ALGEBRAIC TOPOLOGY II

0 : CATEGORIES AND FUNCTORS

<u>Definition</u>. A <u>category</u> $\mathscr{C}$ is the data of

i) A collection of <u>objects</u> $|\mathscr{C}|$

ii) A set $Hom_{\mathscr{C}}(A,B)$ of <u>morphisms</u> or <u>arrows</u> from A to B for any $A, B \in \mathscr{C}$.

iii) A composition law

$$\circ : Hom_{\mathscr{C}}(B,C) \times Hom_{\mathscr{C}}(A,B) \longrightarrow Hom_{\mathscr{C}}(A,C)$$

for every triple $A, B, C \in \mathscr{C}$

iv) An <u>identity</u> morphism $Id_A \in Hom_{\mathscr{C}}(A,A)$ for any $A \in \mathscr{C}$

such that

- $Id_B \circ f = f = f \circ Id_A \qquad \forall f \in Hom_{\mathscr{C}}(A,B)$
- $h \circ (g \circ f) = (h \circ g) \circ f \qquad \forall f, g, h$ composables.

<u>Definition</u> : A morphism $f : A \to B$ in $\mathscr{C}$ is an <u>isomorphism</u> if there exists $g : B \to A$ s.t. $g \circ f = Id_A$ and $f \circ g = Id_B$.

<u>Examples</u> : 1) Sets, category of sets $\begin{cases} \text{obj : sets} \\ \text{arrows : maps of sets} \\ \text{iso : bijective maps} \end{cases}$

2) Grp, category of groups $\begin{cases} \text{obj : groups} \\ \text{arrows : group homomorphisms} \\ \text{iso : group isomorphisms.} \end{cases}$

3) Vect_k, k-v.s. $\begin{cases} \text{obj} : k\text{-v.s.} \\ \text{arrows} : \text{linear maps} \\ \text{isos} : \text{linear isos} \end{cases}$

4) Top, top. spaces category $\begin{cases} \text{obj} : \text{top spaces} \\ \text{arrows} : \text{cont. maps} \\ \text{isos} : \text{homeomorphisms} \end{cases}$

5) $\mathrm{Top}_\bullet$, pointed top spaces $\begin{cases} \text{obj} : \text{pointed top spaces} \\ \text{arrows} : \text{basepoint preserving top cont. maps} \\ \text{isos} : \text{homeomorphisms} \end{cases}$

6) Manfd, category of manifolds, $\begin{cases} \text{obj} : \text{manifolds} \\ \text{arrows} : \text{smooth maps} \\ \text{isos} : \text{diffeomorphisms.} \end{cases}$

<u>Definition</u>: Let $\mathscr{C}$ be a category and $f : A \to B$ a morphism.

a) We say that f is <u>monic</u> (or <u>monomorphism</u>) if for all $g, h : X \to A$ with $f \circ g = f \circ h \Rightarrow g = h$

b) We say that f is <u>epic</u> (or <u>epimorphism</u>) if for all $d, e : B \to Y$ with $d \circ f = e \circ f \Rightarrow d = e$

<u>Lemma</u>: In Sets, monic $\Leftrightarrow$ injective, and epic $\Leftrightarrow$ surjective.

<u>Definition</u>: Let $\mathscr{C}, \mathscr{D}$ be categories. A <u>(covariant) functor</u> $F : \mathscr{C} \to \mathscr{D}$ consists of

i) An assignment $A \longmapsto F(A)$ sending an object A of $|\mathscr{C}|$ to a object $F(A)$ of $|\mathscr{D}|$

(ii) Maps $\mathrm{Hom}_{\mathscr{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ for any pair $A, B \in |\mathscr{C}|$.

such that

- $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$ $\forall A \in |\mathscr{C}|$
- $F(g \circ f) = F(g) \circ F(f)$ $\forall f, g$ composables

- Functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{E}$ compose to a functor $\mathcal{C} \to \mathcal{E}$.

<u>Definition</u>: Let $\mathcal{C}$ be a category. Its <u>opposite category</u> $\mathcal{C}^{op}$ is another category with

i) $|\mathcal{C}^{op}| = |\mathcal{C}|$

ii) $\mathrm{Hom}_{\mathcal{C}^{op}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(B,A)$ with the following composition law: for $f^{op}: A \to B$ and $g^{op}: B \to C$, which correspond with $f: B \to A$ and $g: C \to B$, we define $g^{op} \circ f^{op} := (f \circ g)^{op}$

$$\begin{tikzcd} A \arrow[r, "f^{op}"] \arrow[rr, bend right, "(f \circ g)^{op}"'] & B \arrow[r, "g^{op}"] & C \end{tikzcd}$$

<u>Definition</u>: A <u>contravariant functor</u> $F: \mathcal{C} \to \mathcal{D}$ is a (covariant) functor $\mathcal{C}^{op} \to \mathcal{D}$. In other words, it consists of

i) An assignment $A \longmapsto F(A)$

ii) Maps $\mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$

such that

- $F(Id_A) = Id_{F(A)}$
- $F(g \circ f) = F(f) \circ F(g)$

<u>Examples</u>: 1) $Grp \to Sets$, sending every group to its underlying sets, the "<u>forgetful functor</u>"

2) $\pi_1 : Top_* \to Grp$, $H_n(-,A) : Top \to Grp$

3) $-^* : Vect_K \to Vect_K$, taking dual $\left(-^* = \mathrm{Hom}(-,K)\right)$, it is a contravariant functor.

4) $\mathcal{C} : Top \to Rings$, taking set of cont maps $(X \to \mathcal{C}(X)$ & $\varphi \mapsto \varphi^*)$, contravariant.

Algebraic remark: Let G be an empty set.

a) A binary operation on G is a function $G \times G \to G$. We say that G is a magma in this ca

b) A semigroup is a magma whose operation is associative.

c) A monoid is a semigroup with unity

d) A group is a monoid with inverse for every elmt.

$G \times G \longrightarrow G$ s.t. } magma } semigroup } monoid } group

i) associative

ii) unity

iii) inverse

5) $CRings \to Mon$, $R \mapsto M_n(R)$.

Definition: Let $\mathscr{C}$ be a category and let us fix an object $B \in |\mathscr{C}|$. The representable functor or functor represented by B is

$$\begin{aligned} Hom_{\mathscr{C}}(B,-) : \mathscr{C} &\longrightarrow Set \\ C &\longmapsto Hom(B,C) \\ (f: C \to C') &\longmapsto f \circ - : Hom(B,C) \to Hom(B,C') \\ & \qquad\qquad\qquad g \longmapsto f \circ g. \end{aligned}$$

By setting $Hom_{\mathscr{C}}(-,B)$, we obtain a contravariant functor.

Definition: Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\alpha : F \to G$ is a collection of maps $\alpha_A : F(A) \to G(A)$ in $\mathcal{D}$ (for all $A \in |\mathcal{C}|$) such that for every $f : A \to B$ in $\mathcal{C}$ the following square commutes:

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd} \qquad G(f) \circ \alpha_A = \alpha_B \circ F(f)$$

We say that α is a natural isomorphism if α_A is a isomorphism for all $A \in |\mathcal{C}|$, and that F and G are naturally isomorphic.

Example: Taking double dual is a canonical natural transformation of $\mathrm{Id} : \mathrm{Vect}_K \to \mathrm{Vect}_K$, i.e., we take $\alpha_E := \phi_E : E \to E^{**}$ the natural "inclusion",

$$\begin{tikzcd} \mathrm{Id}(E) = E \arrow[r, "\phi_E"] \arrow[d, "\mathrm{Id}(T) = T"'] & E^{**} \arrow[d, "T^{**}"] \\ \mathrm{Id}(F) = F \arrow[r, "\phi_F"] & F^{**} \end{tikzcd}$$

Definition: A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}} \to G \circ F$ and $\varepsilon : F \circ G \to \mathrm{Id}_{\mathcal{D}}$.

Remark: We do not require $G \circ F = \mathrm{Id}_{\mathcal{C}}$ or $F \circ G = \mathrm{Id}_{\mathcal{D}}$.

Definition: Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is

a) full if $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is surjective $\forall A, B \in |\mathcal{C}|$

b) faithful if " " is injective "

c) essentially injective if for all $D \in |\mathcal{D}|$ $\exists C \in |\mathcal{C}|$ and a iso $F(C) \xrightarrow{\sim} D$.

Theorem: A functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is full, faithful and essentially surjective.

Example: Let $K\text{-Vect}^{fin}$ = cat. of finite-dim v.s.

$$Mat_K = \begin{cases} \text{obj}: \mathbb{N} \\ \text{arrows}: Mat_K(n,m) = n \times m \text{ matrices over } K \\ \text{composition: matrix multiplication} \end{cases}$$

Then there's a equivalence of categories

$$Mat_K \longrightarrow K\text{-Vect}^{fin}$$
$$n \longmapsto K^n$$
$$Mat_K(n,m) \ni A \longmapsto (v \mapsto Av)$$

Definition: Let $\mathcal{C}$ be a category. An <u>initial object</u> is an object $A \in |\mathcal{C}|$ st for any $B \in |\mathcal{C}|$ $\# \operatorname{Hom}(A,B) = 1$.

Ex: In Ring, $\mathbb{Z}$ is an initial object (any morph $\mathbb{Z} \to R$ is determined by the mapping of 1).

. Let $\mathcal{C}$ be a category, and consider the category $Fun(\mathcal{C}^{op}, Set)$: $\begin{cases} \text{obj: functors } \mathcal{C}^{op} \to Set \\ \text{arrows: natural transformations.} \end{cases}$

Definition: The <u>Yoneda functor</u> is

$$h: \mathcal{C} \longrightarrow Fun(\mathcal{C}^{op}, Set)$$
$$C \longmapsto \operatorname{Hom}_{\mathcal{C}}(-, C)$$
$$(f: C \to C') \longmapsto \alpha$$

where α is the natural transformation: $\alpha_A = f \circ - : \operatorname{Hom}(A,C) \to \operatorname{Hom}(A,C')$, $g \mapsto f \circ g$.

Lemma (Yoneda): The Yoneda functor $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ is fully faithfull, i.e., for all A, B in $\mathcal{C}$ the map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(C, C') & \xrightarrow{h} & \mathrm{Hom}_{\mathrm{Fun}}(\mathrm{Hom}_{\mathcal{C}}(-,C), \mathrm{Hom}_{\mathcal{C}}(-,C')) \\ f & \longmapsto & \alpha \end{array}$$

is a bijection.

- One should think this as a sort of injectivity: if two objects C, C become isomorphic via h, i.e, $\mathrm{Hom}_{\mathcal{C}}(-,C) \overset{\alpha}{\cong} \mathrm{Hom}_{\mathcal{C}}(-,C')$, then they where isomorphic in $\mathcal{C}$, i.e,

"Tell me how you relate and I'll tell you who you are"

LIMITS AND COLIMITS

Definition. Let $\mathcal{C}$ be a category. An object $A \in |\mathcal{C}|$ is terminal if for every $B \in |\mathcal{C}|$ there exists a unique map $B \to A$, i.e, $\# \mathrm{Hom}(B, A) = 1$.

Lemma: A terminal object in a category is unique up to unique isomorphism.

Example: $\mathrm{Set} \to \{*\}$; $\mathrm{AbGrp} \to \{e\}$, $\mathrm{Grp} \to \{e\}$, $\mathrm{Top} \to \{*\}$ ...

Definition: Let $\mathcal{C}$ be a category. An object $A \in |\mathcal{C}|$ is initial if for any $B \in |\mathcal{C}|$ there exists unique map $A \to B$, i.e, $\# \mathrm{Hom}(A, B) = 1$.

Examples: $\mathrm{Set} \to \emptyset$, $\mathrm{AbGrp} \to \{e\}$, $\mathrm{Grp} \to \{e\}$, $\mathrm{Top} \to \emptyset$.

Note that initial objects are terminal objects in $\mathcal{C}^{op}$.

Definition: Let $\mathcal{C}$ be a category and let I a category that we will call the *shape category*.

A *diagram of shape* I is a functor $F: I \longrightarrow \mathcal{C}$

$$\begin{tikzcd} & 2 \arrow[d] \\ 1 \arrow[r] & 3 \end{tikzcd} \quad \xrightarrow{\ F\ } \quad \begin{tikzcd} & S_2 \arrow[d] \\ S_1 \arrow[r] & S_3 \end{tikzcd}$$

I $\qquad\qquad\qquad\qquad\qquad \mathcal{C}$

Definition: Given a diagram F of shape I, a *cone* to F consists of an object $N \in |\mathcal{C}|$ and maps $\psi_i : N \longrightarrow F(i)$ for any $i \in |I|$, such that for all $g: i \longrightarrow j$ in I $\psi_j = F(g) \circ \psi_i$

$$\begin{tikzcd} N \arrow[r] \arrow[d] \arrow[dr] & S_2 \arrow[d] \\ S_1 \arrow[r] & S_3 \end{tikzcd}$$

commuting

Definition: Let $(N, \{\psi_i\}_{i\in I})$, $(M, \{\varphi_i\}_{i\in I})$ two cones over F. A *morphism* from $(N, \{\psi_i\})$ to $(M, \{\varphi_i\})$ is a map $h: N \longrightarrow M$ s.t. $\psi_i = \varphi_i \circ h \quad \forall\, i \in I$

$$\begin{tikzcd} & M \arrow[rr] \arrow[dd] \arrow[ddrr] & & S_2 \arrow[dd] \\ N \arrow[ur, "h"] \arrow[urrr] \arrow[dr] \arrow[drrr] & & & \\ & S_1 \arrow[rr] & & S_3 \end{tikzcd}$$

commutes.

All cones to F and morphisms of cones over F form a category!

Definition: A *limit* (inverse limit or projective limit) of F is a terminal object in the category of cones to F

Examples: 1) I: $1 \longrightarrow 2$, $\mathcal{C}$ any cat, $C_1 \xrightarrow[F(1)]{f} C_2 \Rightarrow$ $\begin{tikzcd}[row sep=small] & C_1 \\ C_1 \arrow[ur, "Id"] \arrow[dr, "f"'] & \\ & C_2 \end{tikzcd}$ is the terminal o

2) I = discrete category (category where arrows are only identities). Then a diagram of shape I is just a collection $\{C_i : i \in I\}$. The *product* of $\{C_i\}$ is the limit of F.

One can check that is it exactly $S_1 \times S_2 \begin{array}{l} \xrightarrow{\pi_1} S_1 \\ \xrightarrow{\pi_2} S_2 \end{array}$ this is the limit, the terminal obj., "$\lim F$".

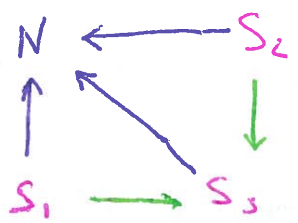
Definition: Let F a diagram of shape $I : \bullet \overset{a}{\underset{b}{\rightrightarrows}} \bullet$. The <u>equalizer</u> of $F(a)$ and $F(b)$ is $\lim F$.

Lemma: In Grp, if F is a diagram of shape I and $F(b)$ is the 0 map, then $\lim F = \mathrm{Ker}\,(F(a))$

Important: The product as in 2) is just the same as the definition given with the universal property!!

The dual version is "colimit": arrows point to the other direction.

Definition: Let $\mathcal{C}$ be a cat. and let $F : I \to \mathcal{C}$ be a diagram of shape I. A <u>cone</u> <u>from</u> F is an object $N \in |\mathcal{C}|$ together with maps $\psi_i : F(i) \to N$ $\forall i \in |I|$, st for all $g : i \to j$ in I then $\psi_i = \psi_j \circ F(g)$. Morphisms of cones are defined analogously.



Definition: The <u>colimit</u> (<u>direct limit or inductive limit</u>) of F is an initial object in the category of cones from F, $\operatorname{colim} F$.

Examples: 1) If I has a terminal object i, then $\operatorname{colim} F = F(i)$ (and the same for limits!!)

2) (<u>Coproducts</u>) I = discrete category. The <u>coproduct</u> of $\{G_i : i \in I\}$ is the colimit of F.

(where $G_i = F(i)$, ie, F is a diagram of shape I).

* In Sets and Top, the colimit is the disjoint union.

* For Gp, the colimit is the direct sum.

Definition: If $I = \begin{tikzcd} \overset{1}{\bullet} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \overset{2}{\bullet} \end{tikzcd}$, then the **coequalizer** of $F(a)$ and $F(b)$ is $\operatorname{colim} F$

Lemma: In AbGps: if $F(b) = 0$, then $\operatorname{Coker} F(a) = \operatorname{colim} F$

FIBRED PRODUCTS AND COPRODUCTS (PULLBACKS AND PUSHOUTS)

Definition: Let $\mathcal{C}$ be a cat, and let F be a diagram of shape

$$I = \begin{tikzcd} & \overset{3}{\bullet} \arrow[d] \\ \underset{2}{\bullet} \arrow[r] & \underset{1}{\bullet} \end{tikzcd}$$

The **fibred product** or **pullback** of $F(2)$ and $F(3)$ over $F(1)$ is $\lim F$.

