps $\check H^k(M, \mathcal{C}^\infty(M,A); \mathcal{U})$. But

<u>Proposition</u>: $\check H^k(M, \mathcal{C}^\infty_{\mathbb{R}}; \mathcal{U}) = 0 \quad \forall k > 0$.

• Since $GL(1,\mathbb{R}) = \mathbb{R}^*$ and $GL(1,\mathbb{C}) = \mathbb{C}^*$, we get that the previous correspondence is rephrased as (revisiting the Prop for $\check{\mathcal{C}}^1(M, \mathbb{R}^*; \mathcal{U})$).

<u>Theorem</u> (Classification of line bundles, 1st version): Let $K = \mathbb{R}$ or $\mathbb{C}$. Then there's a 1-1 correspondence

$$\check H^1(M, \underbrace{\mathcal{C}^\infty(M,K^*)}_{K^*}; \mathcal{U}) = \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{line bundles over } M \text{ that} \\ \text{can be trivialized over } \mathcal{U} \end{array} \right\}$$

• If $\varphi: G \to H$ is a gp hom, then it induces a cochain map

$$\varphi_*: \check{\mathcal{C}}^k(M, \mathcal{C}^\infty(M,G); \mathcal{U}) \to \check{\mathcal{C}}^k(M, \mathcal{C}^\infty(M,H); \mathcal{U})$$

$$\check g = \{g_\alpha\} \longmapsto \varphi_* \check g = \{\varphi \circ g_\alpha\}.$$

• Using the exact sequences $0 \to \mathbb{R} \to \mathbb{R}^* \to \mathbb{Z}_2 \to 0$ and $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0$ and the les we get

<u>Theorem</u> (Classification of line bundles, 2nd version): There are 1-1 correspondences

$$\check H^1(M, \underbrace{\mathcal{C}^\infty(M,\mathbb{Z}_2)}_{\mathbb{Z}_2}; \mathcal{U}) = \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{real line bundles that can} \\ \text{be trivialized over } \mathcal{U} \end{array} \right\},$$

and if $\mathcal{U}$ is a <u>good cover</u> (i.e., every U_α is either contractible or empty) then

$$\check H^2(M, \underbrace{\mathcal{C}^\infty(M,\mathbb{Z})}_{\mathbb{Z}}; \mathcal{U}) = \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{complex line bundles that} \\ \text{can be trivialized over } \mathcal{U} \end{array} \right\}$$

<u>Definition</u>: The <u>first Stiefel-Whitney class</u> of a real line bundle is the class of $\check H^1$ in correspondence with the line bundle; and the <u>first Chern class</u> of a complex line bundle is the class of $\check H^2$ ———

CONSTRUCTIONS WITH VECTOR BUNDLES

• We are going to construct vector bundles out of other ones. We are going to use mainly the construction lemma. For that, note that to give local frames = to give local trivializations; and since the change of coordinate functions in an intersection $U_\alpha \cap U_\beta$ are precisely the transition functions of the corresponding trivializations, they are smooth in the overlapping $\iff$ the transition functions of the trivializations are smooth.

$$\boxed{\begin{array}{ll} \text{local trivializations} = \text{local frames} \\ \text{transition functions} = \text{change of basis matrix} \\ \qquad\qquad (\text{from the old one to the new one}) \end{array}}$$

ie, $s_j^\alpha = \sum_i (g_\beta^\alpha)_{ij} \, s_i^\beta$.

1) <u>Direct sum</u>: If $E, F \to M$, set $E \oplus F := \coprod_{x \in M} E_x \oplus F_x \longrightarrow M$. If $\{s_1 \dots s_n\}$ and $\{t_1 \dots t_m\}$ are local frames of E and F over U, then $\{s_1, \dots, s_n, t_1 \dots t_m\}$ is a local frame over U, and the transition functions (= change of basis matrix) of the two local frames are

$$\begin{pmatrix} g_\beta^\alpha & 0 \\ 0 & h_\beta^\alpha \end{pmatrix}$$

In particular, $\Gamma(E \oplus F) = \Gamma(E) \oplus \Gamma(F)$.

2) <u>Dual vector bundle</u>: If $E \to M$, set $E^* := \coprod_{x \in M} E_x^* \longrightarrow M$. If $\{s_1 \dots s_n\}$ is a local frame, then consider its dual local frame $\{s_1^*, \dots, s_n^*\}$ (where $\{s_i^*(x)\}$ is the dual basis of $\{s_i(x)\}$). Transition functions? By linear algebra, it is $(g_\alpha^\beta)^t$ (from "α" old to "β" new").

3) <u>Tensor product</u>: if $E, F \to M$, set $E \otimes F := \coprod_{x \in M} E_x \otimes F_x \to M$. Trivialization? If $\{s_1 \ldots s_m\}$ and $\{t_1 \ldots t_m\}$ are local frames, then $\{s_i \otimes t_j\}$ is a local frame. Moreover,

$\Gamma(E \otimes F) = \Gamma(E) \otimes \Gamma(F)$.

4) <u>Hom bundle</u>: If $E, F \to M$, then $Hom(E,F) := \coprod_{x \in M} Hom(E_x, F_x)$. Trivialization? If $\{s_i\}$, $\{t_j\}$ are local frames, then $\{f_{ij}(x) :=$ unique lin map sending $s_i(x)$ to $t_j(x)\}_{ij}$ is a local frame. Observe that to give a section of $Hom(E,F)$ is the same as to give a vb morphism $E \to F$, i.e.,

$$\Gamma(Hom(E,F)) = Hom_{vb}(E,F) = Hom_{C^\infty(M)-mod}(\Gamma(E), \Gamma(F)).$$

5) <u>Exterior algebra</u>: If $E \to M$, set $\Lambda^m E := \coprod_{x \in M} \Lambda^m E_x \to M$, where $m =$ rank E. If $\{s_1 \ldots s_m\}$ is a local frame, then $\{s_1 \wedge \cdots \wedge s_m\}$ is a local frame of $\Lambda^m E$. The transition functions are $\det(g^\alpha_\beta)$.