Definition: Let now F be a diagram of shape

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r] \arrow[d] & \overset{3}{\bullet} \\ \underset{2}{\bullet} & \end{tikzcd}$$

The **fibred coproduct** or **pushout** of $F(2)$ and $F(3)$ over $F(1)$ is $\operatorname{colim} F$

Example: In Top, the pullback and pushout are precisely the ones we know.

ADJOINT FUNCTORS

Definition: Let $\mathcal{C}, \mathcal{D}$ be categories and consider functors.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

We say that F and G are **adjoint** (more precisely, that G is **left-adjoint** to F and F is **right-adjoint** to G)

$$\operatorname{Hom}_{\mathcal{C}}(G(-), -) \quad \text{and} \quad \operatorname{Hom}_{\mathcal{D}}(-, F(-)) \qquad (: \mathcal{D}^{op} \times \mathcal{C} \to \mathrm{Set})$$

are naturally isomorphic ("G appears to the left and F to the right")

• That is, $\forall D \in |\mathcal{D}|$, $C \in |\mathcal{C}|$, we have a bijection

$$\mathrm{Hom}_{\mathcal{C}}(G(D), C) = \mathrm{Hom}_{\mathcal{D}}(D, F(C))$$

<u>Examples</u>: 1)
$$\begin{tikzcd} \mathrm{AbGp} \arrow[r, bend left, "\text{forgetful}"] & \mathrm{Set} \arrow[l, bend left, "\mathbb{Z}[-] \text{ the free abelian grp}"] \end{tikzcd}$$

$\mathbb{Z}[-]$ is left-adjoint to forgetful. Forgetful is right-adjoint to $\mathbb{Z}[-]$.

Why? This is the fucking universal property of the free ab grp!!

$$\mathrm{Hom}_{\mathrm{AbGp}}(\mathbb{Z}[S], A) = \mathrm{Hom}_{\mathrm{Sets}}(S, A).$$

2) $R \xrightarrow{f} R'$ ring hom, then
$$\begin{tikzcd} R\text{-mod} \arrow[r, bend left, "- \otimes_R R'"] & R'\text{-mod} \arrow[l, bend left, "\text{restriction}"] \end{tikzcd}$$
are adjoint, giving

$$\mathrm{Hom}_{R'}(M \otimes_R R', P) = \mathrm{Hom}_R(M, P)$$

(P a R'-mod is also a R-mod by setting $r \cdot p = f(r) \cdot p$.)

<u>ADJOINTS AN (CO) LIMITS</u>

<u>Definition</u>: let $T: \mathcal{C} \to \mathcal{D}$ be a functor, let I be a shape category and let $F: I \to \mathcal{C}$ be a diagram.
We say that T <u>commutes with limits</u> when

$$T(\lim F) = \lim(T \circ F),$$

and we say that <u>T commutes with colimits</u> when

$$T(\operatorname{colim} F) = \operatorname{colim}(T \circ F).$$

<u>Theorem</u>: 1) If T is right adjoint to other functor, and $\lim F$ exists, then so $\lim(T \circ F)$ exists and the equality holds.

2) If T is left adjoint to some functor, and $\operatorname{colim} F$ exists, then so $\operatorname{colim}(T \circ F)$ exists and the eq. holds.

SLICE CATEGORY

Definition: Let $\mathcal{C}$ be a category, and let $S \in |\mathcal{C}|$. The slice category $\mathcal{C}/S$ of $\mathcal{C}$ over S is another category with

- objects: arrows $\bullet \longrightarrow S$, i.e. all morphisms with target S
- morphisms: Given objects $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$, a morphism in $\mathcal{C}/S$ is a morphism $X \xrightarrow{\varphi} Y$ in $\mathcal{C}$ such that $g\varphi = f$:

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

DIRECT AND INVERSE LIMITS

Definition: Let I be a poset (partially ordered set). We say that I is a directed set, or a filtered set, if $\forall \alpha, \beta \in I \ \exists \gamma : \alpha, \beta \leq \gamma$.

Definition: Let $\mathcal{C}$ be a category. A direct system, or inductive system over I is a collection of objects $\{X_\alpha : \alpha \in I\}$ of $\mathcal{C}$ and morphisms $i_{\alpha\beta} : X_\alpha \to X_\beta \ \forall \alpha \leq \beta$ such that

i) $i_{\alpha\alpha} = \mathrm{Id}_{X_\alpha} \quad \forall \alpha \in I$

ii) $i_{\beta\gamma} \circ i_{\alpha\beta} = i_{\alpha\gamma} \quad \forall \alpha \leq \beta \leq \gamma$

$$\begin{tikzcd} X_\alpha \arrow[rr, "i_{\alpha\beta}"] \arrow[dr, "i_{\alpha\gamma}"'] & & X_\beta \arrow[dl, "i_{\beta\gamma}"] \\ & X_\gamma & \end{tikzcd}$$

Definition: Let $\mathcal{C}$ be a category and let $\{X_\alpha\}_{\alpha \in I}$ be a direct system. The direct limit (if it exists) of this system is an object $\varinjlim X_\alpha$ of $\mathcal{C}$ together with maps $i_\alpha : X_\alpha \to \varinjlim X_\alpha \ \forall \alpha \in I$ satisfying that $i_\beta \circ i_{\alpha\beta} = i_\alpha \ \forall \alpha \leq \beta$,

$$\begin{tikzcd} X_\alpha \arrow[rr, "i_{\alpha\beta}"] \arrow[dr, "i_\alpha"'] & & X_\beta \arrow[dl, "i_\beta"] \\ & \varinjlim X_\alpha & \end{tikzcd}$$

and the following universal property: for any object Y of $\mathcal{C}$ and maps $\varphi_\alpha : X_\alpha \to Y$ commuting with the $i_{\alpha\beta}$ (ie, $\varphi_\beta \circ i_{\alpha\beta} = i_\alpha$), there exists a unique morphism $\varphi : \varinjlim X_\alpha \to Y$ such that

$\varphi \circ i_\alpha = \varphi_\alpha \quad \forall \alpha \in I$,

$$\begin{tikzcd} X_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "\varphi_\alpha"'] & \varinjlim X_\alpha \arrow[d, "\varphi"] \\ & Y \end{tikzcd}$$

• The direct limit is just a colimit where the shape category is given by I, with as many morphisms as pairs of indexes with $\alpha \leq \beta$. Therefore, if it exists, the direct limit is unique up to isomorphism.

<u>Examples</u>: 1) In **Sets**, given a directed system $\{X_\alpha : \alpha \in I\}$, its direct limit is given by a quotient of the disjoint union:

$$\varinjlim X_\alpha = \frac{\coprod_\alpha X_\alpha}{\equiv}, \qquad x_\alpha \equiv x_\beta \iff \exists \gamma : i_{\alpha\gamma}(x_\alpha) = i_{\beta\gamma}(x_\beta).$$

2) In **AbGps**, given any directed system $\{G_\alpha : \alpha \in I\}$, its direct limit is

$$\varinjlim G_\alpha = \frac{\bigoplus_\alpha G_\alpha}{\langle g_\alpha - i_{\alpha\beta}(g_\alpha) : g_\alpha \in G_\alpha, \alpha \in I \rangle}$$

$\longleftarrow$ smallest subgroup containing the relations $g_\alpha = i_{\alpha\beta}(g_\alpha)$.

3) In **AbGps**, for the directed set $\mathbb{N}$, a directed system is a sequence of groups homomorphisms

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} G_4 \longrightarrow \cdots$$

and its direct limit is

$$\varinjlim G_\alpha = \frac{\bigoplus_\alpha G_\alpha}{\langle (g_1, g_2 - f_1(g_1), g_3 - f_2(g_2), \ldots) : g_\alpha \in G_\alpha, \alpha \in \mathbb{N} \rangle}$$

4) In **Top**, $\varinjlim X_\alpha$ is given by placing the final topology of $\{i_\alpha : X_\alpha \to \varinjlim X_\alpha\}$ to the set-theoretic direct limit of 1).

Definition: Let $\mathcal{C}$ be a category. An <u>inverse system</u> or projective system over a directed set I is a collection of objects $\{X_\alpha : \alpha \in I\}$ of $\mathcal{C}$ and morphisms $\pi_{\beta\alpha} : X_\beta \to X_\alpha$ $\forall \alpha \leq \beta$ s.t.

i) $\pi_{\alpha\alpha} = Id_{X_\alpha}$ $\forall \alpha \in I$

ii) $\pi_{\beta\alpha} \circ \pi_{\gamma\beta} = \pi_{\gamma\alpha}$ $\forall \alpha \leq \beta \leq \gamma$

$$\begin{tikzcd} & X_\gamma \arrow[dl] \arrow[dr] & \\ X_\beta \arrow[rr] & & X_\alpha \end{tikzcd}$$

Definition: Let $\mathcal{C}$ be a category and let $\{X_\alpha : \alpha \in I\}$ be an inverse system. The <u>inverse limit</u> (if it exists) of this system is an object $\varprojlim X_\alpha$ of $\mathcal{C}$ together with morphisms $\pi_\alpha : \varprojlim X_\alpha \to X_\alpha$ $\forall \alpha \in I$ satisfying that $\pi_\beta \circ \pi_{\beta\alpha} = \pi_\alpha$ $\forall \alpha \leq \beta$,

$$\begin{tikzcd} & \varprojlim X_\alpha \arrow[dl] \arrow[dr] & \\ X_\beta \arrow[rr] & & X_\alpha \end{tikzcd}$$

and the following <u>universal property</u>: given any object Y of $\mathcal{C}$ and morphisms $\varphi_\alpha : Y \to X_\alpha$ commuting with the $\pi_{\alpha\beta}$ (ie, $\varphi_\beta \circ \pi_{\beta\alpha} = \varphi_\alpha$), there exists a unique morphism $\varphi : Y \to \varprojlim X_\alpha$ such that $\pi_\alpha \circ \varphi = \varphi_\alpha$:

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[dr, "\varphi_\alpha"'] & \varprojlim X_\alpha \arrow[d, "\pi_\alpha"] \\ & X_\alpha \end{tikzcd}$$

As before, inverses limits are just an special case of limits, thus if it exists, it is unique up to isomorphism.

Examples: 1) In **Sets**, given a inverse system $\{X_\alpha\}$ its inverse limit is

$$\varprojlim X_\alpha = \{ (x_\alpha) \in \prod_\alpha X_\alpha : \pi_{\beta\alpha}(x_\beta) = x_\alpha \ \forall \alpha \leq \beta \}$$

2) In **Top**, the inverse limit is given by endowing $\varprojlim X_\alpha$ with the subspace topology of the product topology, or, what is the same, by inducing the initial topology given by $\pi_\alpha : \varprojlim X_\alpha \to X_\alpha$.

• The universal properties can be expressed as

$$\mathrm{Hom}\left(\varinjlim X_\alpha, Y\right) = \varprojlim \mathrm{Hom}(X_\alpha, Y),$$

$$\mathrm{Hom}\left(Y, \varprojlim X_\alpha\right) = \varprojlim \mathrm{Hom}(Y, X_\alpha).$$

I. SINGULAR COHOMOLOGY

- Let R be a commutative ring, and M, N R-modules. Module homomorphisms $g: M_1 \to M_2$ and $h: M_1 \to M_2$ induce, for an R-module N,

$$g^*: \mathrm{Hom}_R(M_2, N) \longrightarrow \mathrm{Hom}_R(M_1, N) \qquad , \qquad h_*: \mathrm{Hom}_R(N, M_1) \longrightarrow \mathrm{Hom}_R(N, M_2)$$
$$f \longmapsto g^* f := f \circ g \qquad\qquad\qquad f \longmapsto h_* f = h \circ f .$$

which define functors

$$\mathrm{Hom}_R(-, N): \mathrm{Mod}_R^{op} \to \mathrm{Mod}_R \qquad , \qquad \mathrm{Hom}_R(N, -): \mathrm{Mod}_R \to \mathrm{Mod}_R$$

Definition: a) A <u>chain complex</u> of R-modules C_* is a sequence of R-modules C_n, $n \geq 0$ and module homomorphisms $\partial_n: C_n \to C_{n-1}$ s.t. $\partial_{n-1} \circ \partial_n = 0$.

b) A <u>cochain complex</u> of R-modules C^* is a sequence of R-modules C^n, $n \geq 0$ and module homomorphisms $\delta^n: C^n \to C^{n+1}$ s.t. $\delta^{n+1} \circ \delta^n = 0$

$$C^* = (C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots)$$
$$C_* = (C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \dots)$$

Definition: a) A <u>chain map</u> $f: C_* \to D_*$ is a sequence of R-module homomorphisms $f_n: C_n \to D_n$ such that

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ f_n \downarrow & & \downarrow f_{n-1} \\ D_n & \xrightarrow{\partial_{n-1}} & D_{n-1} \end{array}$$

commutes

b) A (co)chain map $f: C^* \to D^*$ is a sequence of R-module homomorphisms $f^n: C^n \to D^n$ s.t.

$$\begin{tikzcd} C^n \arrow[r, "\delta^n"] \arrow[d, "f^n"'] & C^{n+1} \arrow[d, "f^{n+1}"] \\ D^n \arrow[r, "\delta^n"] & D^{n+1} \end{tikzcd}$$

commutes

· We obtain two categories $Ch_{\geq 0}$ of chain complexes (over R) and $Ch^{\geq 0}$ of cochain complexes (over R). Observe that the categories $Ch^{\geq 0}$ and $Ch_{\leq 0}$ are equivalent, $C^* \mapsto C_*$ where $C_n := C^{-n}$, $n \leq 0$.

Definition: a) Given a chain complex of R-modules C_*, the n-th homology group of C_* is

$$H_n(C_*) := \frac{\operatorname{Ker} \partial_n : C_n \to C_{n-1}}{\operatorname{Im} \partial_{n+1} : C_{n+1} \to C_n}, \quad n \geq 1$$

and $H_0(C_*) := \frac{C_0}{\operatorname{Im} \partial_1}$.

b) Given a cochain complex of R-modules, the n-th cohomology group of C^* is

$$H^n(C^*) := \frac{\operatorname{Ker} \delta^n : C^n \to C^{n+1}}{\operatorname{Im} \delta^{n-1} : C^{n-1} \to C^n}, \quad n \geq 1$$

and $H^0(C^*) := \operatorname{Ker} \delta^0$.

· CAUTION: They are not just "groups", but R-modules!

Eg: Given a chain complex C_*, and M a R-module, setting $C^n := \operatorname{Hom}(C_n, M)$ and $\delta^n := (\partial_{n+1})^*$ gives a cochain complex.

SINGULAR COHOMOLOGY

• Quick review : Let X be a top space, R a comm ring and M a R-module. Set

$$\mathcal{S}(X)_n := \{ \sigma : \Delta^n \to X \text{ cont} \} \quad \text{the singular complex}$$

Now set $C_n(X;R) := R[\mathcal{S}(X)_n]$ the R-linearization, ie, the free R-module generated by $\mathcal{S}(X$ and form $C_*(X;R)$ the singular chain complex with differential $\partial_n := \sum (-1)^j \delta_j^*$.

Definition : Let X be a top space. The singular cochain complex with coefficients in a R-module M is

$$C^*(X;M) := \mathrm{Hom}_R (C_*(X;R), M) ,$$

and the n-th singular cohomology group of X with coefficients in M is

$$H^n(X;M) := H^n(C^*(X;M)).$$

• We obtain functors

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "\mathcal{S}(-)"] \arrow[rrrr, bend right, "H^n(-{,}M)"'] & s\mathrm{Set} \arrow[r, "C_*(-{,}R)"] & Ch_{\geq 0} \arrow[r, "\mathrm{Hom}_R(-{,}M)"] & Ch^{\geq 0} \arrow[r, "H^n(\,)"] & \mathrm{Mod}_R \end{tikzcd}$$

$H^n(-,M)$ a contravariant functor

Theorem (Universal Property of the Free module) : Let S be a set and M a R-module. Given $\varphi : S \to M$ a map of sets, there exist a unique R-module homomorphism $\overline{\varphi} : R[S] \to M$ st. $\overline{\varphi} \circ i = \varphi$

$$\mathrm{Hom}_R(R[S], M) \xrightarrow{\sim} \mathrm{Map}(S, M)$$
$$f \longmapsto (s \mapsto f(1 \cdot s))$$

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "i"'] & M \\ R[S] \arrow[ur, dashed, "\overline{\varphi}"'] & \end{tikzcd}$$

Corollary : There is an R-module isomorphism

$$C^n(X;M) \xrightarrow{\sim} \mathrm{Map}(\mathcal{S}(X)_n, M)$$
$$f \longmapsto f(1 \cdot \sigma)$$

Under this description, $\delta^n : C^n(X;M) \to C^{n+1}(X;M)$ has the form, for $\alpha : S(X)_n \to M$,

$$(\delta^n \alpha)(\sigma) = \sum_{i=1}^{n+1} (-1)^i \alpha(\delta_i^* \sigma),$$

re $\delta_i^* \sigma$ is the composite $\Delta^n \xrightarrow{(\delta_i)_*} \Delta^{n+1} \xrightarrow{\sigma} X$

How to relate homology & cohomology? Are they the same? Let C_* a chain complex and define

$$\Phi : H^n(\mathrm{Hom}_R(C_*;M)) \longrightarrow \mathrm{Hom}_R(H_n(C_*), M)$$

$$[f : C_n \to M] \longmapsto ([c] \longmapsto f(c))$$

which is well defined. Is it iso? NO, ie, first form the cochain complex and then take cohomology is not the same as first take the homology and then form the cochain complex.

• What results we studied for singular homology are true for singular cohomology?

Definition. Let (X, X') a pair of spaces. The relative chain complex is

$$C_*(X, X';R) := \frac{C_*(X;R)}{C_*(X';R)}$$

and the relative cochain complex is

$$C^*(X, X';R) := \mathrm{Hom}_R(C_*(X, X';R), M)$$

Definition: The n-th relative cohomology group is

$$H^n(X, X';M) := H^n(C^*(X, X';M))$$

Lemma: Let $0 \to M' \xrightarrow{j} M \xrightarrow{q} \overline{M} \to 0$ be a short exact sequence of R-modules. The following are equivalent:

1) There is a section $s : \overline{M} \to M$: $q \circ s = \mathrm{Id}_{\overline{M}}$

2) There is a retraction $r : M \to M'$: $r \circ j = \mathrm{Id}_{M'}$

3) There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & \overline{M} & \to & 0 \\ & & \| & & \downarrow\wr & & \| & & \\ 0 & \to & M' & \to & M' \oplus \overline{M} & \to & \overline{M} & \to & 0 \end{array}$$

A ses which satisfies this properties is called a split ses.

Lemma: The functor $\mathrm{Hom}_R(-, N)$ preserves split ses: if $0 \to M' \to M \to \overline{M} \to 0$ is a split ses, then

$$0 \to \mathrm{Hom}_R(\overline{M}, N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N) \to 0$$ also is a split ses.

Theorem: Let (X, X') be a pair of spaces and consider the split ses

$$0 \to C_*(X'; R) \to C_*(X, R) \to C_*(X, X'; R) \to 0$$

then the induced sequence

$$0 \to C^*(X, X'; M) \to C^*(X; M) \to C^*(X'; M) \to 0$$

also is a split ses.

Important: Every ses of free modules (eg, $R[-]$ or vector spaces) always splits. More explicitly, if $\overline{M}$ is free, then it splits.

Corollary: There's a long exact sequence

$$
\begin{tikzcd}
 & & 0 \arrow[dll, out=0, in=180, looseness=0.5] \\
H^0(X, X'; M) \arrow[r] & H^0(X; M) \arrow[r] & H^0(X'; M) \arrow[dll, out=0, in=180, looseness=0.5] \\
H^1(X, X'; M) \arrow[r] & H^1(X; M) \arrow[r] & H^1(X'; M) \arrow[dll, out=0, in=180, looseness=0.5] \\
H^2(X, X'; M) \arrow[r] & \ldots &
\end{tikzcd}
$$

Definition: a) A chain homotopy from $f: C_* \to D_*$ to $g: C_* \to D_*$ is a sequence of R-module homomorphisms $P_n: C_n \to D_{n+1}$ s.t. $\partial^D_{n+1} \circ P_n + P_{n-1} \circ \partial^C_n = f_n - g_n$.

b) Let $f, g: C^* \to D^*$ be cochain maps. A cochain homotopy from f to g is a sequence of R-mod hom $P^n: C^n \to D^{n-1}$ s.t.

$$
\begin{tikzcd}
C^{n-1} \arrow[r] \arrow[d, "\delta_C^{n-1}"'] & D^{n-1} \arrow[d, "\delta_D^{n-1}"] \\
C^n \arrow[r, "f^n - g^n"] \arrow[ur, "P_n"] \arrow[d, "\delta_C^n"'] & D^n \arrow[d, "\delta_D^n"] \\
C^{n+1} \arrow[r] \arrow[ur, "P_{n+1}"] & D^{n+1}
\end{tikzcd}
$$

$$P^{n+1} \circ \delta_C^n + \delta_D^{n-1} \circ P^n = f^n - g^n$$

Theorem: $f \simeq g: C^* \to D^*$ homotopic $\Rightarrow$ $f^* = g^*: H^n(D^*; M) \to H^n(C^*; M)$

Lemma: A chain homotopy between two chain maps $f, g: C_* \to D_*$ induces a cochain homotopy between $f^*, g^*: \mathrm{Hom}_R(D_*, M) \to \mathrm{Hom}_R(C_*, M)$

Corollary: Homotopic maps $f \simeq g: X \to Y$ of top spaces induce the same maps in cohomology.
$f^* = g^*: H^n(Y; M) \to H^n(X; M)$.

Lemma: Let $f: C_* \to D_*$ be a chain map between chain complexes of R-modules that induces isomorphisms

$$f_*: H_n(C_*) \xrightarrow{\simeq} H_n(D_*) \quad \forall n \geq 0.$$

If C_n, D_n are free R-modules $\forall n$, then the induced maps in cohomology are also isomorphisms

$$f^*: H^n(\mathrm{Hom}_R(D_*; M)) \xrightarrow{\simeq} H^n(\mathrm{Hom}_R(C_*, M)) \quad \forall n \geq 0, \forall M.$$

- If C_n, D_n are not free it is not true !!

Theorem (Excision for cohomology): Let $Y \subset X' \subset X$ be top spaces with $\overline{Y} \subset \mathring{X}'$. The inclusion induces isomorphisms

$$H^n(X, X'; M) \xrightarrow{\simeq} H^n(X - Y, X' - Y; M) \quad \forall n, \forall M.$$

Lemma: $H^n(*; M) = \begin{cases} M, & n = 0 \\ 0, & n > 0 \end{cases}$

The 4 properties which remains from homology:

1. Long exact sequence
2. Excision
3. Homotopy invariance
4. Values of the one-point space

give axioms to construct a "cohomology theory". It will follow.

II : HOMOLOGICAL ALGEBRA

<u>Definition</u>: a) A functor $F : Mod_R \to Mod_R$ is called right exact / left exact / exact when it takes a ses $0 \to M' \to M \to \bar{M} \to 0$ to a exact sequence

$$0 \to F(M') \to F(M) \to F(\bar{M}) \to 0$$

b) A functor $F : Mod_R^{op} \to Mod_R$ is right exact / left exact / exact when it takes a ses $0 \to M' \to M \to \bar{M} \to 0$ to a exact sequence

$$0 \to F(\bar{M}) \to F(M) \to F(M') \to 0$$

<u>Lemma</u>. The functor $Hom_R(-, N) : Mod_R^{op} \to Mod_R$ is left exact $\forall N$ (but not right exact in general

<u>Theorem</u>: There exist functors

$$Ext_R^n : Mod_R^{op} \times Mod_R \to Mod_R \qquad , \quad n \geq 0$$

such that

E1) $Ext_R^0 = Hom_R$

E2) For a R-module N and a ses $0 \to M' \to M \to \bar{M} \to 0$, there's a natural les

$$\begin{aligned} &0 \to \\ &\to Hom_R(\bar{M}, N) \to Hom_R(M, N) \to Hom_R(M', N) \to \\ &\to Ext_R^1(\bar{M}, N) \to Ext_R^1(M, N) \to Ext_R^1(M', N) \to \\ &\to Ext_R^2(\bar{M}, N) \to \dots \end{aligned}$$

(E3) If M is free and $n \geq 1$, then $\mathrm{Ext}^n_R(M,N) = 0 \quad \forall N$.

The "naturality" means that it is natural in N and in the ses. This functor is called the n-th derived functor of Hom_R

Theorem: Any two families of functors satisfying (E1) – (E3) are naturally isomorphic.

Remarks: 1) K field, every K-module = vs is free $\Rightarrow \mathrm{Ext}^n_K = 0 \quad \forall n \geq 1$

2) If R is a commutative ring, $a \in R$ is a non-zero divisor, then for a R-module N, call $\mathrm{Ker}\, a = \mathrm{Ker}\,(N \xrightarrow{a\cdot} N) = \{n \in N : an = 0\}$; and denote $N/a = N/aN$. Then
$\underset{\text{"}}{}{}_aN$

$$\mathrm{Hom}_R(R/a, N) \simeq {}_aN \qquad , \qquad \mathrm{Ext}^1_R(R/a, N) \simeq N/a .$$

where R/a is $R/(a)$.

3) If R is a PID (eg. $\mathbb{Z}$, k, $k[x]$, ...), then submodules of free modules are also free.

Therefore $\mathrm{Ext}^n_R = 0 \quad \forall n \geq 2$

Corollary: 1) $\mathrm{Ext}^n_{\mathbb{Z}}(M,N) = 0 \quad , \; n \geq 2$

2) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

3) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/\gcd(m,n)\mathbb{Z}$

PROJECTIVE MODULES

Definition: An R-module P is projective if for any pair of module maps $f : P \to N$, $p : M \to N$ with p surjective, $\exists\, g : P \to M : p \circ g = f$.

$$\begin{tikzcd} & M \arrow[d, "p"] \\ P \arrow[ur, dashed, "g"] \arrow[r, "f"] & N \end{tikzcd}$$

, ie, f admits a lift.

Proposition: Every free module is projective.

Lemma: Projective modules are direct summands of free modules.

Concretely, P projective $\iff \exists Q$ module : $P \oplus Q$ is free.

Proposition: P projective $\Rightarrow \mathrm{Hom}_R(P, -)$ is exact.

Definition: A projective resolution of an R-module M is an exact sequence of the following form:

$$\dots \to P_n \to P_{n-1} \to \dots \to P_0 \to M \to 0$$

with P_i projective.

The deleted resolution is the chain complex $P_M := (\dots \to P_n \to P_{n-1} \to \dots \to P_1 \to P_0)$.

Lemma: If P_M is a deleted resolution, then

$$H_n(P_M) = \begin{cases} M, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Theorem: Every R-module admits a projective resolution.

Definition: Let M, N be R-modules, and choose a projective resolution of M, P_M. Define

$$\mathrm{Ext}^n_R(M,N) := H^n(\mathrm{Hom}_R(P_M, N)).$$

It is functorial on N, and we'll see it is also in M. The big theorem will tell us it does not depend on the chosen proj resolution.

Definition: a) A projective chain complex $P_* = (\dots \to P_2 \to P_1 \to P_0)$ is a chain complex with all P_i projective modules.

b) An acyclic chain complex is a chain complex C_* such that $H_n(C_*) \simeq 0$ $\forall n \geq 1$ (i.e. C_* is an exact sequence).

Theorem: If P_* is a proj chain complex, C_* is acyclic and

$$\varphi : H_0(P_*) \longrightarrow H_0(C_*)$$

is a module map, there exists a chain map $f : P_* \to C_*$ such that $f_* = \varphi : H_0(P_*) \to H_0(C_*)$.

Moreover, such f is unique up to chain homotopy.

Corollary: Any two deleted projective resolutions P_M and P'_M of M are chain homotopy equivalent.

Corollary: $Ext^n_R(M,N) = H^n(Hom_R(P_M, N))$ is also functorial on M (and on N) and does not depend on the choice of the projective resolution.

Corollary (E3): If M is projective (or free) $\Rightarrow Ext^n_R(M,N) \simeq 0$, if $n \geq 1$.

Corollary (E1): $Ext^0_R = Hom_R$

Corollary: Ext^n_R commutes with finite direct sums:

$$Ext^n_R(M \oplus M', N) \simeq Ext^n_R(M,N) \oplus Ext^n_R(M', N).$$

Definition: Let $f : C_* \to D_*$ be a chain map. The mapping cone $C(f)_*$ is a chain complex with $C(f)_n := D_n \oplus C_{n-1}$ and

$$\partial^{C(f)}_n : D_n \oplus C_{n-1} = C(f)_n \longrightarrow C(f)_{n-1} = D_{n-1} \oplus C_{n-2}$$

$$(x, y) \longmapsto (\partial^D_n x + f_{n-1}(y), -\partial^C_n y)$$

Definition: For a chain complex C_*, define $C[1]_*$ as the chain complex resulting by shifting everything a degree up:

$$C[1]_n := C_{n-1} \quad , \quad \partial^{C[1]}_n := -\partial_{n-1}$$

In particular, $H_n(C[1]_*) = H_{n-1}(C_*)$.

Proposition (E2) : The long exact sequence for Ext_R holds.

III : UNIVERSAL COEFFICIENTS FORMULA

• Let R be a commutative ring, C_* chain complex and N an R-module. We considered a map

$$\Phi : H^n(\mathrm{Hom}_R(C_*, N)) \longrightarrow \mathrm{Hom}_R(H_n(C_*), N)$$

which was well-defined but wasn't iso. We can say something about this map:

<u>Theorem</u> (Universal Coefficients Formula): Let R be a PID and let C_* be a levelwise free chain complex. Then there's a split ses

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C_*), N) \longrightarrow H^n(\mathrm{Hom}_R(C_*, N)) \xrightarrow{\Phi} \mathrm{Hom}_R(H_n(C_*), N) \to 0$$

which is natural in C_*, but the splitting is not.

<u>Corollary</u>: Let R be a PID, N R-module and (X, X') a pair of spaces. Then there's a natural ses

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X, X'; R), N) \to H^n(X, X'; N) \to \mathrm{Hom}_R(H_n(X, X'; R), N) \to 0$$

which splits not naturally. In particular,

$$H^n(X; N) \simeq \mathrm{Ext}^1_R(H_{n-1}(X; R), N) \oplus \mathrm{Hom}_R(H_n(X; R), N)$$

whatever R ring we take!!!

<u>Examples</u>: 1) From the homology of the projective plane $H_n(\mathbb{RP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}_2, & n=1 \\ 0 & n \geq 2 \end{cases}$,

we get $H^n(\mathbb{RP}^2; A) = \begin{cases} A, & n=0 \\ {}_2A, & n=1 \quad \text{(the 2-torsion)} \\ A/2A, & n=2 \\ 0, & n \geq 3 \end{cases}$

2) $H^n(S_m; N) = \begin{cases} N, & n = m \\ 0, & \text{else} \end{cases}$, $n, m \geq 1$

• <u>Recall</u>: The tensor product of two R-modules M and N, $M \otimes_R N$, has the following <u>universal property</u>: given a bilinear map $\phi : M \times N \to Q$, there exists a unique R-module hom. $\bar{\phi} : M \otimes_R N \to Q$ such that

$$\begin{tikzcd} M \times N \arrow[r, "\phi"] \arrow[d, "\pi"'] & Q \\ M \otimes_R N \arrow[ur, dashed, "\bar{\phi}"'] & \end{tikzcd} \qquad \bar{\phi} \circ \pi = \phi .$$

(explicitly, $M \otimes_R N = \dfrac{R[M \times N]}{\langle \cdots \rangle}$, and it is the unique space satisfying the universal property).

It has the following properties:

1) $M \otimes_R N \simeq N \otimes_R M$, 2) $M \otimes_R R \simeq M$

3) $(M \otimes_R N) \otimes_R Q \simeq M \otimes_R (N \otimes_R Q)$, 4) $\left(\bigoplus_i M_i\right) \otimes_R N = \bigoplus_i M_i \otimes_R N$

5) Module maps $f : M \to M'$, $g : N \to N'$ induce $f \otimes g : M \otimes N \to M' \otimes N'$.

6) $\mathrm{Hom}(M \otimes_R N, Q) \simeq \mathrm{Hom}(M, \mathrm{Hom}(N, Q))$, $f \longmapsto (m \mapsto f(m \otimes -))$

"$- \otimes_R N$ is left adjoint to $\mathrm{Hom}(N, -)$".

<u>Lemma</u>: The functor $- \otimes_R N : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is right exact (but not exact in general)

• In the previous lemma (and in what follows) we will need two universal prop. of the kernel and cokernel of an R-module (one can define this in some categories but in our language :). Recall that the <u>cokernel</u> of an R-mod hom $f : M \to N$ is $\mathrm{Coker}\, f := \dfrac{N}{\mathrm{Im}\, f}$.

Proposition (Universal Property of the Kernel): Let $f: M \to N$ be an R-mod hom. For any R-module hom $g: P \to M$ such that $f \circ g = 0$, there exists a unique R-mod hom $g': P \to \operatorname{Ker} f$ s.t. $i \circ g' = g$.

$$\begin{tikzcd} & \operatorname{Ker} f \arrow[d, "i"] & \\ P \arrow[ur, dashed, "\exists g'"] \arrow[r, "g"] & M \arrow[r, "f"] & N \end{tikzcd}$$

Proposition (Universal Property of the Cokernel): Let $f: M \to N$ be an R-mod hom. For any R-mod hom $g: N \to P$ st $g \circ f = 0$, there exist a unique R-mod hom $\bar{g}: \operatorname{Coker} f \to P$ st $\bar{g} \circ \pi = g$.