<u>Definition</u>: Let $E \to M$ be a real vector bundle of rank m. The <u>first Stiefel-Whitney class</u> of E is the first SW class of $\Lambda^m E$. If it is a complex vb, we define the <u>first Chern class</u> as the first C class of $\Lambda^m E$.

6) <u>Pullback bundle</u>: Let $E \to N$ vb and $f: M \to N$ smooth. Define $f^* E := \coprod_{x \in M} E_{f(x)}$, i.e., such that the diagram

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ M \arrow[r] & N \end{tikzcd}$$

is a bundle map. Actually $f^* E = M \times_N E$. Trivialization? If $\{s_1 \ldots s_m\}$ is a local frame over $U \subset N$, $\{f^* s_1, \ldots f^* s_m\}$ is a local frame over $f^{-1}(U) \subset M$. The transition functions are given by $f^* g^\alpha_\beta$.

Proposition: let $E \to N$ vb. If f^*E is not trivializable for some $f: M \to N$ $\Rightarrow$ E is not trivializable.

Ex: $\tau_n \to \mathbb{RP}^n$ is not trivializable because for the inclusion $i: \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$, $i^*\tau_n = \tau_1$, which is not trivializable.

Theorem: Homotopic maps induce isomorphic vector bundles.

Ie, if $f \simeq g: M \to N$ $\Rightarrow$ $f^*E \simeq g^*E$.

II : CONNECTIONS ON VECTOR BUNDLES

DIFFERENTIAL FORMS WITH VALUES IN VECTOR BUNDLES

• There are canonical isomorphisms $\mathrm{End}(E) = \mathcal{T}^1_1(E) = E^* \otimes E$, where E is a vector space. This extends to say that an element $(E^*)^{\otimes p} \otimes E^{\otimes q}$ can be seen as a linear map $E^{\otimes r} \otimes (E^*)^{\otimes s} \longrightarrow (E^*)^{\otimes (p-r)} \otimes E^{\otimes (q-s)}$.

• For instance, an element of $E^* \otimes E^* \otimes E = E^* \otimes \mathrm{End}(E)$ can be seen as a lin map $E \to \mathrm{End}(E)$, ie, a matrix with 1-forms in its entries.

• This extends to sections: recall that $\Omega^k(M) := \Gamma(\Lambda_k TM) = \{ \mathfrak{X}(M)^k \longrightarrow \mathcal{C}^\infty(M) \text{ skew} \}$.

<u>Definition</u>: Let $E \to M$ be a v.b. A <u>k-form with coefficients in E</u> is a section of $\Lambda_k TM \otimes E$,

$$\Omega^k(M, E) := \Gamma(\Lambda_k TM \otimes E) = \Omega^k(M) \otimes \Gamma(E);$$

ie, it can be seen as a map $\mathfrak{X}^k(M) \to \Gamma(E)$. skew.

• The standard 1-forms are recovered by setting $E = \underline{1} = M \times \mathbb{R}$, since $\Gamma(\underline{1}) = \mathcal{C}^\infty(M)$.

CONNECTIONS

<u>Definition</u>: Let $E \to M$ be a v.b. A <u>connection</u> on E is an $\mathbb{R}$-linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E) = \Omega^1(M) \otimes \Gamma(E) = \Omega^1(M, E)$$

satisfying the following Leibniz rule:

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s, \qquad f \in \mathcal{C}^\infty(M),\ s \in \Gamma(E).$$

<u>Warning</u>: It is just $\mathbb{R}$-lin, it is <u>not</u> a $\mathcal{C}^\infty(M)$-module homomorphism!!!

• The previous discussion says that a connection is the same thing as a $\mathbb{R}$-bilinear map

$$\begin{aligned} \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (D, s) &\longmapsto D^\nabla s \end{aligned}$$

satisfying the following Leibniz rule:

$$D^{\nabla}(fs) = Df \cdot s + f \cdot D^{\nabla}s.$$

Both descriptions are related by $D^{\nabla}s := i_D \nabla s \in \Gamma(E)$.

Example: For the trivial line bundle $\underline{1} = M \times \mathbb{R}$, $\Gamma(\underline{1}) = \mathcal{C}^{\infty}(M)$ is a rank 1 $\mathcal{C}^{\infty}(M)$-mod, whose basis is given by the section $s: M \to M \times \mathbb{R}$, $x \mapsto (x, 1)$. Since every other section is fs, the Leibniz rule determines the connexion if we define ∇s. Set $\nabla^{\circ} s = 0$. This is a connexion, which is called the trivial connection. Similarly, for $\underline{m} = M \times \mathbb{R}^m \to M$, there is a global frame (s_i) and we can also set $\nabla^{\circ} s_i = 0$, and $\underline{\underline{\nabla^{\circ} = d}}$.

Proposition: Let $E \to M$ be a vb

1) ∇, ∇' connections $\Rightarrow$ $\nabla - \nabla'$ is a $\mathcal{C}^{\infty}(M)$-mod map, thus can be identified with an element of $\Gamma(T^*M \otimes \mathrm{End}\, E)$, ie, a matrix whose coefficients are 1-forms.

2) ∇ connection, $A \in \Gamma(T^*M \otimes \mathrm{End}) \Rightarrow \nabla + A$ connection.

$$\Gamma(T^*M \otimes \mathrm{End}) = \mathrm{Hom}_{\mathcal{C}^{\infty}(M)\text{-mod}}(\Gamma(E), \Gamma(T^*M \otimes E))$$

Theorem: Every vector bundle admits connections.

Proposition: Let $E, F \to M$ be vb, and let ∇^E, ∇^F be connections on E and F.

1) $\nabla^{E \oplus F}(s^E + s^F) := \nabla^E(s^E) + \nabla^F(s^F)$ is a connection on $E \oplus F$.

2) $\nabla^{E \otimes F}(s^E \otimes s^F) := \nabla^E(s^E) \otimes s^F + s^E \otimes \nabla^F(s^F)$ is a connection on $E \otimes F$.

3) $\nabla^{E^*}(s^*)(s) := d(s^* s) - s^*(\nabla^E s)$ defines a connexion on E^*

4) $(f^*\nabla)(f^*s) := f^*\nabla s$ defines a connection on f^*E ($f: N \to M$).

(Expression in coordinates). If ∇ is a connection and U is a trivializing open subset, then on a local frame ∇ is

$$\boxed{\nabla = d + A}$$

where $A \in \Omega^1(U; \mathrm{End}(\mathbb{R}^m))$, ie, a matrix of 1-forms.

That is, if $s = (f_1 \dots f_m)$ is a local frame, then

$$\nabla \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_m \end{pmatrix} + A \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

<u>Definition</u>: The matrix A obtained on a local trivialization is called the <u>connection 1-form</u> or <u>connection matri</u>

- So if we have a cover of M, how do different connection matrices relate each other?

<u>Proposition</u>: Let $E \to M$ be a vb and $\mathcal{U} = \{U_\alpha\}$ an open cover of trivializable open sets, and let $\{g^\alpha_\beta\}$ be the transition functions. If ∇ is a connection on E and A_α is the connection matrix on U_α, then

$$\boxed{A_\alpha - (g^\alpha_\beta)^{-1} \cdot A_\beta \cdot (g^\alpha_\beta) = (g^\alpha_\beta)^{-1} \cdot (d g^\alpha_\beta)}$$

Conversely, if $\{A_\alpha\}$ satisfies the previous expression, then there exists a unique connection ∇ s.t. A_α are its connection matrices.

- <u>(Relation with Čech cohomology)</u>. For line bundles, the expression becomes $A_\alpha - A_\beta = \dfrac{d g^\alpha_\beta}{g^\alpha_\beta} = d(\log g^\alpha_\beta)$

But $g^\alpha_\beta \in \check{\mathcal{C}}^1(M; K^*, \mathcal{U})$, thus $\log g^\alpha_\beta \in \check{\mathcal{C}}^1(M, \mathcal{C}^\infty_K, \mathcal{U})$, thus $d(\log g^\alpha_\beta) \in \check{\mathcal{C}}^1(M, \Omega^1_K; \mathcal{U})$, which is a cocyc

But $A_\alpha, A_\beta \in \check{\mathcal{C}}^0(M, \Omega^1_K; \mathcal{U})$ and the expression is

$$\underline{\delta \check{A} = d(\log g^\alpha_\beta)}$$

That is, a connection for a line bundle corresponds precisely to a choice of primitive for the cocycle $d(\log \check{g})$, i.e., a choice of $\check{A}$ s.t. $\delta \check{A} = d(\log \check{g})$.

CURVATURE

• Given a connection $\nabla : \Omega^0(M,E) \to \Omega^1(M,E)$, we wonder: can we extend ∇ to a $\mathbb{R}$-linear operator $\nabla : \Omega^p(M;E) \to \Omega^{p+1}(M;E)$ as with the differential? Does $\nabla^2 = 0$?..

<u>Theorem</u>: There exists a unique extension of a connection ∇ to a $\mathbb{R}$-linear operator

$$\nabla : \Omega^p(M,E) \longrightarrow \Omega^{p+1}(M;E)$$

such that

$$\boxed{\nabla(\omega_p \otimes s) = d\omega_p \otimes s + (-1)^p \omega_p \wedge \nabla s}$$

• In the previous expression, if $\nabla s = \xi \otimes t$, then $\omega_p \wedge \nabla s := (\omega \wedge \xi) \otimes t$.

<u>Example</u>: For the trivial line bundle $M \times \mathbb{R} \to M$, the trivial connection is $\nabla^0 = d : \overset{\Gamma(1)}{\overset{\shortparallel}{\mathcal{C}^\infty(M)}} \to \overset{\Omega^1(M,1)}{\overset{\shortparallel}{\Omega^1(M)}}$. If A is a $\mathcal{C}^\infty$-mod map $\mathcal{C}^\infty(M) \to \Omega^1(M)$, then by the iso it corresponds to a matrix of 1-forms, which in this case it's just a 1-form, ie, $A = \xi \cdot -$, ie, $A = (\xi)$, $\xi \in \Omega^1(M)$. Ie, $(\nabla^0 + A)(f) = df + f \cdot \xi$.

By the Leibniz rule, $\nabla^0 + A$ extends to $(\nabla^0 + A)(\omega_p \otimes s) = d\omega_p + (\xi \wedge \omega_p) \otimes s$, ie,

$$\underline{\nabla^0 + A \equiv d + \xi \wedge}.$$

<u>Proposition</u>: Let $E \to M$ be a vb over M and ∇ a connection on E.