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, "g"] \arrow[d, "\pi"] & P \\ & \operatorname{Coker} f \arrow[ur, dashed, "\exists \bar{g}"'] & \end{tikzcd}$$

***Theorem**: There exist functors

$$\operatorname{Tor}_n^R : \operatorname{Mod}_R \times \operatorname{Mod}_R \longrightarrow \operatorname{Mod}_R$$

such that

(T1) $\operatorname{Tor}_0^R = \otimes$

(T2) For an R-mod N on a ses $0 \to M' \to M \to \bar{M} \to 0$ there is a natural les

$$\cdots \to \operatorname{Tor}_n^R(M', N) \to \operatorname{Tor}_n^R(M, N) \to \operatorname{Tor}_n^R(\bar{M}, N) \to$$

$$\to \operatorname{Tor}_{n-1}^R(M', N) \to \operatorname{Tor}_{n-1}^R(M, N) \to \operatorname{Tor}_{n-1}^R(\bar{M}, N) \to \cdots$$

(T3) $\mathrm{Tor}_n^R(M,N) = 0$ if $n \geq 1$ and M free, $\forall N$.

Such family of functors Tor_n^R is unique up to isomorphism.

Remark: 1) K field, every K-mod $=$ vs is free, thus $\mathrm{Tor}_n^K = 0 \quad \forall n \geq 1$.

2) If $a \in R$ is a non zero divisor, then $\mathrm{Tor}_1^R(R/a, N) = {}_aN$ the a-torsion.

(and $R/a \otimes_R N = N/aN$)

3) If R is PID, then $\mathrm{Tor}_n^R = 0 \quad \forall n \geq 2$, $\forall M, N$.

Corollary: 1) $\mathrm{Tor}_n^{\mathbb{Z}}(M,N) = 0 \quad \forall n \geq 2$

2) $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/\gcd(m,n)\mathbb{Z}$

3) $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/\gcd(m,n)\mathbb{Z}$.

Theorem (Universal Coefficient Formula for Homology). Let (X, X') be a pair of spaces, R a PID and N an R-mod. There exists a split ses

$$0 \to H_n(X, X'; R) \otimes_R N \to H_n(X, X'; N) \to \mathrm{Tor}_1^R(H_{n-1}(X, X'; R), N) \to 0$$

which splits non-naturally (for absolute hom just take $X' = \emptyset$).

- It tells how to compute homology for general ab. groups $=$ $\mathbb{Z}$-modules: take $R = \mathbb{Z}$ and apply the thm.

(CO) HOMOLOGY FOR PRODUCT SPACES

Definition: Let C_*, D_* be chain complexes. The tensor product of C_* and D_* is another chain complex $C_* \otimes D_*$ with

$$(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and with differential $\partial_n(c \otimes d) = \partial_p^C(c) \otimes d + (-1)^p c \otimes \partial_q^D(d)$, or in other words,

$$\partial_n = \partial_p^C \otimes \mathrm{Id} + (-1)^p \, \mathrm{Id} \otimes \partial_q^D .$$

<u>Definition</u>: The <u>algebraic cross product</u> is the map

$$\times_{alg} : H_p(C_*) \otimes H_q(D_*) \longrightarrow H_{p+q}(C_* \otimes D_*)$$
$$[c] \otimes [d] \longmapsto [c \otimes d]$$

- For a chain complex C_*, denote as $H(C_*)$ the chain complex $H(C_*)_n = H_n(C_*)$ (with $\partial = 0$), and $H(C_*) * H(D_*)$ for $(H(C_*) * H(D_*))_n = \bigoplus_{p+q=n} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*))$.

<u>Theorem</u> (Künneth, algebraic version): Let C_*, D_* be chain complexes over a PID R, with C_* (D_*) levelwise free. There is a levelwise natural split ses

$$0 \to H(C_*) \otimes H(D_*) \longrightarrow H(C_* \otimes D_*) \longrightarrow \big(H(C_*) * H(D_*)\big)[1] \longrightarrow 0$$

(ie, $0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \xrightarrow{\oplus \times_{alg}} H_n(C_* \otimes D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*)) \to 0$)

which splits non-naturally.

IV : THE CUP PRODUCT

ACYCLIC MODELS

Definition: Let $\mathcal{C}$ be a category and let $F, G : \mathcal{C} \to \mathrm{Mod}_R$ be functors. A natural transformation $\eta : F \to G$ is a family of R-module maps $\{\eta_C : F(C) \to G(C)\}_{C \in ob(\mathcal{C})}$ such that for all $f : C \to D$ the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{\eta_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(D) & \xrightarrow{\eta_D} & G(D) \end{array}$$

Lemma: 1) A functor $F : \mathcal{C} \to \mathrm{Ch}_{\geq 0}$ is the same as a family of functors $F_n : \mathcal{C} \to \mathrm{Mod}_R$ and natural transformations $\partial_n : F_n \to F_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ ("functorial version of chain complex")

2) A natural transformation of functors $F, G : \mathcal{C} \to \mathrm{Ch}_{\geq 0}$ is the same as a family of natural transformations $\varphi_n : F_n \to G_n$ st $\partial_n^G \varphi_n = \varphi_{n-1} \partial_n^F$ ("functorial version of a chain map").

Definition: Let $F, G : \mathcal{C} \to \mathrm{Ch}_{\geq 0}$ be functors and let φ, ψ be nat. transformations (= "chain maps") of F and G. A natural chain homotopy $s : \varphi \to \psi$ is a family of natural transformations $s_n : F_n \to G_{n+1}$ such that

$$\partial_{n+1}^G s_n + s_{n-1} \partial_n^F = \psi_n - \varphi_n .$$

Definition: Let $\mathcal{C}$ be a category. For each $n \geq 0$, let $(B_{n,j})_{j \in J(n)}$ be a family of objects in $\mathcal{C}$, called **models**. A functor $F_n : \mathcal{C} \to Mod_R$ is **free** if there exist elements $b_{n,j} \in F_n(B_{n,j})$, $\forall j \in J(n)$, such that

$$\{ F_n(f)(b_{n,j}) : j \in J(n), f \in Hom_{\mathcal{C}}(B_{n,j}, X) \}$$

is a R-basis of $F_n(X)$ for all $X \in ob(\mathcal{C})$.

We say that a functor $F : \mathcal{C} \to Ch_{\geq 0}$ is **free** when F_n is free $\forall n$.

We say that $G : \mathcal{C} \to Ch_{\geq 0}$ is **acyclic** when $H_m(G(B_{n,j})) = 0$ $\forall m \geq 1$.

Examples: 1) The functor $C_n(-;R) : Top \to Mod_R$ is free with models $(B_{n,j}) = \{\Delta^n\}$, i.e., $|J(n)| = 1$ $\forall n$, and basis $Id_{\Delta^n} \in C_n(\Delta^n; R)$. Therefore $C_*(-;R) : Top \to Ch_{\geq 0}$ is also free, and acyclic.

2) $C_*(- \times -; R) : Top \times Top \to Ch_{\geq 0}$ is free with models $B_{n,j} = \{\Delta^n \times \Delta^n\}$, i.e., $|J(n)| = 1$ $\forall n$; and basis $Id_{\Delta^n} \times Id_{\Delta^n} \in C_n(\Delta^n \times \Delta^n; R)$, and also acyclic.

3) $C_*(-;R) \otimes C_*(-;R) : Top \times Top \to Ch_{\geq 0}$ is free with models $(B_{n,j}) = \{\Delta^p \times \Delta^q : p+q = n\}$ $\forall n$, and basis $1 \cdot Id_{\Delta^p} \otimes 1 \cdot Id_{\Delta^q} \in C_p(\Delta^p; R) \otimes C_q(\Delta^q; R) \subset$
$\subset (C_*(\Delta^p; R) \otimes C_*(\Delta^q; R))_n$. It is also acyclic.

Theorem (Acyclic models): Let $\mathcal{C}$ be a category with models, let $F, G : \mathcal{C} \to Ch_{\geq 0}$ be functors with F free and G acyclic, and let

$$\{ \overline{\varphi}_X : H_0(F(X)) \to H_0(G(X)) : X \in ob\, \mathcal{C} \}$$

be a natural transformation between $H_0(F(-)), H_0(G(-)) : \mathcal{C} \to Mod_R$.

Then there exists a natural transformation $\varphi : F \to G$ inducing $\overline{\varphi}$ on H_0, and it is unique up to chain homotopy.

Theorem (Eilenberg–Zilber): For spaces X, Y there are natural chain homotopy equivalences

$$C_*(X\times Y;R) \underset{B}{\overset{A}{\rightleftarrows}} C_*(X;R)\otimes C_*(Y;R)$$

Theorem (Künneth, topological version): Let X, Y be top spaces and let R be a PID. There is natural split ses <u>lowkey</u>

$$0 \to H(X;R)\otimes H(Y;R) \longrightarrow H(X\times Y;R) \longrightarrow \big(H(X;R) * H(Y;R)\big)[1] \to 0$$

$\Big($ ie, $0 \to \bigoplus_{p+q=n} H_p(X;R)\otimes H_q(Y;R) \longrightarrow H_n(X\times Y;R) \longrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(X;R), H_q(Y;R)) \to 0 \Big)$

which splits non naturally.

In real computation, how to get A, B?

Definition: Let $\sigma: \Delta^n \to X$ be a singular complex and let $0 \le p, q \le n$. The <u>front p</u> of σ is the singular complex ${}_p\sigma : \Delta^p \to X$, $(t_0 \dots t_p) \mapsto \sigma(t_0, \dots, t_p, 0 \dots 0)$; and the <u>back q</u> of σ is $\sigma_q : \Delta^q \to X$, $(t_0 \dots t_q) \mapsto \sigma(0, \dots, 0, t_0, \dots, t_q)$.

Definition: The <u>Alexander–Whitney map</u> is the chain map

$$\begin{aligned} C_*(X\times Y;R) &\longrightarrow C_*(X;R)\otimes C_*(Y;R) \\ \sigma &\longmapsto \sum_{p+q=n} {}_p(p_X\circ\sigma)\otimes(p_Y\circ\sigma)_q \end{aligned}$$

where $p_X : X\times Y \to X$, $p_Y : X\times Y \to Y$.

Proposition: The Alex.–Whit. map is a homotopy equivalence.

• For a chain complex C_*, denote $C^* := \mathrm{Hom}_R(C_*, R)$, which is both a cochain complex and the "dual" of C_*.

<u>Definition</u>: The <u>algebraic cross product for cohomology</u> is the map

$$\times^{alg}: H^p(C^*) \otimes H^q(D^*) \longrightarrow H^{p+q}(\mathrm{Hom}(C_* \otimes D_*), R))$$
$$[\alpha] \otimes [\beta] \longmapsto \left(\sum_{i=1}^{n} c_i \otimes d_i \longmapsto \sum_{i=1}^{n} \alpha(c_i)\beta(d_i) \right)$$

Here we take $\alpha(c_i) = 0$ if degrees don't match.

<u>Definition</u>: Let $A: C_*(X \times Y, R) \to C_*(X;R) \otimes C_*(Y;R)$ be a hom. equivalence (eg A.W.).
The <u>cohomology cross product</u> is the composite

$$\begin{tikzcd} H^p(X;R) \otimes H^q(Y;R) \arrow[r, "\times^{alg}"] \arrow[rrr, bend right, "\times"'] & H^{p+q}(\mathrm{Hom}(C_*(X;R) \otimes C_*(Y;R)), R) \arrow[r, "A^*"] & H^{p+q}(\mathrm{Hom}(C_*(X \times Y;R), R)) \arrow[r, equal] & H^{p+q}(X \times Y;R) \end{tikzcd}$$

$a \otimes b \longmapsto a \times b$

<u>Definition</u>: Let X be a top space. The <u>cup product</u> is the map defined as the composite

$$\begin{tikzcd} H^p(X;R) \otimes H^q(X;R) \arrow[r, "\times"] \arrow[rr, bend right, "\cup"'] & H^{p+q}(X \times X;R) \arrow[r, "\Delta^*"] & H^{p+q}(X;R) \end{tikzcd}$$

$a \otimes b \longmapsto a \cup b$.

where $\Delta: X \to X \times X$ is the diagonal map.

$*$) Pg 17.2)

Lemma: Cup and cross product determine each other, and they are natural in the following sense:

1) $f^*(a \cup b) = f^*(a) \cup f^*(b)$

2) $(f \times g)^*(a \times c) = f^*(a) \times g^*(c)$

3) $a \cup b = \Delta^*(a \times b)$

4) $a \times c = p_X^*(a) \cup p_Y^*(c)$

for $a \in H^p(X;R)$, $b \in H^q(X;R)$, $c \in H^q(Y;R)$ and $f: X' \to X$ and $g: Y' \to Y$.

Explicitly, with the AW map, the cup product is

$$H^p(X;R) \otimes H^q(X;R) \longrightarrow H^{p+q}(X;R)$$

$$[\alpha: S(X)_p \to R] \otimes [\beta: S(X)_q \to R] \longmapsto [\ S(X)_{p+q} \xrightarrow{\alpha \cup \beta} R\ ]$$

$$(\sigma: \Delta^{p+q} \to X) \longmapsto \alpha({}_p\sigma) \cdot \beta(\sigma_q).$$

This says that with the AW map we can give a meaning to $\alpha \cup \beta$, for $\alpha \in C^p(X;R)$ and $\beta \in C^q(X;R)$.

Theorem (Properties): For the cup product it holds:

1) $1 \cup a = a = a \cup 1$, for $1 = [\mathrm{const}_1] \in H^0(X;R)$

2) $(a \cup b) \cup c = a \cup (b \cup c)$

3) $a \cup b = (-1)^{|a||b|}\, b \cup a$, for $|a|, |b|$ the degrees of a, b.

If one uses the AW map, 1) and 2) hold even before passing to cohomology, at the level of chain complexes.

IV.2 ORIENTABILITY AND POINCARÉ DUALITY

CAP PRODUCT

• Recall the R-module map $\Phi : H^n(\mathrm{Hom}(C_*, N)) \to \mathrm{Hom}_R(H_n(C_*), N)$. By the universal

$$[f : C_n \to N] \longmapsto \left([c] \mapsto f(c) \right).$$

prop of $\otimes$, this is the same as a bilinear map

$$\langle \cdot, \cdot \rangle : H^n(\mathrm{Hom}(C_*, N)) \otimes H_n(C_*) \to N.$$

This can be seen as induced by a map of R-modules $\langle \cdot, \cdot \rangle : \mathrm{Hom}(C_*, N) \otimes C_* \to N$

$\left((f : C_n \to N) \otimes c \right) \mapsto f(c)$ when degrees match, 0 o.w.

<u>Definition</u>: Let C_* be a chain complex of R-modules. The <u>Kronecker pairing</u> is

$$\langle \cdot, \cdot \rangle : \mathrm{Hom}_R(C_*, R) \otimes C_* \to R$$

<u>Definition</u>: Let X be a space. The <u>cap product</u> is the map defined as the composite

$$\begin{tikzcd} C^*(X;R) \otimes C_*(X;R) \arrow[r, "Id \otimes \Delta_*"] \arrow[rrr, bend right, "\cap"'] & C^*(X;R) \otimes C_*(X \times X;R) \arrow[r, "Id \otimes A"] & C^*(X;R) \otimes C_*(X;R) \otimes C_*(X;R) \arrow[r, "\langle \cdot,\cdot \rangle \otimes Id"] & C_*(X) \end{tikzcd}$$

$$a \otimes z \longmapsto a \cap z$$

where $\Delta : X \to X \times X$ is the diagonal map and A is the Eilenberg-Zilber map (or A.W map).

• Observe that if $a \in C^p(X;R)$ and $z \in C_q(X;R)$, $p \leq q$, then $a \cap z \in \underline{C_{q-p}(X;R)}$.

(if $p > q$ then it's 0)

Lemma: For $a \in C^p(X;R)$ and $z \in C_q(X;R)$, $p \leq q$, it holds

$$\partial(a \cap z) = (-1)^p \, \delta a \cap z + a \cap \partial z .$$

Corollary: The cap product descent to a well-defined product

$$\cap : H^p(X;R) \otimes H_q(X;R) \longrightarrow H_{q-p}(X;R)$$

Proposition: The following properties hold:

1) $\langle a, b \cap z \rangle = \langle a \cup b, z \rangle$

2) $a \cap (b \cap z) = (a \cup b) \cap z$

3) $\langle a, z \rangle = \varepsilon_*(a \cap z)$, where $a \in H^p(X;R)$, $z \in H_p(X;R)$ and $\varepsilon_* : H_0(X;R) \to R$ is induced by the augmentation map $\varepsilon : C_0(X;R) \to R$, $\sum r_i \sigma_i \longmapsto \sum r_i$.

4) $a \cap f_*(z) = f_*(f^*(a) \cap z)$, where $f: X \to Y$, $f^*: H^p(Y;R) \to H^p(X;R)$ and $f_*: H_q(X;R) \to H_q(Y;R)$.

ORIENTATIONS IN TOPOLOGICAL MANIFOLDS

- If M is a top manifold of dim n, then by excision we have

$$H_i(M, M-\{x\};R) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - 0; R) \cong \widetilde{H}_i(\mathbb{R}^n - 0; R) \cong \begin{cases} R, & i = n \\ 0, & \text{else} . \end{cases}$$

A choice of generator of $H_n(M, M-\{x\};R) \cong R$ (ie, an invertible element) is a local orientation of M. This encodes the usual notion of orientability of $\mathbb{R}^n$.