1) $\nabla^2 : \Omega^0(M,E) \overset{\Gamma(E)}{=} \to \Omega^2(M,E) \overset{\Gamma(\Lambda^2 T^*M \otimes E)}{=}$ is a map of $\mathcal{C}^\infty(M)$-mod, thus it corresponds to a $\mathrm{End}\,E$-valued 2-form $F_\nabla \in \Omega^2(M, \mathrm{End})$, called the <u>curvature</u> of ∇, which is a matrix of 2-forms, ie, in a local frame $s = (f_1 \dots f_m)$ and

$$\nabla^2 s = F_\nabla s = \begin{pmatrix} w_{ij} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}, \quad w_{ij} \in \Omega^2(M)$$

2) $\nabla^2 : \Omega^p(M,E) \to \Omega^{p+2}(M,E)$ is a $\mathcal{C}^\infty(M)$-mod map. thus it corresponds to a elmt

$F_\nabla \in \mathrm{Hom}(\Gamma(\Lambda_p TM \otimes E), \Gamma(\Lambda_{p+2} TM \otimes E)) = \Gamma(\Lambda^p TM \otimes \Lambda_{p+2} TM \otimes \mathrm{End}\, E)$, ie, in

a local frame it is a matrix $\begin{pmatrix} \omega_{ij} \wedge - \end{pmatrix}$, $\omega_{ij} \in \Omega^2(M)$, ie, if

$\omega_p \otimes s = (\xi_1, \dots, \xi_n)$, $\xi_i \in \Omega^p(M)$, then

$$\nabla^2(\omega_p \otimes s) = \begin{pmatrix} \omega_{ij} \wedge - \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \overset{\text{not}}{=} F_\nabla \wedge \omega_p \otimes s .$$

Setting $\rho = \omega_p \otimes s$, one writes

$$\boxed{\nabla^2 \rho = F_\nabla \wedge \rho}$$

For the prop one needs

<u>Lemma</u>: Let ∇ be a connection on $E \to M$. If $\rho \in \Omega^p(M,E)$ then

$$\boxed{\nabla(f\rho) = df \wedge \rho + f \cdot \nabla \rho}$$

<u>Lemma</u>: If ∇ is a connection on $E \to M$, then it induces a connection on $\mathrm{End}\, E$ determined by

$$\nabla(Ts) = (\nabla T)s + T(\nabla s),$$

$T \in \Gamma(\mathrm{End}\, E) = \mathrm{Hom}(\Gamma(E), \Gamma(E))$, and where if $\nabla T = \omega \otimes S$, then $(\nabla T)s := \omega \otimes Ss$ and if $\nabla s = \omega \otimes t$ then $T(\nabla s) = \omega \otimes Tt$.

<u>Theorem</u> (Bianchi's Identity): $\boxed{\nabla F_\nabla = 0}$.

• (<u>Expression in coordinates of F_∇</u>): In a trivializing open set, $\nabla = d + A$ where A is a matrix of 1-forms. Then

$$\boxed{F_\nabla = dA + A \wedge A}$$

where if $A = (\omega_{ij})$ matrix of 1-forms then $dA = (d\omega_{ij})$ and $A \wedge A = \left(\sum_k \omega_{ik} \wedge \omega_{kj}\right)$.

<u>Theorem</u> (Curvature class):

1) $d(\operatorname{tr} F_\nabla) = 0$.
2) For a vector bundle $E \to M$, the class $[\operatorname{tr} F_\nabla] \in H^2_{dR}(M)$ does not depend on the connection ∇.

<u>Definition</u>: The <u>curvature class</u> of a vb $E \to M$ is the dR cohomology class

$$\kappa(E) := \frac{1}{2\pi i}[\operatorname{tr} F_\nabla].$$

<u>Ex</u>: 1) $\kappa(\underline{n}) = 0$, since $F_{\nabla^0} = (\nabla^0)^2 = d^2 = 0$.

2) $\kappa(\mathbb{CP}^1) \neq 0$, because $\int_{\mathbb{CP}^1} \kappa(\mathbb{CP}^1) = -1$.

3) If $E \to M$ admits trivializations for which g^α_β ~~is~~ a matrix of const functions, then $\kappa(E) = 0$, as in 1)

4) $\kappa(M) = 0$, by 3).

<u>CECH - DE RHAM RELATION</u>

<u>Theorem</u> (de Rham): Let $\mathcal{U}$ be a good cover of M. Then there is an isomorphism

$$\check{H}^p(M, \underline{\mathbb{R}}; \mathcal{U}) \cong H^p_{dR}(M)$$

Thus Čech cohomology does not depend on the choice of the good cover $\mathcal{U}$ of M.

- The proof starts by defining a double complex $E^{p,q} = \check{C}^p(M, \Omega^q(M); \mathcal{U}) := \prod_{\alpha=(\alpha_0,\dots,\alpha_p)} \Omega^q(U_\alpha)$, with differentials $\delta : E^{p,q} \to E^{p+1,q}$ and $d : E^{p,q} \to E^{p,q+1}$. Setting $\left(E^k := \bigoplus_{p+q=k} E^{p,q}, D := \delta + (-1)^p d\right.$ for $\alpha \in E^{p,q}$ gives rise to a cochain complex. The claim is that the cochain maps

$$i : \Omega^p(M) \longrightarrow E^{0,p} \subset E^p \qquad , \qquad j : \check{C}^p(M; \underline{\mathbb{R}}) \longrightarrow E^{p,0}$$
$$\omega \longmapsto (i\omega)_\alpha := \omega|_{U_\alpha} \qquad\qquad \check{f} \longmapsto j(\check{f}) := \check{f}$$

induce isos in cohomology, $\check{H}^p(M, \underline{\mathbb{R}}; \mathcal{U}) \overset{j^*}{\cong} H^p(E) \overset{i^*}{\cong} H^p_{dR}(M)$.

<u>Corollary</u>: If M admits a finite good cover (e.g. if M is compact), then H^p_{dR} is fin. dim.

- What relation do the first Chern class and the curvature class have?

<u>Theorem</u>: The image of the first Chern class under the following composite is the curvature class:

$$\check{H}^2(M; \mathbb{Z}) = \check{H}^2(M; \underline{\mathbb{Z}}) \longrightarrow \check{H}^2(M; \underline{\mathbb{R}}) \xrightarrow{\sim} H^2_{dR}(M)$$

III : METRICS ON VECTOR BUNDLES

- If V is a vs, then call $S_2(V) = \{$ multilin maps $V \times V \to K$ symmetric $\} \leq \mathcal{T}_2^0(V)$.