- An orientation for M will be a choice of orientation "varying continuously". Let us make this precise.

Definition: Given an n-manifold M, an R-fundamental class of M at $A \subseteq M$ is a homology class $\mu_A \in H_n(M, M-A; R)$ such that

$$\rho_x^A(\mu_A) = \mu_x \in H_n(M, M-x; R) \cong R$$

is a generator for all $x \in A$, where $\rho_x^A : H_n(M, M-A; R) \to H_n(M, M-x; R)$ is the map induced by the inclusion of pairs $(M, M-A) \hookrightarrow (M, M-x)$. If $A = M$, then μ_M is called an R-fundamental class of M.

Definition: An R-orientation of M is a choice of R-fundamental class $(\mu_{U_i})_{i \in I}$ for an open cover $\{U_i\}$ of M with the coherence condition that if $U_i \cap U_j \neq \emptyset$, then

$$\rho^{U_i}_{U_i \cap U_j}(\mu_{U_i}) = \rho^{U_j}_{U_i \cap U_j}(\mu_{U_j}),$$

where $\rho^{U_k}_{U_i \cap U_j}$ is induced by the inclusion of pairs $(M, M-U_k) \hookrightarrow (M, M-U_i \cap U_j)$.

A manifold M is R-orientable if there exists an orientation. For $R = \mathbb{Z}$ we drop the "R".

- Note: The "gluing condition" of the definition implies the consistence condition

$$\rho_x^{U_i}(\mu_{U_i}) = \rho_x^{U_j}(\mu_{U_j}) \qquad \forall x \in U_i \cap U_j.$$

Moreover, this condition is equivalent to the given one.

- (Connection with differential geometry): The classes $\mu_x \in H_n(M, M-x; R)$ play an analogous role as n-forms $\omega_x \in \Lambda^n T_x M$ in a smooth manifold. In particular, our definition is the analogous statement of the fact that if $(U_i, [\omega_i])$ is an open cover of oriented open subsets and in the intersections the orientations agree, then there's a unique orientation restricting to the $[\omega_i]$'s. Here we cannot define orientation as $[\omega]$ globally in general. The analogous is the existence of a fundamental class $\mu_M \in H_n(M; R)$.

If these exist, then M is R-orientable trivially. But the converse only holds if M is compact or $R = \mathbb{Z}/2$.

Moreover, similarly to the fact that an orientation can be seen as a non-vanishing section $s \in \Gamma(\Lambda^n TM)$, one can construct a covering space $p: M_R \to M$ with fibres $p^{-1}(x) = H_n(M, M-x; R) \cong R$, and a section s of p s.t. $s(x)$ is a gen of $H_n(M, M-x; R)$ is the same as an orientation.

- A simplification of the covering $M_R \to M$ gives rise to a 2-sheeted covering $\tilde{M} \to M$: this is, as a set, $\tilde{M} = \{\mu_x \text{ generator of } H_n(M, M-x; \mathbb{Z}) \cong \mathbb{Z} : x \in M\}$ with the obvious projection, endowed with a suitable topology where a basis is given by $U_B := \{\mu_x = \rho_x^B(\mu_B) : x \in B\}$, B ball. By definition, $\tilde{M}$ is orientable.

__Proposition__: Let M be connected. Then M is orientable $\iff$ $\tilde{M}$ has two components.
In particular, if M is simply-connected or $\pi_1(M)$ has no subgp of index 2, then M is orientable.

- The first statement is a formulation of the intuitive notion of non-orientability as being able to go around a loop and come back with the opposite orientation, lifting the loop to a path $\tilde{M} \to M$ connecting the two opposite orientabilities.

*** __Theorem__**: Let M be a connected n-manifold.

1) $H_i(M; R) \cong 0$ for all $i > n$.

2) $$H_n(M; R) \cong \begin{cases} R & \text{, if } M \text{ compact and orientable} \\ {}_2R & \text{, if } M \text{ compact and non-orientable} \\ 0 & \text{, if } M \text{ non compact} \end{cases}$$

Here, ${}_2R = \mathrm{Ker}(R \xrightarrow{\cdot 2} R)$ is the __2-torsion__ of R, and orientable means "R-orientable".
In particular, in the first case the iso is induced by $\rho_x^M : H_n(M; R) \to H_n(M, M-x; R) \cong R$ for any $x \in M$, and in the second case ρ_x^M is inj with image ${}_2R$.

Corollary: If M is compact connected, then M is $\mathbb{Z}$-orientable if $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ and non $\mathbb{Z}$-orientable if $H_n(M;\mathbb{Z}) \cong 0$.

Corollary: Let M be connected n-manifold. Then M has an R-fundamental class if and only if M is R-orientable and compact. In particular, if M is non-compact but R-orientable, for any K compact there is a unique R-fundamental class μ_K at K.

- Observe that if M is compact then $H_n(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$ independently of the orientation, so any (compact) n-manifold is $\mathbb{Z}/2$-orientable.

Corollary: Let M be a compact connected n-manifold. Then

$$\mathrm{Tors}(H_{n-1}(M;\mathbb{Z})) \cong \begin{cases} 0, & M \text{ orientable} \\ \mathbb{Z}/2, & M \text{ non-orientable.} \end{cases}$$

DUALITY

* Theorem (Poincaré Duality): Let M be a compact connected R-orientable n-manifold with a fundamental class μ_M. Then the map

$$D: H^k(M;R) \longrightarrow H_{n-k}(M;R)$$
$$[\alpha] \longmapsto [\alpha] \cap \mu_M$$

is an isomorphism for all $k \geq 0$

- We also care about duality in the non-compact case, but we will have to replace cohomology by cohomology with compact support:

<u>Definition</u>: A cochain $\alpha \in C^p(X;R)$ has <u>compact support</u> if there exists $K \subset X$ compact such that $\alpha(c) = 0$ for any $c \in C_p(X;R)$ with image in $X - K$. If $C^p_c(X;R)$ denotes the R-module of compactly supported cochains, then $C^*_c(X;R) \subset C^*(X;R)$ is a sub-cochain complex and its cohomology is called <u>cohomology with compact support</u> and denoted by $H^p_c(X;R)$.

• Alternatively, $C^p_c(X;R) = \coprod_{K \in \mathcal{K}} C^p(X, X-K;R)$, where $\mathcal{K}$ is the class of compact subsets of X.

For $K \subset L$ compact, the inclusion induces $C^p(X, X-K;R) \hookrightarrow C^p(X; X-L;R)$, which in turn induces map $H^p(X, X-K;R) \to H^p(X, X-L;R)$. Note that $\mathcal{K}$ is a directed system.

<u>Proposition</u>: $H^p_c(X;R) \cong \operatorname{colim}_{\mathcal{K}} H^p(X, X-K;R)$.

Moreover, if X is compact, then $H^p_c(X;R) \cong H^p(X;R)$.

• As with the cup product, there is also a relative version of cap product: if $A, B \subset X$ are either open or subcomplexes of CW complex X, then there is a relative cap product

$$\frown : H^p(X,A;R) \otimes H_q(X, A \cup B;R) \to H_{p-q}(X,B;R)$$

In our case, we get a relative version

$$\frown : H^p(X, X-K;R) \otimes H_n(X, X-K;R) \to H_{n-p}(X;R).$$

The pairing $[\alpha] \frown \mu_K$ induces a "pairing map"

$$D_M : H^p_c(X;R) \longrightarrow H_{n-p}(X;R)$$
$$\omega = [[\alpha]] \longmapsto [\alpha] \frown \mu_K$$

that doesn't depend on the repr. $[\alpha]$ of ω = colim ....

*Theorem (General Poincaré Duality): let M be an R-orientable n-manifold. Then

$$D_M : H^p_c(X;R) \xrightarrow{\cong} H_{n-p}(X;R)$$

is an isomorphism for all $p \geq 0$. When M is compact, this is the previous duality theorem.

Corollary: A compact manifold of odd dimension has Euler characteristic zero.

DUALITY FOR MANIFOLDS WITH BOUNDARY

Definition: An n-manifold with boundary is a second countable, Hausdorff space st every point has a neighbourhood homeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n = \{(x_1, \dots x_n) \in \mathbb{R}^n : x_n \geq 0\}$ with the point corresponding to the origin.

Points with a nbhd homeomorphic to $\mathbb{R}^n$ form the interior, and points with a nbhd homeomorphic to $\mathbb{H}^n$ as before the boundary, ∂M.

• By excision, $H_n(M, M-x; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } x \text{ interior} \\ 0, & \text{if } x \in \partial M \end{cases}$, so they form well-defined subspaces.

• We want to extend orientability to manifolds with boundary.

Lemma: If M is a compact manifold with boundary, then ∂M has a collar neighbourhood $\partial M \times [0,1)$.

Definition: A compact manifold with boundary M is orientable if $M - \partial M$ is orientable as (non compact) manifold without boundary.

• let M as before. If it is R-orientable, since by excision we have $\underset{\text{manifold}}{H_n(M, \partial M; R)} \xleftarrow{\cong} \underset{\text{manifold - compact}}{H_n(M - \partial M, \partial M \times (0,\varepsilon); R)}$

so we get a relative fundamental class $\mu_{M,\partial M} \in H_n(M, \partial M; R)$.

Theorem: Let M be a compact R-orientable n-manifold with boundary $\partial M = A \cup B$ union of two $(n-1)$-dim manifolds with a common boundary $\partial A = A \cap B = \partial B$. Then there is an iso

$$D_M : H^p(M, A; R) \xrightarrow{\cong} H_{n-p}(M, B; R)$$
$$[\alpha] \longmapsto [\alpha] \cap \mu_{M, \partial M}.$$

Theorem: Let $K \subset M$ be a compact, locally contractible subspace of a compact orientable n-manifold (without boundary). Then

$$H_p(M, M-K; \mathbb{Z}) \cong H^{n-p}(K; \mathbb{Z}) \qquad \forall p.$$

*Theorem (Alexander Duality): Let $K \subset S^n$ be a compact, locally contractible subspace of S^n ($\neq \phi$ & S^n). Then

$$\tilde{H}_p(S^n - K; \mathbb{Z}) \cong \tilde{H}^{n-p-1}(K; \mathbb{Z}) \qquad \forall p.$$

- An important observation is that the homology of the complement does not depend on the embedding $K \hookrightarrow S^n$

*Corollary: If $X \subset \mathbb{R}^n$ is compact and locally contractible, then $H_i(X; \mathbb{Z}) \cong 0$ for $i \geq n$ and $H_{n-1}(X; \mathbb{Z})$, $H_{n-2}(X; \mathbb{Z})$ are torsion free.

Corollary: A non-orientable compact n-manifold cannot be embedded in $\mathbb{R}^{n+1}$.

- We finish with the following comments about cohomology of manifolds: it is a topological fact that the homology groups of a compact manifold are finitely-generated. By Universal Coeff we get

Proposition: Let M be a connected compact n-manifold. Then

$$H^n(M; R) \cong \begin{cases} R, & M \text{ orientable} \\ R/2, & M \text{ non-orientable.} \end{cases}$$

V : THE COHOMOLOGY RING

Definition: The <u>cohomology ring</u> of a top space X is

$$H^\bullet(X;R) := \bigoplus_{p\geq 0} H^p(X;R).$$

• The properties of the previous thm make it into a graded anticommutative ring (wrt the cup product).

Example:

i	0	n
$H^i(S^n;R)$	R	R
gen	1	x

$1 \cup 1 = 1$
$1 \cup x = x$
$x \cup x = 0 \in H^{2n}$

$$\Rightarrow H^\bullet(S^n;R) = \frac{R[x]}{(x^2)}.$$

Proposition: If $A, B \subseteq X$ are open, then the obvious map $C^*(A;R) + C^*(B;R) \hookrightarrow C^*(A\cup B;R)$ is a chain homotopy equivalence. Moreover, if $\alpha \in C^p(X,A;R)$, $\beta \in C^q(X,B;R)$, then $\alpha \cup \beta$ vanishes on $C^*(A;R) + C^*(B;R)$.

Definition: Let $A, B \subseteq X$ be open subsets. The <u>relative cup product</u> is the map

$$H^p(X,A;R) \otimes H^q(X,B;R) \xrightarrow{\cup} H^{p+q}(X,A\cup B;R).$$

Definition: Let (X,A), (Y,B) be pairs of spaces. The <u>relative cross product</u> is

$$H^p(X,A;R) \otimes H^q(Y,B;R) \xrightarrow{\times} H^{p+q}(X\times Y, X\times B \cup A\times Y;R)$$
$$\alpha \otimes \beta \longmapsto (p_X^*\alpha) \cup (p_Y^*\beta).$$

Remark: There is also a relative version of Künneth thm, stating that if $\mathrm{Tor} = 0$ then

$$\left(H(X,A;R) \otimes H(Y,B;R)\right)_n \cong H^n(X\times Y, X\times B \cup A\times Y;R)$$

Proposition: The cohomology rings of the projective spaces are

1) $H^{\bullet}(\mathbb{C}P^n;\mathbb{Z}) = \dfrac{\mathbb{Z}[x]}{(x^{n+1})}$, $|x|=2$

2) $H^{\bullet}(\mathbb{C}P^{\infty};\mathbb{Z}) = \mathbb{Z}[x]$, $|x|=2$

3) $H^{\bullet}(\mathbb{R}P^n;\mathbb{Z}_2) = \dfrac{\mathbb{Z}_2[x]}{(x^{n+1})}$, $|x|=1$

4) $H^{\bullet}(\mathbb{R}P^{\infty};\mathbb{Z}_2) = \mathbb{Z}_2[x]$, $|x|=1$.

Corollary:

4bis) $H^{\bullet}(\mathbb{R}P^{\infty};\mathbb{Z}) = \dfrac{\mathbb{Z}[y]}{(2y)}$, $|y|=2$

3bis) $H^{\bullet}(\mathbb{R}P^{2n};\mathbb{Z}) = \dfrac{\mathbb{Z}[y]}{(2y,\, y^{n+1})}$, $|y|=2$

$H^{\bullet}(\mathbb{R}P^{2n+1};\mathbb{Z}) = \dfrac{\mathbb{Z}[y,z]}{(2y,\, y^{n+1},\, z^2,\, yz)}$, $|y|=2$, $|z|=2n+1$.

Example: $\mathbb{C}P^2$ and $S^2 \vee S^4$ have the same cohomology groups, but not the same cohomology rings, thus they cannot be hty equivalent. Why? For the wedge of two spaces, the classes coming from each summand are not going to interact, they behave independently (wrt the cup product)

Remark (*): There are also Künneth theorems for cohomology, that we need: given by the equivalence of categories $Ch_{\geq 0}$ and $Ch^{\leq 0}$:

Theorem (Künneth, alg version for cohomology): Let C_*, D_* be free finite chain complexes over a PID R, and suppose that $H_*(C_*)$, $H_*(D_*)$ are levelwise finite-generated. If $C^* = \mathrm{Hom}(C_*, R)$, $D^* = \mathrm{Hom}(D_*, R)$, then there is a split ses

$$0 \to \bigoplus_{p+q=n} H^p(C^*)\otimes H^q(D^*) \xrightarrow{\times^{alg}} H^n(C^*\otimes D^*) \to \bigoplus_{p+q=n+1} \mathrm{Tor}_1^R(H^p(C^*), H^q(D^*))$$

Theorem (Künneth, top version for cohomology): let R be a PID and let X, Y be spaces. If $H_*(X;R)$, $H_*(Y;R)$ are finite-generated, then there is a split ses

$$0 \to \bigoplus_{p+q=n} H^p(X;R) \otimes H^q(Y;R) \xrightarrow{\times} H^n(X \times Y;R) \longrightarrow \bigoplus_{p+q=n+1} \mathrm{Tor}_1^R(H^p(X;R), H^q(Y;R)) \to 0$$

• Defining $H^*(X;R) \otimes H^*(Y;R)$ as graded ring (levelwise $\bigoplus_{p+q=n} \dots$) with multiplication $(a \otimes b)(c \otimes d) := (-1)^{|b||c|} ac \otimes bd$, we have that the cohomology cross product is a ring homomorph and we conclude

Theorem: let R be a PID and suppose that $H^*(Y;R)$ is levelwise free and finite-generated. Then the cohomology cross product induces a ring isomorphism

$$H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \xrightarrow{\sim} H^\bullet(X \times Y;R).$$

Example: let Σ_g be the genus g compact orientable surface. At the level of groups we have

i	0	1	2	3	4
$H^i(\Sigma_g, \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}$	0	0 ....
gen	1	$\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$	σ		

and the ring structure of $H^\bullet(\Sigma_g;\mathbb{Z})$ is given by (the only non-trivial products are between degree 1 elements)

$$\alpha_i \alpha_j = 0 \quad , \quad \beta_i \beta_j = 0 \quad , \quad \alpha_i \beta_j = \delta_{ij} \sigma.$$

For that we collapse $\Sigma_g = \mathbb{T} \# \mathbb{T} \# \cdots \# \mathbb{T}$ into $\mathbb{T} \vee \cdots \vee \mathbb{T}$, and we know the cohomology product on $\mathbb{T}$ since by Künneth ($\mathbb{T} = S^1 \times S^1$)

$$H^\bullet(\mathbb{T};\mathbb{Z}) = \frac{\mathbb{Z}[x,y]}{(x^2,y^2)}$$ with anticommutative product, $xy = -xy$.