Definition: A <u>metric</u> on a real vector bundle $E \to M$ is a section $g \in \Gamma(S_2 E)$ which is pointwise Euclidean.

Proposition: Every real vector bundle admits a metric.

Proposition: Given $(E,g) \to M$ vb with metric, there is a vector bundle isomorphism

$$\phi : E \simeq E^* \\ e \longmapsto i_e g$$

Definition: A local frame $\{s_1, \dots, s_n\}$ over U is called <u>orthonormal</u> if $g(s_i, s_j) = \delta_{ij}$.

Notation: $g(s_1, s_2) := s_1 \cdot s_2$.

Lemma: Every vb with metric (E,g) admits local orthonormal frames; or what is the same, local trivializations for which the transition functions take values in $O(n) \subset GL(n,\mathbb{R})$.

Corollary: Every real line bundle admits a connection with zero curvature.

Corollary: κ (real vector bundle) $= 0$.

- Let V be a vs and $g : V \times V \to K$ a metric, for instance (in gen a tensor). Then g induces a bilinear map $(V^* \otimes V) \times V \xrightarrow{g} V^*$ by $g(\omega \otimes v, w) := \omega \cdot g(v,w)$; ie, it comes as the composition $V^* \times V \times V \xrightarrow{\mathrm{Id} \times g} V^* \times K \to V^*$, using the Univ. Prop of $\otimes$. More gen with $T_p^q : V^p \times V^{*q} \to K$.

- Given $(E,g) \to M$, a connection on E induces a connection on $S_2 E$, by the rule

$$d(g(s_1,s_2)) = (\nabla g)(s_1, s_2) + g(\nabla s_1, s_2) + g(s_1, \nabla s_2).$$

Definition: Let $(E,g) \to M$. We say that a connection ∇ is compatible with g or metric if $\nabla g = 0$, i.e., if

$$d(s_1 \cdot s_2) = (\nabla s_1)\cdot s_2 + s_1 \cdot (\nabla s_2), \qquad \forall s_1, s_2 \in \Gamma(E).$$

Example: Let $M \times \mathbb{R}^n \to M$, with global frame $\{s_1 \dots s_n\}$. An Euclidean metric is given by

$$g\left(\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \right) = \sum f_i g_i \text{ ; and the trivial connection } \nabla^0 = d \text{ is compatible with } g.$$

Proposition: Every vb with metric $(E,g) \to M$ admits a metric connection.

More precisely, if $\mathcal{A}$ is the space of all connections and $\tilde{\mathcal{A}} \subset \mathcal{A}$ metric connections, then the map

$$\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{A}$$
$$\nabla \longmapsto \mathcal{F}(\nabla) := \frac{1}{2}\left(\nabla + \phi^{-1} \nabla^* \phi \right)$$

is a retraction to $\tilde{\mathcal{A}}$, i.e., $\mathcal{F}$ has $\tilde{\mathcal{A}}$ as image and $\mathcal{F}|_{\tilde{\mathcal{A}}} = \mathrm{Id}$.

Corollary: Given $(E,g) \to N$ and $f: M \to N$ smooth, consider $(f^*E, f^*g) \to M$. Then

$$\mathcal{F}_{f^*E}(f^*\nabla) = f^* \mathcal{F}_E(\nabla).$$

Definition: Let (V,g) be an euclidean vs, and $T: V \to V$ an endomorphism. We say that T is symmetric (or self-adjoint) if $(Te)\cdot v = e \cdot (Tv)$ $\forall e, v \in V$. We say that T is skew-symmetric if $(Te)\cdot v = -e\cdot(Tv)$.

Lemma: Let $(E,g) \to M$ and ∇ metric connection, and let ∇' be a connection on E.

$$\nabla' \text{ is metric} \iff A = \nabla - \nabla' \text{ is skew-symmetric}, \ (As_1)\cdot s_2 = -s_1 \cdot (As_2)$$

In a orthonormal local frame, A skew-sym means that the matrix of A is skew-sym.

Lemma: let $(E,g) \to M$ with ∇ metric connection. The curvature of ∇ is a skew-sym operator,

$$(F_\nabla s_1)\cdot s_2 = -s_1 \cdot (F_\nabla s_2).$$

THE EULER CLASS

Definition: A vector bundle $E \to M$ is <u>orientable</u> if it admits a trivialization such that the transition functions $g^\alpha_\beta \in SL(n,\mathbb{R}) \subset GL(n,\mathbb{R})$, and such trivialization is said to be <u>oriented</u>.

- We define the following eq. relation in the set of oriented trivializations:

$$\{\phi_\alpha\} \equiv \{\psi_\alpha\} \iff \det h_\alpha > 0 \qquad \left(\psi_\alpha \circ \phi_\alpha^{-1}(x,v) = (x, h_\alpha v)\right)$$

Definition: An <u>orientation</u> on E is an equivalence class of the previous relation.

- Let $(E,g) \to M$ be a rank $2k$ oriented vector bundle with metric and a metric connection ∇.

Define $\omega \in \Omega^2(M;\Lambda_2 E)$ by $\omega(s_1,s_2) := g(F_\nabla s_1, s_2) \in \Omega^2(M)$.