VI THE BOCKSTEIN HOMOMORPHISM

THE HOPF INVARIANT

In general it's a highly non trivial issue to give non-nullhomotopic maps between spheres of different dimensions. We'll give one: consider $\eta : S^3 \to S^2$ the attaching map for the 4-cell in $\mathbb{CP}^2$, i.e. $\eta : \partial D^4 = S^3 \to \mathbb{CP}^1 = S^2$ ($\mathbb{CP}^1$ the 3-skeleton = 2-skeleton), i.e. $\mathbb{CP}^2 = S^2 \cup_\eta D^4$.

This map is called the __Hopf map__.

- Recall from AT I that homotopic attaching maps of cells induce homotopy equivalent spaces. So we have

__Lemma__: The Hopf map $\eta : S^3 \to S^2$ is not null-homotopic.

- For homology we had an iso $H_n(X;A) = H_{n+1}(SX;A)$ $\forall n \geq 1$ given by the Mayer-Vietoris sequence. Since we have also MV for cohomology, we have as well

$$H^n(X;R) = H^{n+1}(SX;R) \qquad \forall n \geq 1.$$

The latter argument fails. This motivates the "cohomology operations" that follow.

- Is $S\eta : S^4 \to S^3$ not nullhotopic?

- Consider a map $f : S^{2n-1} \to S^n$ that we'll think as an attaching map of the CW-complex $X = S^n \cup_f D^{2n}$.

From cellular homology we deduce that

i	0	...	n	...	$2n$	...
$H^i(X;\mathbb{Z})$	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
gen	1		x		y	

$(n \geq 2)$

so we'll have $x \cup x = ky$ for some $k \in \mathbb{Z}$.

__Definition__: The Hopf invariant of f is $H(f) := k$. Note that it is well defined up to a sign (since the generator could be $+1$ or -1).

- By the same argument as the lemma, if $H(f) \neq 0$, then f is not nullhomotopic $\overset{\text{later}}{\Longrightarrow}$ $S^i f$ not nullhopic

COHOMOLOGY OPERATIONS

Definition: Let A, B be abelian groups. A <u>cohomology operation</u> is a natural transformation

$$\theta : H^m(-;A) \longrightarrow H^n(-;B), \qquad \text{for some } m, n \in \mathbb{N}$$

that is, gp hom $\theta_X : H^m(X;A) \to H^n(X;B)$ which are natural in the sense that if $f: X \to Y$,

$$\begin{tikzcd} H^m(Y;A) \arrow[r, "\theta_Y"] \arrow[d, "f^*"'] & H^n(Y;B) \arrow[d, "f^*"] \\ H^m(X;A) \arrow[r, "\theta_X"] & H^n(X;B) \end{tikzcd}$$

commutes.

- A ses $0 \to A \to B \to C \to 0$ of ab gp induces $0 \to C^*(X;A) \to C^*(X;B) \to C^*(X;C) \to 0$ a ses of cochain complexes which induce a les in cohomology, $H^n(X;A) \to H^n(X;B) \to H^n(X;C) \to H^{n+1}(X;A) \to$

Definition: The <u>Bockstein homomorphism</u> associated to a ses $0 \to A \to B \to C \to 0$ is the connecting homomorphism of the previous les,

$$H^n(X;C) \longrightarrow H^{n+1}(X;A)$$

Since the connecting hom is natural, this is a cohomology op.

Examples: Let p be prime

1) $0 \to \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ induces $H^n(X;\mathbb{Z}_p) \xrightarrow{\beta} H^{n+1}(X;\mathbb{Z}_p)$,

$[n] \longmapsto [pn]$
$[m] \longmapsto [m]$

usually "<u>the Bockstein operation</u>"

2) $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}_p \to 0$ induces $H^n(X;\mathbb{Z}_p) \xrightarrow{\tilde\beta} H^{n+1}(X;\mathbb{Z})$

3) Moreover,

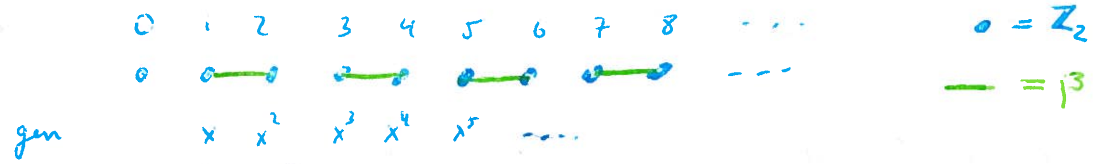
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[d, "\pi"] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & \mathbb{Z}_{p^2} \arrow[r] & \mathbb{Z}_p \arrow[r] & 0 \end{tikzcd}$$

induces

$$\begin{tikzcd} H^n(X;\mathbb{Z}_p) \arrow[r, "\tilde\beta"] \arrow[dr, "\beta"'] & H^{n+1}(X;\mathbb{Z}) \arrow[d, "\rho = \pi^*"] \\ & H^{n+1}(X;\mathbb{Z}_p) \end{tikzcd}$$

Example: For $\beta: \underset{\mathbb{Z}_2}{\underset{\|}{H^n(\mathbb{RP}^\infty;\mathbb{Z}_2)}} \longrightarrow \underset{\mathbb{Z}_2}{\underset{\|}{H^{n+1}(\mathbb{RP}^\infty;\mathbb{Z}_2)}}$, this is $\beta = \begin{cases} \text{iso}, & n \text{ odd} \\ \text{zero map}, & n \text{ even.} \end{cases}$

This is represented as



Since the Bockstein is natural, the representation of $\mathbb{RP}^n$ is just the restriction up to degree n.

Definition. A cohomology operation is *stable* if it is compatible with the suspension homomorphism, i.e., if the following diagram commutes:

$$\begin{tikzcd}
\widetilde{H}^m(X;A) \arrow[r, "\theta_X"] \arrow[d, "\wr\,\sigma"'] & \widetilde{H}^m(X;B) \arrow[d, "\wr\,\sigma"] \\
\widetilde{H}^{m+1}(SX;A) \arrow[r, "\theta_{SX}"] & \widetilde{H}^{m+1}(SX;B)
\end{tikzcd}$$

where σ is the suspension isomorphism.

- For a stable cohomology operation, the picture of the Bockstein for SX is just the shift of the picture of X.
- The Bockstein will tell us whether two spaces are htpy eq. or aren't;

Example. $\mathbb{RP}^3/\mathbb{RP}^1$ and $S\mathbb{RP}^2$ have the same cohomology groups, same ring cohomology, but their Bockstein are

so they cannot be htpy eq.

Lemma. The Bockstein hom. $\beta: H^n(X;\mathbb{Z}_p) \to H^{n+1}(X;\mathbb{Z}_p)$ is an antiderivation,

$$\boxed{\beta(x \cup y) = \beta x \cup y + (-1)^{|x|} x \cup \beta y}.$$

Lemma: $\beta^2 = 0$, i.e., β behaves as a differential.

• Therefore, the Bockstein defines a cochain complex $\cdots \xrightarrow{\beta} H^{n-1}(X;\mathbb{Z}_p) \xrightarrow{\beta} H^n(X;\mathbb{Z}_p) \xrightarrow{\beta} H^{n+1}(X;\mathbb{Z}_p) \to$

Definition: The n-th <u>Bockstein cohomology group</u> is the n-th coh. gp defined by the previous cochain complex, and we will denote it as $BH^n(X;\mathbb{Z}_p)$.

Theorem: Let X be a space with $H^n(X;\mathbb{Z})$ finite-generated $\forall n \geq 0$, so that

$$H^n(X;\mathbb{Z}) = (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \oplus \left(\bigoplus_{i,j} \mathbb{Z}/p_i^{n_{ij}}\mathbb{Z} \right)$$

1) Every summand $\mathbb{Z}$ on $H^n(X,\mathbb{Z})$ contributes one copy of $\mathbb{Z}_p$ on $BH^n(X;\mathbb{Z}_p)$.

2) Every summand $\mathbb{Z}/p^k\mathbb{Z}$, $\underline{k>1}$, on $H^n(X,\mathbb{Z})$ contributes one copy of $\mathbb{Z}_p$ on $BH^n(X;\mathbb{Z}_p)$ and one copy of $\mathbb{Z}_p$ in $BH^{n-1}(X;\mathbb{Z}_p)$

3) Every $\mathbb{Z}/p\mathbb{Z}$ on $H^n(X;\mathbb{Z})$ contributes nothing to $BH^*(X;\mathbb{Z}_p)$.

• That is, one can construct $BH^n(X;\mathbb{Z}_p)^{\forall p}$ from $H^n(X;\mathbb{Z})$ $\forall n$, but the reverse is not true in general. Anyway, it gives valuable information:

Corollary: If $BH^n(X;\mathbb{Z}_p) = 0$, then $H^n(X;\mathbb{Z}) = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$.

Lemma: If $H^n(X;\mathbb{Z})$ has not elements of order p^k, $\underline{k>1}$ (i.e., it has not p^k-torsion)$^{\forall k}$, then

$$\rho: H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}_p)$$

is injective in the p-torsion (i.e., it "embeds" the p-torsion), and it is precisely the image of β.

Example: Künneth, the lemma and the Bockstein allow us to conclude that (Künneth only $H^*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[x,y]$)

$$H^*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty;\mathbb{Z}) = \frac{\mathbb{Z}[x,y,z]}{(2x, 2y, 2z, z^2 + x^2y + xy^2)}$$

VII : STEEROD SQUARES I

Theorem: let (X, X') be a pair of spaces. There are unique natural homomorphisms

$$Sq^i : H^n(X, X'; \mathbb{Z}_2) \longrightarrow H^{n+i}(X, X'; \mathbb{Z}_2) \quad , \quad i \geq 0$$

called the Steenrod squares, satisfying

1) $Sq^0 = Id$

2) $Sq^i(x) = x \cup x$ for $i = |x| = n$

3) $Sq^i(x) = 0$ for $i > |x|$

4) $Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)$ (Cartan formula)

If $Sq := Sq^0 + Sq^1 + Sq^2 + \cdots$ is the total Steenrod square (finitely non-zero by 3)), then this can be rewritten as

$$Sq(x \cup y) = Sq(x) \cup Sq(y) ,$$

saying that

$$Sq : H^*(X, X'; \mathbb{Z}_2) \longrightarrow H^*(X, X'; \mathbb{Z}_2)$$

is a ring homomorphism.

Proposition (More properties):

5) Sq^i are stable cohomology operations, $\sigma \circ Sq^i = Sq^i \circ \sigma$.

6) $Sq^1 = \beta$, the Bockstein operation.

Example: $\overset{0}{\circ}$ $\qquad \overset{2}{\circ}\xrightarrow{Sq^2}\overset{4}{\circ}$ is the Sq's of $\mathbb{C}P^2$ with $\mathbb{Z}_2$ coefficients. One deduces that $S\mathbb{C}P^2$ and $S^3 \vee S^5$ are not htpy eq.

Corollary: If $f: S^{2n-1} \to S^n$ has odd Hopf invariant, then $S^i f$, $i \geq 0$ is not nullhomotopic.

Example: Consider $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[x]$ with $|x|=1$. By 1), 2), 3) we see that $Sq(x) = Sq^0(x) + Sq^1(x) = x + x^2 = x(x+1)$, and since degreewise the gen is x^n, we have

$$Sq(x^n) \overset{\text{ring hom}}{=\!=} Sq(x)^n = x^n(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n+k}, \quad \text{so} \quad Sq^i(x^n) = \binom{n}{i} x^{n+i}.$$

• "Easy" way to compute binomial coefficients mod a prime p: let $n = \sum n_i p^i$ and $k = \sum k_i p^i$, $n_i, k_i = 0, \dots, p-1$. Then

$$\binom{n}{k} = \prod_{i \geq 0} \binom{n_i}{k_i} \pmod p,$$

where $\binom{0}{0} = \binom{1}{0} = \binom{1}{1} = 1$ and $\binom{0}{1} = 0$.

Example: $S^2(\mathbb{R}P^3)$ and $\mathbb{R}P^5/\mathbb{R}P^2$ have the same cohomology gps, cohomology rings, same Bockstein but different Steenrod, so they are not homotopy equivalent.

ADEM'S RELATIONS

What relations have the Steenrods with each other?

Proposition (Adem relations): Denote by $\lfloor q \rfloor$ the integer part of $q \in \mathbb{Q}$. Then

$$Sq^i Sq^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

Corollary:

1) $Sq^1 \circ Sq^1 = 0$

2) $Sq^1 \circ Sq^2 = Sq^3$

3) $Sq^1 \circ Sq^3 = 0$

4) $Sq^2 \circ Sq^3 = Sq^5 + Sq^4 Sq^1$.

Consider the $\mathbb{Z}_2$-algebra generated by the Steenrod operations, i.e., as $\mathbb{Z}_2$-v.s. this is spanned by

$$\{ Sq^{i_1} \circ \dots \circ Sq^{i_k} = Sq^I \ : \ I = (i_1 \dots i_k) \in \mathbb{N}_0^k, \ k \in \mathbb{N} \}$$

and multiplication is given by concatenation.

Definition: The Steenrod algebra A^* is the quotient of the previous $\mathbb{Z}_2$-algebra by the two-sided ideal generated by the Adams relations.

Call a monomial $Sq^{i_1} \dots Sq^{i_k}$ "admissible" when $i_j \geq 2 i_{j+1}$ $\forall j$. Eg.: $Sq^5 Sq^2 Sq^1$ is, whereas $Sq^2 Sq^3 Sq^1$ is not. The monomials from the Adams relations are always admissible.

Theorem: Admissible monomials are linearly independent and span A^*, so they form a basis of A^*.

Lemma: If i is not a power of 2, then Sq^i is decomposable, i.e.,

$$Sq^i = \sum_{j=1}^{i-1} a_j \, Sq^j \, Sq^{i-j}$$

Theorem: If $H^*(X; \mathbb{Z}_2) = \mathbb{Z}_2[x]$ or $\mathbb{Z}_2[x]/(x^m)$, $m > 2$, then the generator x satisfies that $|x| = 2^k$ for some k.

• Analogously to the Steenrod squares (or more exactly to Sq^{2i}) we have

<u>Theorem</u>: Let $p \geq 3$ be prime. There are cohomology operations

$$P^i : H^n(X;\mathbb{Z}_p) \longrightarrow H^{n+2i(p-1)}(X;\mathbb{Z}_p) \qquad , \quad i \geq 0,$$

called the <u>Steenrod powers</u>, satisfying

1) $P^0 = \mathrm{Id}$
2) $P^i(x) = x^p$ if $|x| = 2i$
3) $P^i(x) = 0$ if $2i > |x|$
4) $P(x \cup y) = P(x) \cup P(y)$ for $P = P^0 + P^1 + P^2 + \ldots$ the <u>total Steenrod power</u> (<u>Cartan formula</u>)

<u>Proposition</u>: 5) The Steenrod powers are stable cohomology operations, $\sigma P^i = P^i \sigma$.

• For the powers, the Bockstein is not one of these anymore!

<u>Theorem</u>: If $H^\bullet(X;\mathbb{Z}_p) = \mathbb{Z}_p[x]$ or $\mathbb{Z}_p[x]/(x^m)$, $m > p$, then $|x| = 2p^k \cdot \ell$, with ℓ a divisor of $p-1$.

<u>Corollary</u>: If $H^\bullet(X;\mathbb{Z}) = \mathbb{Z}[x]$ or $\mathbb{Z}[x]/(x^m)$, $m > 3$, then $|x| = 2$ or 4.

VIII STEENROD SQUARES II

The lecture is devoted to prove the existence of the Steenrod squares (and to construct them!). Some results and ideas that were used:

Definition: Let G be a group. The group ring of G is $\mathbb{Z}[G]$, which is a ring with product $(ng)(n'g') := (nn')(gg')$.

For $G = \mathbb{Z}_2 = \{1, \tau\}$ (multiplicative notation), $R := \mathbb{Z}[G] = \{a + b\tau : a, b \in \mathbb{Z}\}$, and it is commutative.

The functor $C_*(-,\mathbb{Z}) : \mathrm{Top} \to \mathrm{Ch}_{\geq 0}$. To fix this, consider the acyclic chain complex

$$W_* = (\cdots \to R \xrightarrow{\tau+1} R \xrightarrow{\tau-1} R \xrightarrow{\tau+1} R \xrightarrow{\tau-1} R).$$

and we see that $C_*(-,\mathbb{Z}) \otimes W_*$ is free with models $\{\Delta^j : j \in J\}$. By the acyclic models thm, we get a natural transformation of functors $\mathrm{Top} \to \mathrm{Ch}_{\geq 0}$ $\tilde{\Delta} : W_* \otimes C_*(-) \to C_*(-) \otimes C_*(-)$.