Lemma: 1) $\nabla \omega = 0$

2) $\nabla \omega^k = 0 \quad \forall k \quad \left(\omega^k = \omega \wedge \overset{k}{\dots} \wedge \omega \in \Omega^{2k}(M;\Lambda_{2k}E)\right)$

Lemma: If $E \to M$ is an orientable vb, then exists a nowhere vanishing section $\sigma \in \Gamma(\Lambda^{2k}E)$ st $\nabla\sigma = 0$

Definition: The <u>Euler form</u> of E is the closed $2k$-form

$$e(E) := \frac{1}{(2\pi)^k}\frac{1}{k!}\,\omega^k(\sigma),$$

and its cohomology class $\varepsilon(E) := [e(E)] \in H^{2k}_{dR}(M)$ is the <u>Euler class</u> of E.

Theorem: The Euler class does not depend on the choice of metric or metric connection

Proposition (Properties): 1) $f^*\mathcal{E}(E) = \mathcal{E}(f^*E)$, $f: M \to N$

2) $\mathcal{E}(E \oplus F) = \mathcal{E}(E) \cup \mathcal{E}(F)$, $\cup$ cup product : $[\omega] \cup [\omega'] = [\omega \wedge \omega']$.

3) If E has a nowhere vanishing section, $\mathcal{E}(E) = 0$.

- How to relate the Chern class and the Euler class? (For rank 2 bundles):

Definition: Let (V, g, Ω) be an oriented euclidean space. Recall that the polarity $\phi: V \xrightarrow{\sim} V^*$ induces isos $\phi: \Lambda^k V \xrightarrow{\sim} \Lambda^k V^* = \Lambda_k V$. We write $i_{v_1 \wedge \dots \wedge v_k} \omega_p := i_{v_1} i_{v_2} \cdots i_{v_k} \omega_p$.

The Hodge star operator is the map

$$\begin{aligned} * : \Lambda^k V &\longrightarrow \Lambda^{n-k} V \\ \xi_k &\longmapsto *\xi_k := \phi^{-1}(i_{\xi_k} \Omega). \end{aligned}$$

Lemma: If $\{e_1, \dots e_n\}$ is a positive orthonormal basis, $I \subseteq \{1, \dots, n\}$ and I^c its complementary, then

$$* e_I = \pm e_{I^c}$$

where the sign is determined by the condition $(*e_I) \wedge e_I = e_1 \wedge \dots \wedge e_n$

If E is a real vb of rank 2, then it can be endowed with a str. of complex vb by $(a+bi)s := as + b(*s)$ because $*^2 = -\mathrm{Id}$ on $\Lambda^1 E$.

Theorem: Let $(E, g) \to M$ be a rank 2 real vb with metric. Then endowing E with the complex vb str. coming from the Hodge star operator, the Chern class (= curvature class) is the Euler class

$$\kappa(E) = \mathcal{E}(E) \qquad \left(= \frac{1}{2\pi}[dA_{21}] \right).$$

TRANSVERSAL MANIFOLDS

<u>Definition</u>: Let $N, P \hookrightarrow M$ be embedded submanifolds. We say that N and P <u>intersect transversally</u> if

$$T_x N + T_x P = T_x M \quad , \qquad \forall x \in N \cap P .$$

<u>Lemma</u>: If $N_n, P_p \subseteq M_m$ are submanifolds intersecting transversally, then $N \cap P$ is an embedded submanifold of dim $n + p - m$, ie,

$$\operatorname{codim}(N \cap P) = \operatorname{codim} N + \operatorname{codim} P .$$

<u>Definition</u>: $N, P \subset M$ have <u>complementary dimensions</u> if $\dim N + \dim P = \dim M$.

- In this case, $\dim(N \cap P) = 0$, and if any of the manifolds is compact, this is just a bunch of points

<u>Definition</u>: Let $N, P \hookrightarrow M$ be two compact, oriented submanifolds of complementary dim intersecting transversally. The <u>sign</u> of $x \in N \cap P$ is

$$\varepsilon(x) := \begin{cases} 1 , & \text{if } T_xN \oplus T_xP = T_xM \text{ is orientation preserving, ie,} \\ & \{\underbrace{e_1 - e_n}_{\text{positive in } T_xN} ; \underbrace{v_1 - v_p}_{\text{positive in } T_xP}\} \text{ is positive in } T_xM \\ -1 , & \text{else} \end{cases}$$

The <u>intersection number</u> of N and P is $\displaystyle I(N,P) = \sum_{x \in N \cap P} \varepsilon(x)$.

<u>Theorem</u>: Let $E \to M$ oriented vb with M compact, of rank $2k$ and M oriented. Identify M with the 0-section. Let s be a section transverse to the 0-section and let $P = s(M) \cap M \subset M$ be the zero locus of s. Given $i : N^{2k} \hookrightarrow M$ oriented compact submanifold of M transverse to P with compl. dim, then $\int_N i^* e(E) = I(N,P)$.

*<u>Theorem</u> (Gauss-Bonnet): Let M be a orientable compact manifold. Then

$$\boxed{\int_M e(TM) = \chi(M)}$$

IV. PARALLEL TRANSPORT AND CONECTIONS ON TM

- One is usually interested in solving $\begin{cases} df=0 \\ f(x_0)=c \end{cases}$ on a manifold M. This is just $f \equiv \text{const}_c$ ($\exists!$ sol). Can we replace d by ∇ in a vb $E \to M$? We now want to solve $\begin{cases} \nabla s=0 \\ s(x_0)=v \in E_{x_0} \end{cases}$. Can we? Not always, as $v \in \ker F_\nabla$.

<u>Definition</u>: A section $s: M \to E$ is <u>parallel</u> if $\nabla s = 0$.

- When can we solve this?

<u>Theorem</u>: The problem $\begin{cases} \nabla s=0 \\ s(x_0)=v \in E_{x_0} \end{cases}$ admits a local solution $\forall v \in E_{x_0} \iff F_\nabla = 0$ in a nbhd of x_0.

- For 1-dim mnflds, $F_\nabla = 0$ as $F_\nabla \in \Omega^2(M, \operatorname{End} E)$. In particular, given M a mnfld and $\gamma: I \to M$ path we can form $(\gamma^* E, \gamma^* \nabla) \to I$ vb over I, and

<u>Theorem</u> (Parallel transport): Given $(E, \nabla) \to M$, a path $\gamma: I \to M$ and a vector $v \in E_{x_0}$, there is a unique section $s \in \Gamma(\gamma^* E)$ solving the problem with constrains $\begin{cases} (\gamma^* \nabla) s = 0 \\ s(\gamma(t_0)) = v \end{cases}$

<u>Definition</u>: The previous solution is called the <u>parallel transport of v along γ</u> and it is denoted $T_\gamma^{t,t_0}(v) := s(\gamma(t$

<u>Proposition</u>: 1) $T_\gamma^{t,t_0} = T_\gamma^{t,t'} \circ T_\gamma^{t',t_0}$

2) $\gamma = \text{const}_{x_0} \Rightarrow T^{t,t_0} = Id$

3) $T_\gamma^{t,t_0}: E_{x_0} \to E_{\gamma(t)}$ is a linear map

4) $T_\gamma^{b,a} = T_{\gamma'}^{d,c}$ for $\gamma' = \gamma \circ \varphi$, $\varphi: [c,d] \xrightarrow{\sim} [a,b]$ reparametrization.

- The last prop. says that the par. transp. doesn't depend on the parametrization, so we will put $I = [0,1]$.

5) $T_{\sigma \cdot \gamma}^{1,0} = T_\gamma^{1,0} \circ T_\sigma^{1,0}$

6) $T_{\gamma^{-1}}^{1,0} \circ T_\gamma^{1,0} = Id$, so $T_\gamma^{1,0}: E_{x_0} \to E_{\gamma(1)}$ is a linear isomorphism.

• **Theorem**: Every vector bundle over a disk $\mathbb{D}^n$ is trivializable.

• The problem about finding solutions to $\begin{cases} \nabla s = 0 \\ \text{constrains} \end{cases}$ is related to the Frobenius theorem:

Definition: A rank k distribution on M is a rank k vector subbundle $\mathcal{D} \subset TM$

Definition: Given a rank k distribution $\mathcal{D}$, and $x_0 \in M$, an integral submanifold of $\mathcal{D}$ through x_0 is an immersion of a k-dim manifold $i : L \hookrightarrow M$ st $x_0 \in i(L)$ and $i_*(T_x L) = \mathcal{D}_{i(x)}$ $\forall x \in L$. If for every $x \in M$ there is such a integral submfd, $\mathcal{D}$ is called integrable.

Definition: A distribution $\mathcal{D}$ is involutive if $D_1, D_2 \in \Gamma(\mathcal{D}) \Rightarrow [D_1, D_2] \in \Gamma(\mathcal{D})$

Theorem (Frobenius): $\mathcal{D}$ involutive $\Leftrightarrow$ $\mathcal{D}$ integrable

CONECTIONS ON TM

• From now on we'll focus on the vb $TM \to M$ an conections on it. Moreover we'll regard a conection ∇ as a map $\mathcal{D}(M) \times \mathcal{D}(M) \to \mathcal{D}(M)$, $(D, \bar{D}) \longmapsto D^\nabla \bar{D} = i_D(\nabla \bar{D})$. Leibniz rule becomes $D^\nabla f\bar{D} = Df \cdot \bar{D} + f \cdot D^\nabla \bar{D}$. This is similar to $[D, f\bar{D}] = Df \cdot \bar{D} + f \cdot [D, \bar{D}]$. Are they the same? No since $-^\nabla-$ is not skew. But we can skew-it by $D^\nabla \bar{D} - \bar{D}^\nabla D$ and now measure the difference with $[-,-]$:

Definition. The torsion of a conection ∇ on TM is the $(2,1)$-tensor (skew in the first two indeces)

so $Tor_\nabla \in \Omega^2(M, TM)$

$$Tor_\nabla(D_1, D_2) = D_1{}^\nabla D_2 - D_2{}^\nabla D_1 - [D_1, D_2]$$

and ∇ is torsion free or symmetric when $Tor_\nabla = 0$, ie, $[D_1, D_2] = D_1{}^\nabla D_2 - D_2{}^\nabla D_1$.

*Theorem (Generalized Fundamental thm of Riemannian Geometry): Let (M,g) be a Riemannian (or pseudo-Riem) manifold, and let $T \in \Omega^2(M, TM)$. Then there exists a unique metric connection ∇ such that $\mathrm{Tor}_\nabla = T$, and it is completely determined by the identity

$$(D_1 {}^\nabla D_2) \cdot D_3 = \frac{1}{2}\Big(D_1(D_2 \cdot D_3) + D_2(D_3 \cdot D_1) - D_3(D_1 \cdot D_2)$$

$$+ D_2 \cdot [D_3, D_1] + D_3 [D_1, D_2] - D_1 [D_2, D_3]$$

$$+ D_3 \cdot T(D_1, D_2) + D_2 \cdot T(D_3, D_1) - D_1 \cdot T(D_2, D_3) \Big).$$

Definition: The Levi-Civita connection of a Riemannian manifold M is the unique metric, torsion free connection ∇.

• (Christoffel symbols): Let M be a manifold and ∇ a connection. On a chart $(U; u_1 \dots u_m)$ the connection is given by the Christoffel symbols, $\boxed{\partial_i {}^\nabla \partial_j = \sum_k \Gamma_{ij}^k \partial_k}$, $\Gamma_{ij}^k \in C^\infty(U)$.

The trivial connection $\nabla^0 = d$ on M, $\nabla^0(f\partial_i) = df \otimes \partial_i$, is $D^\nabla(\sum f_i \partial_i) = \sum Df_i \cdot \partial_i$.