Definition: Write $d_i \in W_i = R$ for the generator $1 \in R$. Let $\varphi \in C^p(X;A)$, $\psi \in C^q(X,A)$ be cochain complexes, where A is an ab. gp. The cup-i product is the $(p+q-i)$-th cochain

$$\varphi \cup_i \psi : C^{p+q-i}(X;A) \longrightarrow A$$
$$x \longmapsto (\varphi \otimes \psi)\big(\tilde{\Delta}(d_i \otimes x)\big).$$

Eg: $\varphi \cup_0 \psi = \varphi \cup \psi$.

Lemma (Annoying): Write δ for the differential of $C^*(X;A)$, and $\varphi \in C^p(X;A)$, $\psi \in C^q(X;A)$. Then

$$\delta(\varphi \cup_i \psi) = (-1)^i \delta\varphi \cup_i \psi + (-1)^{i+p} \varphi \cup_i \delta\psi - \left((-1)^i \varphi \cup_{i-1} \psi + (-1)^{pq} \psi \cup_{i-1} \varphi\right)$$

In particular, for $A = \mathbb{Z}_2$, we get

$$\boxed{\delta(\varphi \cup_i \psi) = \delta\varphi \cup_i \varphi + \varphi \cup_i \delta\psi} \qquad \text{(Leibniz rule)}$$

Lemma:

1) If φ is a cocycle, so is $\varphi \underset{i}{\cup} \varphi$.

2) ——— coboundary, ———.

Definition: Let $Z^p(X;\mathbb{Z}_2) \leq C^p(X;\mathbb{Z}_2)$ be the subgroup of degree p cocycles, $n \geq 0$. Then

$$Sq_i : Z^p(X;\mathbb{Z}_2) \longrightarrow H^{2p-i}(X;\mathbb{Z}_2)$$
$$\varphi \longmapsto [\varphi \underset{i}{\cup} \varphi]$$

Lemma. Sq_i is a group homomorphism.

- The lemma at the top allows us to extend Sq_i to $Sq_i : H^p(X;\mathbb{Z}_2) \longrightarrow H^{2p-i}(X;\mathbb{Z}_2)$.

Note that $Sq_0(\varphi) = \varphi \cup \varphi$.

Definition: The Steenrod operations are

$$Sq^i := Sq_{p-i} : H^p(X;\mathbb{Z}_2) \to H^{p+i}(X;\mathbb{Z}_2) \quad , \quad \underline{0 \leq i \leq p},$$

and for other values of i $Sq^i = 0$ by definition.

Theorem: The properties stated in the previous lecture hold.

IX : FIBERS BUNDLES

<u>Definition</u>: A (locally trivial) <u>fiber bundle</u> with typical fiber F (a top space) is a continuous map $p: E \to B$ such that every point $b \in B$ has a nbhd $U \subseteq B$ and a homeomorphism $\psi_U : p^{-1}(U) \simeq U \times F$ over U, ie, such that

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\psi", "\sim"'] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

commutes.

<u>Examples</u>:

1) If F is a discrete space, one recovers the notion of <u>covering map</u>.

2) If F is a vector space, and we ask maps between overlapping nbhds to be vector space homomorphisms, we recover the notion of <u>vector bundle</u>.

3) If $F = G$ is a top group acting on E, with the trivial action on B, and we ask ψ to be G-invariant, then we recover the notion of <u>principal G-bundle</u>.

Set $K = \mathbb{R}$ or $\mathbb{C}$, and set $S(K^n) = \{x \in K^n : |x| = 1\}$, ie, $S(\mathbb{R}^n) = S^{n-1}$ and $S(\mathbb{C}^n) = S^{2n-1}$. Call $G(K) = S(K^1)$, which are groups in both cases. Note that $KP^n = S(K^{n+1})/G(K)$ with the action $(x_0 \dots x_n) \sim \lambda (x_0 \dots x_n)$, $\lambda \in G(K)$.

<u>Lemma</u>: The quotient map $S(K^{n+1}) \to KP^n$ is a fiber bundle with typical fiber $G(K)$.

<u>Definition</u>: Let $p: E \to B$ be a cont map and let (X, A) be a pair of spaces. We say that p has the <u>homotopy lifting property</u> (HLP) wrt the pair (X, A) if for any solid commutative diagram as the following there is a dashed arrow H doing both triangles commutative.

$$\begin{tikzcd} X \times \{0\} \cup A \times I \arrow[r, "g"] \arrow[d] & E \arrow[d] \\ X \times I \arrow[r, "G"] \arrow[ur, dashed, "\exists H"] & B \end{tikzcd}$$

We say that $p: E \to B$ has the HLP wrt X (absolute version) if it has the HLP wrt $(X, \emptyset)$

i.e., such that there's a commutative diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d] \\ X \times I \arrow[r, "G"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

- In other words, p has HLP wrt (X, A) if whenever you have a homotopy $X \times I \to B$ and a lift to E at $t=0$ and for A, then you can lift it to all $X \times I$. Absolute version: whenever you have a homotopy $X \times I \to B$ and a lift at $t=0$, you can lift it $\forall t$.

<u>Warning</u>: In general, the lifts <u>will not be unique</u>, in contrast to covering theory.

<u>Definition</u>: Let $p: E \to B$ be a cont. map.

a) p is a <u>Hurewicz fibration</u> (or just a <u>fibration</u>) if it has the HLP wrt any space X

b) p is a <u>Serre fibration</u> if it has the HLP wrt all $\mathbb{D}^n$, $n \geq 0$.

<u>Lemma</u>: The following are equivalent:

1) p is a Serre fibration

2) p has the HLP wrt $(\mathbb{D}^n, \partial \mathbb{D}^n)$ $\forall n \geq 0$.

3) p has the HLP wrt (X, A) relative CW-complex.

<u>Definition</u>. A topological space B is <u>paracompact</u> if it is Hausdorff and every open cover admits partitions of unity subordinated to the cover.

Eg: CW-complexes, smooth manifolds, and Hausdorff and compact spaces are paracompact.

<u>Theorem</u>: Every fiber bundle $p: E \to B$ is a Serre fibration; and if B is paracompact, then it is a Hurewicz fibration.

Recall: $\pi_n(X,A,x_0) := [(\mathbb{D}^n, \partial\mathbb{D}^n, s_0),(X,A,x_0)]$ where the homotopies H satisfy $H_t(\partial\mathbb{D}^n) \subseteq$ For $n=0$ it is not defined, for $n=1$ is just a based set, for $n=2$ is a group, and for $n\geq 3$ is an abelian group.

Lemma: Let $p: E \to B$ be a Serre fibration, let $Y \subseteq B$ and $x \in p^{-1}(Y) \subseteq E$. Then p induces an isomorphism (bijection for $n=1$)

$$p_*: \pi_n(E, p^{-1}(Y), x) \xrightarrow{\sim} \pi_n(B, Y, p(x)).$$

Theorem (Long exact sequence of a Serre fibration): Let $p: E \to B$ be a Serre fibration and let $x \in E$. Set $F := p^{-1}(p(x))$. Then there is a les

$$\cdots \to \pi_n(F,x) \to \pi_n(E,x) \to \pi_n(B,p(x)) \to \pi_{n-1}(F,x) \to \cdots \to \pi_0(B,p(x)).$$

Examples: 1) $\exp: \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ is a covering map, thus a fiber bundle, with fiber $\mathbb{Z}$; which is a discrete space, thus $\pi_n(S^1) = 0$ $\forall n \geq 2$.

2) $p: S^n \to \mathbb{R}P^n$ same, with fiber $\{\pm 1\}$, discrete, thus $p_*: \pi_i(S^n) \xrightarrow{\sim} \pi_i(\mathbb{R}P^n)$, $i \geq 2$.

3) $\eta: S^3 = S(\mathbb{C}^2) \to \mathbb{C}P^1 = S^2$ is a fiber bundle with fiber S^1. The les of the Serre fib. gives that $\mathbb{Z} = \pi_3(S^3) \xrightarrow{\sim} \underline{\pi_3(S^2)}$.

The argument of 1) and 2) generalize to

Theorem: A covering map $p: (X,x_0) \to (S,s_0)$ induces isomorphisms

$$p_*: \pi_n(X,x_0) \xrightarrow{\sim} \pi_n(S,s_0) \qquad \forall n \geq 2$$

(for $n=1$ it is only injective in general).

Definition: Consider the following diagram of top spaces and cont maps:

$$X \xrightarrow{f} B \xleftarrow{g} Y$$

The **pullback** of this diagram is

$$X \times_B Y := \{ (x,y) \in X \times Y : f(x) = g(y) \}.$$

• It comes with two canonical maps $X \times_B Y \xrightarrow{p_1} X$ and $X \times_B Y \xrightarrow{p_2} Y$.

Theorem (Universal Property of the Pullback): For any solid commutative diagram there is a unique dashed map $\varphi : Z \to X \times_B Y$,

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists ! \varphi"] \arrow[drr, bend left, "\varphi_2"] \arrow[ddr, bend right, "\varphi_1"'] & & \\
& X \times_B Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & B
\end{tikzcd},$$

ie,

$$\mathrm{Hom}_B(Z, X \times_B Y) = \mathrm{Hom}_B(Z, X) \times \mathrm{Hom}_B(Z, Y)$$

where Hom_B denotes the morphisms in the slice category Top/B. Ie, the pullback is the direct product in the slice category over B.

• A square $\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$ is a **pullback square** when the previous $\varphi : Z \xrightarrow{\sim} X \times_B Y$ is homeomorphism

Proposition (Change of basis of fibrations): The notion of fibration is stable under change of basis: consider the pullback diagram

$$\begin{tikzcd} E \times_B B' \arrow[r, "g"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

1) If p is a fiber bundle with fiber F, then so is p' (with fiber F too)

2) If p is a Serre fibration, then so is p'

3) If p is a Hurewicz fibration, then so is p'.

Lemma: If

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

is a pullback square with p a Serre fibration and f a weak homotopy equivalence, then g is a weak homotopy equivalence as well.

Theorem: Let $p: E \to B$ be a Serre fibration and let $b_0, b_1 \in B$ be points in the same path-component. Then there is a weak homotopy equivalence relating $p^{-1}(b_0)$ and $p^{-1}(b_1)$. If the fibers have the hty type of a CW-complex, then (by Whitehead thm) they are hty eq. For Hurewicz fibration, the fibers are hty eq. directly.

$$\boxed{\begin{array}{lcl} \text{Serre fibration} & \longrightarrow & \text{fibers (over the same path-component) are weak homotopy equivalent} \\ \text{Hurewicz fibration} & \Longrightarrow & \text{———————— homotopy equivalent.} \end{array}}$$

X : THE COMPACT-OPEN TOPOLOGY

• As usual for Serganova, compact = quasi-compact + Hausdorff.

Recall: - A base $\mathcal{B}$ of a topology is a subset $\mathcal{B} \subseteq \tau$ st every open is arbitrary union of basic elements.

- A subbase $\mathcal{S}$ of a topology is a subset $\mathcal{S} \subseteq \tau$ that generates the topology, ie, every open is arbitrary union of finite intersections of elements of $\mathcal{S}$.

- Given a family of subsets whose union is the total space, it generates a topology (ie, it is a subbase of a topology): take finite intersections and arbitrary unions, and get a topology.

• Let X, Y be top spaces and set $F(X,Y) := \mathrm{Hom}_{\mathrm{Top}}(X,Y) = \{\text{cont. maps } X \to Y\}$. Can we endow $F(X,Y)$ with a useful topology?

Definition: Let X, Y be top spaces. For $K \subseteq X$ compact and $O \subseteq Y$ open, set

$$W(K,O) := \{f \in F(X,Y) : f(K) \subseteq O\}.$$

The compact-open topology is the topology generated by the $W(K,O)$, ie, such that

$$\{W(K,O) : K \subseteq X \text{ compact and } O \subseteq Y \text{ open}\}$$

is a subbasis.

Lemma: If $\mathcal{S}$ is a subbasis of the topology of Y, then

$$\{W(K,O) : K \subseteq X \text{ compact and } O \in \mathcal{S}\}$$

is a subbasis for the compact-open topology of $F(X,Y)$.

Lemma: Let X, Y be spaces with X locally compact ($=$ every point has a compact nbhd). Then

$$ev : F(X,Y) \times X \longrightarrow Y \qquad \text{is continuous.}$$
$$(f \; , x) \longmapsto f(x)$$

Theorem (Adjoint thm): Let X, Y, Z be top spaces, and consider the map

$$\phi : F(X \times Y, Z) \longrightarrow F(X, F(Y,Z))$$
$$f \longmapsto (x \mapsto (y \mapsto f(x,y)))$$

1) ϕ is a well-defined continuous map,
2) Y loc. compact $\Rightarrow$ ϕ is bijective
3) Y loc compact and X Hausdorff $\Rightarrow$ ϕ is homeomorphism.

In the language of categories, "$- \times Y$ is left adjoint to $F(Y,-)$", if Y is loc compact.

Corollary: If $f : X \to X'$, $g : Y \to Y'$ are continuous maps between Hausdorff spaces, then all arrows in the following diagram are continuous:

$$\begin{array}{ccc} F(X,Y) & \xrightarrow{g_*} & F(X,Y') \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ F(X',Y) & \xrightarrow{g_*} & F(X',Y') \end{array}$$

THE CATEGORY OF K-SPACES

If we want the product of two CW-complexes to be another one with the product topology, we need to move to another category (a subcategory of Top);

Definition: Let X be a top space. A subset $A \subseteq X$ is called compactly closed if for any compact space K and any continuous map $t : K \to X$, $t^{-1}(A) \subseteq K$ is closed.

Definition: A **K-space** is a space X where every compactly closed subset is closed (the converse always holds).

- What if a space is not a K-space? We can always turn it into one:

Definition: We denote as KX to X with the topology given by setting the closed subsets as the compactly closed ones.

- Note that the new topology is finer, since we are adding new closed subsets (the previous ones also are).

Proposition (Properties):

1) $Id: KX \to X$ is cont, and it is homeomorphism $\Leftrightarrow$ X is a K-space.
2) If X is a K-space, $f: X \to Y$ cont $\Leftrightarrow$ $f \circ t: K \to Y$ cont $\forall t: K \to X$, K cmpt.
3) $f: X \to Y$ cont $\to$ $Kf: KX \to KY$ cont.
4) Every compact space is a K-space.

- This construction defines a subcategory $\mathcal{K} \subseteq Top$ of K-spaces, and a functor

$$K: Top \to \mathcal{K}$$

which is right-adjt to the inclusion,

$$\mathcal{K}(Y, KX) = F(Y, X),$$

saying that the algebraic invariants H_n, H^n, π_n, ... are preserved.

- The product of two K-spaces may fail to be a K-space. Solution? To define a new one.

Definition: For two spaces X, Y, set the K-product $X \times_K Y := K(X \times Y)$.

Theorem: The K-product $X \times_K Y$ of two CW-complexes is a CW-complex again.

FUNCTION SPACES

Let X, Y be K-spaces. For $h: K \to X$ cont, K compact and $O \subseteq Y$ open, set

$$N(K, O, h) := \{ f: X \to Y \text{ cont} : (fh)(K) \subseteq O \}$$

<u>Definition</u>: We write $Map(X,Y) := k(F(X,Y))$ for two k-spaces X, Y; where $F(X,Y)$ is considered with the topology generated by $N(K,O,h)$.

<u>Theorem</u> (Adjunction, extra):

1) For k-spaces X, Y, Z, there's a homeomorphism

$$Map(X \times_k Y, Z) \xrightarrow{\sim} Map(X, Map(Y,Z))$$

2) For $(X,x_0), (Y,y_0), (Z,z_0)$ pointed k-spaces, if Map_* denotes pointed maps with the k-subspace top, and $X \wedge_k Y := X \times_k Y / X \times \{y_0\} \cup \{x_0\} \times Y$ denotes the k-smash product, then there is a homeomorphism

$$Map_*(X \wedge_k Y, Z) \xrightarrow{\sim} Map_*(X, Map_*(Y,Z)).$$

BACK TO FIBRATIONS IN Top

<u>Lemma</u>: If $A \subseteq B$ is closed, and (B,A) has the HEP $\Rightarrow (X \times A, X \times B)$ has HEP for all X.

<u>Proposition</u>: If $A \subseteq B$ is closed with the HEP, and A, B are locally compact, then for any Z

$$i^* : F(B,Z) \longrightarrow F(A,Z) \quad \text{is a Hurewicz fibration.}$$

<u>Notation</u>: $Y^X \overset{not}{=} F(X,Y)$, thus the adjunction can be written as $Z^{X \times Y} = (Z^Y)^X$.

Let (Y, y_0) be a pointed space, and set

$Y^I = F(I, Y)$

$P_{y_0}Y := \{\sigma \in Y^I : \sigma(0) = y_0\}$ = paths starting at y_0

$\Omega Y = \Omega_{y_0} Y := \{\sigma \in Y^I : \sigma(0) = y_0 = \sigma(1)\}$ = loops based at y_0, <u>the loop space</u> at y_0

Theorem :

1) $p : Y^I \longrightarrow Y$, $\sigma \longmapsto \sigma(1)$ is a Hurewicz fibration

2) $p : P_{y_0} Y \longrightarrow Y$, $\sigma \longmapsto \sigma(1)$ is a Hurewicz fibration with fiber over y_0 $\Omega_{y_0} Y$.