By Leibniz, $A = \nabla - \nabla^0$ is $(\nabla - \nabla^0)(f\partial_i) = f_i \nabla \partial_i$

Then, viewing $A = (\omega_{ij})$ as a matrix of 1-forms, it is $\boxed{\omega_{ij} = \sum_k \Gamma_{kj}^i dx_k}$

GEODESICS

Definition: Let ∇ be a connection on M. A curve $\sigma : I \to M$ is a geodesic if $(\sigma^* \nabla)\dot\sigma = 0$.

Observe that if $\sigma : I \to M$, $\dot\sigma \in \Gamma(\sigma^* TM)$, so the previous makes sense. This is analogous to $\dot\sigma {}^\nabla \dot\sigma = 0$, i.e., "constant speed".

Theorem: Given $D_p \in T_pM$, there exists a geodesic $\sigma: I \to M$ passing through p at $t=0$ with speed D_p, and any two agree on the overlapping. It is denoted as σ_{D_p}.

• if $\sigma = (\sigma_1, \dots, \sigma_n)$ in a chart, then the condition of geodesic is

$$\boxed{\sigma_k'' + \sum_{i,j} \sigma_i' \sigma_j' (\Gamma_{ij}^k \circ \sigma) = 0}, \quad k = 1, \dots, n.$$

Lemma (Homogenity): $\sigma_{\lambda v}(t) = \sigma_v(\lambda t)$, when both terms are defined.

Definition: We call domain of the exp map at p to $\mathcal{E}_p := \{ v \in T_pM : \sigma_v \text{ is defined in } [0,1] \}$, and the exponential map at p is

$$\begin{aligned} \exp_p : \mathcal{E}_p \subset T_pM &\longrightarrow M \\ v &\longmapsto \exp_p(v) := \sigma_v(1) \end{aligned}$$

Proposition: $\mathcal{E}_p$ is an open nbhd of 0 in T_pM, $\exp_p$ is smooth and $\sigma_v(t) = \exp_p(tv)$, ie, exp transforms lines in T_pM in geodesics in M.

Lemma: Given $p \in M$, there is $V \subset T_pM$ nbhd of 0 and $U \subset M$ nbhd of p st $\exp_p : V \xrightarrow{\sim} U$ is diffeomorphism. Such nbhd U is called normal nbhd.

TUBULAR NEIGHBOURHOOD

• let $i: N \hookrightarrow M$ be an embedded submnfd, so that $i_*(TN) \simeq TN$ can be viewed as a subbundle of $i^*TM \overset{\text{not}}{=} TM_{|N}$.

Definition: The normal bundle of a submanifold $N \hookrightarrow M$ is the quotient bundle $\mathcal{N}_N := \dfrac{i^*TM}{i_*TN}$, ie, fiberwise it is the quotient vs $T_{i(p)}M / i_*(T_pN)$

Definition: Let (M,g) be Riem, and $N \hookrightarrow M$ submfd. The orthogonal complement of TN in TM is the vector subbundle $T^\perp N \leq TM|_N$ defined by $(T^\perp N)_p = T_p^\perp N$.

Lemma: $\mathcal{N}_N \simeq T^\perp N$ via $\pi : TM|_N \to \mathcal{N}_N$.

Definition: Let $N \hookrightarrow M$. A tubular nbhd of N in M is a diffeomorphism $\varphi : \mathcal{U} \subset \mathcal{N}_N \xrightarrow{\simeq} V \subset M$ between a nbhd $\mathcal{U}$ of the 0-section of $\mathcal{N}_N$ and a nbhd V of N in M, st

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\varphi", "\sim"'] & & V \\ & M \arrow[ul, "0"] \arrow[ur, "i"'] & \end{tikzcd}$$

Theorem (Tubular nbhd): Every embedded submanifold has a tubular neighbourhood.

Corollary (Jordan): Let M be simply-connected and N an embedded, connected, compact, orientable submfd of codim 1. Then $M - N$ has exactly two connected components.

GEODESICS AS MINIMIZING CURVES

- Fix a Riemann mnfd (M, g, ∇), ∇ metric.

Definition: Given C an oriented curve on M (ie, fixed a direction), let ω_C be the volume form of the Riem. mnfd $(C, g|_C = i^*g)$. Given $\Omega \subset C$ segment (mnfd with boundary), the length of Ω is

$$\boxed{\operatorname{len} \Omega := \int_\Omega \omega_C}$$

So given a parametrization $\sigma : I \xrightarrow{\simeq} C$, $\sigma = \sigma(t)$, $\Omega = [a,b] \subset I$,

$$\boxed{\operatorname{len} \sigma|_{[a,b]} = \int_a^b |\dot{\sigma}(t)|\, dt.}$$

Definition: let (M,g) Riem. We say that a metric connexion ∇ on M has skew-sym. torsion if the tensor $H \in \Gamma(\Lambda_2 TM \otimes T^\vee M)$,

$$H(D_1 D_2 D_3) := g(\mathrm{Tor}_\nabla(D_1, D_2), D_3)$$

is skew-sym, $H \in \Omega^3(M)$.

Theorem: let (M,g) Riem and let ∇ be metric with skew-sym torsion. let $\mathcal{U} \subset T_pM$ be a normal nbhd. if $v \in \mathcal{U}$, then for any path σ connecting p and $q = \exp_p(v) = \sigma_v(1)$, we have that

$$\mathrm{len}\ \sigma_v \leq \mathrm{len}\ \sigma$$

$(\sigma_v(t) = \exp_p(tv))$ and the equality holds $\Leftrightarrow$ σ is a reparametrization of σ_v.

Proposition: let (M,g) be Riem. Then

$$d(p,q) := \inf\{\sigma : \sigma \text{ curve joining } p \text{ and } q\}$$

is a distance on M.