3) $i : Y \longrightarrow Y^I$, $y \longmapsto \mathrm{const}_y$ is a homotopy equivalence

4) $P_{y_0} Y$ is contractible.

Corollary : There are isomorphisms for $n \geq 0$ (bijection for $n = 0$)

$$\pi_n (\Omega_{y_0} Y, \mathrm{const}_{y_0}) = \pi_{n+1} (Y, y_0)$$

XI : PUPPE SEQUENCES

• A gap to fill at the end of IX:

<u>Theorem</u>: Let $p : E \to B$ be a Hurewicz fibration and let $b_0, b_1 \in B$ points in the same path compt. Then $F_0 := p^{-1}(b_0)$ and $F_1 = p^{-1}(b_1)$ are homotopy equivalent.

<u>FACTORIZATIONS</u>

• In AT I we saw that every cont. map $f : X \to Y$ factors through the mapping cone as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\text{closed inclusion}"'] & Y \\ & M_f \arrow[u, "\text{htpy equivalence}"'] \end{tikzcd}$$

<u>Theorem</u>: Every continuous map $f : X \to Y$ factors through $P_f := X \times_Y Y^I$ as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"', "\text{htpy equivalence}"] & Y \\ & P_f \arrow[u, "p"', "\text{Hurewicz fibration}"] \end{tikzcd}$$

i.e. $f = p \circ h$ with p a Hurewicz fibration and h a htpy equivalence.

In other words, up to homotopy equivalence, every continuous map is a Hurewicz fibration.

Explicitly, $P_f = X \times_Y Y^I = \{ (x, \sigma) \in X \times Y^I : f(x) = \sigma(0) \}$, and

$$p : P_f \to Y, \quad (x, \sigma) \longmapsto \sigma(1) \qquad , \qquad h : X \to P_f, \quad x \longmapsto (x, \mathrm{cont}_{f(x)})$$

Definition: Let $f: X \to Y$ be cont. The homotopy fiber of f at $y \in Y$ is

$$\mathrm{hofib}_y(f) := p^{-1}(y) = \{(x,\sigma) \in X \times Y^I : \sigma(0) = f(x),\ \sigma(1) = y\}.$$

Corollary: For any $y, y' \in Y$ in the same path-component, we have a homotopy equivalence

$$\mathrm{hofib}_y(f) \simeq \mathrm{hofib}_{y'}(f)$$

- The previous cor. is a consequence of the first thm of the lecture. The les of the fibration $p: P_f \to Y$ can be rewritten as

Corollary: For any continuous map $f: X \to Y$, the induced maps $f_*: \pi_n(X, x_0) \to \pi_n(Y, y_0)$ fit in a les

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(\mathrm{hofib}_{y_0}(f), \bar{x}_0) \arrow[r] & \pi_n(X, x_0) \arrow[r, "f_*"] & \pi_n(Y, y_0) \arrow[lld, bend left] \\ & \pi_{n-1}(\mathrm{hofib}_{y_0}(f), \bar{x}_0) \arrow[r] & \cdots \end{tikzcd}$$

where $\bar{x}_0 = (x_0, \mathrm{const}_{y_0})$.

- This says that hofib is measuring the failure of f_* of being isomorphism.

Lemma. If $f: X \to Y$ is already a Hurewicz fibration, the hty equivalence $h: X \xrightarrow{\simeq} P_f$ restricts to hty equivalences fiberwise,

$$f^{-1}(y) \overset{h}{\simeq} \mathrm{hofib}_y(f).$$

- With more generality,

Proposition: A homotopy eq. between Hurewicz fibrations restricts to a hty eq. fiberwise:

$$\begin{tikzcd} E \arrow[dr, "p"] \arrow[d, "h"', "\wr"] & \\ E' \arrow[r, "p'"'] & B \end{tikzcd} \quad \Longrightarrow \quad p^{-1}(b) \overset{h}{\simeq} (p')^{-1}(b) \qquad \forall b \in B$$

hty equivalence; Hurew. fibration; Hurewicz fibration

Corollary (Fiberwise uniqueness of factorization): If $f: X \to Y$ factors as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \\ & Z \arrow[u, "q"'] \end{tikzcd} \quad \Longrightarrow \quad \mathrm{hofib}_y(f) \simeq q^{-1}(y)$$

hty eq; Hurewicz fibration

- As usual $[X, Y] = \{$ hty classes $f: X \to Y\}$. If Y is path-connected, there is a preferred element $[\mathrm{const}_{y_0}]$.

Lemma: Let $p: E \to B$ be a Hurewicz fibration, B path-connected, $b_0 \in B$ and $F = p^{-1}(b_0)$.

For any space Y, the sequence

$$[Y, F] \xrightarrow{i_*} [Y, E] \xrightarrow{p_*} [Y, B]$$

is an exact sequence of based sets (where $i: F \hookrightarrow E$).

Theorem: Let $q: E \to B$ be a Hurewicz fibration, let $e_0 \in E$ and $b_0 = q(e_0)$, and set $F = q^{-1}(b_0)$.

Consider

$$\begin{tikzcd} \mathrm{hofib}_{b_0}(q) \arrow[r, hook, "i"] & P_q \arrow[r, "p"] & B \\ F \arrow[u, "h_F", "\wr"'] \arrow[r, "j"'] & E \arrow[u, "h"', "\wr"] \arrow[ur, "q"'] \arrow[ul, dashed, "g"] & \end{tikzcd}$$

g obtained by composite with the inverse

Then g is a fibration, $g^{-1}(e_0) = \Omega B$ and therefore $\mathrm{hofib}_{e_0}(j) \simeq \Omega_{b_0} B$.

Theorem (Puppe sequence): Let $q: E \to B$ be a Hurewicz fibration, $e_0 \in E$, $b_0 = q(e_0)$ and $F = q^{-1}(b_0)$, and $j: F \hookrightarrow E$.

Any two consecutive maps in the sequence

$$\cdots \to \Omega^n F \to \Omega^n E \to \Omega^n B \to \Omega^{n-1} F \to \cdots \to \Omega B \to F \to E \to B$$

are homotopic to

$$(\text{inclusion of fiber}) \circ (\text{fibration}),$$

and therefore we obtain a long exact sequence of based sets

$$\cdots \to [Y, \Omega^n F] \to [Y, \Omega^n E] \to [Y, \Omega^n B] \to [Y, \Omega^{n-1} F] \to \cdots$$

$$\cdots \to [Y, F] \to [Y, E] \to [Y, B].$$

COFIBRATIONS

- We saw the HEP for a pair (X, A). This generalizes for a map $i: A \to X$ in an analogous way to the fibrations:

Definition: A map $i: A \to X$ is a Hurewicz cofibration if for any solid diagram there is a dashed arrow $X \times I \to Z$ doing both triangles commutative:

$$\begin{tikzcd}
A \times \{0\} \arrow[r] \arrow[d, hook] & X \times \{0\} \arrow[d] \arrow[ddr, bend left] & \\
A \times I \arrow[r] \arrow[drr, bend right] & X \times I \arrow[dr, dashed] & \\
& & Z
\end{tikzcd}$$

In other words, $i: A \to X$ is a cofibration if given a space Z, a cont map $X \to Z$ and a homotopy $A \times I \to Z$ starting "in the restriction of $X \to Z$ to A", there is an extension of $A \times I \to Z$ starting from $X \to Z$.

In an analogous way we have:

Theorem: Every continuous map $f: X \to Y$ factors through M_f as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{cofibration}"'] & & Y \\ & M_f \arrow[ur, "\text{hty equivalence}"'] & \end{tikzcd}$$

One can show that every Hurewicz cofibration is always an embedding, ie, $A \xrightarrow{\sim} i(A) \subseteq X$ is a homeomorphism. If X is Hausdorff, we can identify A with a closed subspace of X.

Definition: Let $i: A \to X$ a cofibration. The cofiber of i is the pushout of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & X/A \end{tikzcd}$$

that we'll denote by X/A.

Definition: Let $f: X \to Y$ be cont. The homotopy cofiber of f is the cofiber of $X \to M_f$, ie,

$M_f / X = C_f$ the mapping cone of f.

Proposition: If $i: A \to X$ is a cofibration, and Y path-connected, $[X/A, Y] \xrightarrow{f^*} [X, Y] \xrightarrow{i^*} [A, Y]$ is exact

Theorem (Puppe sequence): If $i: A \to X$ is a cofibration, any two consecutive arrows in

$$A \to X \to X/A \to SA \to SX \to S(X/A) \to S^2 A \to \dots$$

is homotopic to (cofibration) $\circ$ (projection onto the cofiber) giving rise to a les

$$\cdots \to [S^i(X/A), Y] \to [S^i X, Y] \to [S^i A, Y] \to \cdots$$

$$\cdots \to [X/A, Y] \to [X, Y] \to [A, Y].$$

And exactly the same using Σ.

XII : EILENBERG - MACLANE SPACES

Definition : Let $n \geq 1$ and A be a group which is abelian for $n \geq 2$. An Eilenberg - MacLane space of type (A, n) is a path-connected space X such that

$$\pi_i(X) = \begin{cases} A, & i = n \\ 0, & i \neq n. \end{cases}$$

We say that X is a $K(A,n)$. More precisely, if $\varphi : \pi_n(X, x_0) \xrightarrow{\sim} A$ is the iso, we refer to the EM as (X, φ).

Examples :

1) S^1 is $K(\mathbb{Z}, 1)$

2) X is $K(A,n)$, Y is $K(B,m)$ $\Rightarrow$ $X \times Y$ is $K(A \times B, n+m)$.

3) $\mathbb{RP}^\infty$ is $K(\mathbb{Z}_2, 1)$

4) $\mathbb{K}$ is $K(\langle a, b \mid aba^{-1}b \rangle, 1)$

5) X is $K(A,n)$ $\Rightarrow$ ΩX is $K(A, n-1)$.

6) $\mathbb{CP}^\infty$ is $K(\mathbb{Z}, 2)$

Theorem (Killing of homotopy groups) : Let $n \geq 0$ and let (Y, y_0) be a pointed space. There exists a relative CW-complex (X, Y) such that

1) Relative cells have dim $\geq n+1$ (ie, the n-skeleton is still Y)..

2) The inclusion $i : Y \hookrightarrow X$ induces isomorphisms $\pi_i(Y, y_0) \xrightarrow{\sim} \pi_i(X, y_0)$ for $i < n$.

3) $\pi_i(X, y_0) = 0$ for $i \geq n$.

Recall: The Hurewicz thm states that if X is n-connected ($\pi_i(X)=0\ \forall i\leq n$), $n\geq 2$, then $H_i(X;\mathbb{Z})=0\ \forall\ 0<i<n$ and the Hurewicz map $h_n:\pi_n(X,x_0)\xrightarrow{\sim} H_n(X;\mathbb{Z})$ is isomorphism.

Theorem (Existence of EM spaces): If $n\geq 2$ and A is an abelian group, then there is a $K(A,n)$ space, which is a CW-complex.

The proof is very artificial. A more down-to-Earth construction is as follows. Recall that a simplicial set is a functor $K:\Delta^{op}\to \mathrm{Set}$ (ie, a family of sets (K_n) with maps $\alpha^*:K_m\to K_n$ for any $\alpha:[n]\to[m]$ order pres.

Definition: Let K be a simplicial set. The <u>geometric realization</u> of K is

$$|K| := \frac{\coprod_{n\geq 0} K_n\times\Delta^n}{(\alpha^*(k),t)\sim(k,\alpha_*(t))},$$

for all $k\in K_m$, $t\in\Delta^n$ and $\alpha:[n]\to[m]$. Here we consider K_n with the discrete topology.

This basically consists in glueing n-simplices by their boundaries. Logically, this turns to be a CW-complex. There is a simplicial set $\underline{\Delta}^n/\partial\underline{\Delta}^n$ with $|\underline{\Delta}^n/\partial\underline{\Delta}^n|\cong\Delta^n/\partial\Delta^n\cong S^n$. If $\tilde{A}[\underline{\Delta}^n/\partial\underline{\Delta}^n]=A[\underline{\Delta}^n/\partial\underline{\Delta}^n]/A(*)$ is the "degreewise residual A-linearization", then

Theorem: If $n\geq 2$ and A abelian, $|\tilde{A}[\underline{\Delta}^n/\partial\underline{\Delta}^n]|$ is a $K(A,n)$. For $n=1$, and G a group, setting $B(G)$ a simplicial set with $B(G)_p := G\times\overset{p}{\cdots}\times G$, we have $|B(G)|$ is $K(G,1)$.

Lemma (Extension): Let (X,Y) be a relative CW-complex, and let Z be a space such that $\pi_{n-1}(Z,z_0)=0$ $\forall z_0\in Z$ whenever (X,Y) has a relative cell of dimension $n\geq 1$.

Then every continuous map $Y\to Z$ extends to a map $X\to Z$.

Theorem: Let $n \geq 1$, let X be a CW-complex with $X_{n-1} = \{x_0\}$ and let (Y, y_0) be a pointed space with $\pi_i(Y, y_0) = 0$ for $i > n$.

If $\rho : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is a group homomorphism, there exists a continuous map $f : (X, x_0) \to (Y, y_0)$, unique up to based homotopy, such that $\rho = f_*$.

In other words,

$$\begin{array}{ccc} [(X, x_0), (Y, y_0)]_* & = & \mathrm{Hom}\left(\pi_n(X, x_0), \pi_n(Y, y_0)\right) \\ [f] & \longmapsto & f_* \end{array}$$

Corollary: Let $n \geq 1$ and let A be a group which is abelian if $n \geq 2$. If (X, φ), (Y, ψ) are two CW-complexes which are $K(A, n)$ spaces, then there is a homotopy equivalence $f : X \to Y$ such that

$$\begin{tikzcd} \pi_n(X, x_0) \arrow[rr, "f_*"] \arrow[dr, "\varphi"', "\wr"] & & \pi_n(Y, y_0) \arrow[dl, "\psi", "\wr"'] \\ & A & \end{tikzcd}$$

commutes.

In other words, "Eilenberg-MacLane spaces are unique up to homotopy equivalence"

XIII: (CO)HOMOLOGY THEORIES

Definition: A reduced homology theory on pointed CW-complexes is a sequence of covariant functors $\{\tilde{h}_n : CW_* \to AbGps : n \geq 0\}$, satisfying the Eilenberg–Steenrod axioms:

i) (Homotopy invariance): If $f \simeq g : X \to Y$, then $f_* = g_* : \tilde{h}_n(X) \to \tilde{h}_n(Y)$.

ii) (Long exact sequence): For any CW pair (X,A), there are natural group homomorphisms, called boundary maps, $\delta : \tilde{h}_n(X/A) \to \tilde{h}_{n-1}(A)$, fitting in a les

$$\cdots \to \tilde{h}_n(A) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X/A) \xrightarrow{\delta} \tilde{h}_{n-1}(A) \to \cdots$$

iii) (Sums) For a wedge sum $X = \bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow X$ induce an iso

$$\bigoplus_\alpha i_{\alpha *} : \bigoplus_\alpha \tilde{h}_n(X_\alpha) \xrightarrow{\sim} \tilde{h}_n(X) \qquad \forall n.$$

iv) (Dimension) $\tilde{h}_n(S^0) = 0 \quad \forall n \neq 0$, and $\tilde{h}_0(S^0)$ is called the coefficients of the hom. theory.

There is a version for absolute homology, but more intricate, by defining homology for pairs of spaces $h_n(X,A)$ and setting $h_n(X) := h_n(X, \emptyset)$. One should also modify ii) by the obvious version and iii) by excision. Actually, reduced and unreduced homology theories are equivalent: from an unreduced theory one recovers $\tilde{h}$ as $\tilde{h}_n(X) := \mathrm{Ker}(h_n(X) \to h_n(*))$; and from $\tilde{h}$, one recovers h by setting $h_n(X,A) := \tilde{h}_n(X/A)$.
(and $X/\emptyset = X \sqcup *$)

Lemma: For a homology theory, it is true

1) Mayer-Vietoris sequence (for h).

2) Suspension isomorphism: $\tilde{h}_n(X) \simeq \tilde{h}_{n+1}(SX) \quad \forall n \geq 0$.

• Duality.

Definition : A reduced cohomology theory on pointed CW-complexes is a sequence of contravariant functors $\{\tilde{h}^n : CW_* \to AbGps : n \geq 0\}$ satisfying the Eilenberg-Steenrod axioms :

i) (Homotopy invariance) : If $f \simeq g : X \to Y$, then $f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)$.

ii) (Long exact sequence) : For any CW pair (X,A), there are natural group homomorphisms called coboundary maps $\delta : \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A)$, fitting in a les

$$\cdots \to \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A) \to \cdots$$

iii) (Sums) For a wedge sum $X = \bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow X$ induce an iso

$$\prod_\alpha i_\alpha^* : \tilde{h}^n(X) \xrightarrow{\simeq} \prod_\alpha \tilde{h}^n(X_\alpha)$$

iv) (Dimension) : $\tilde{h}^n(S^0) = 0 \ \forall n \neq 0$, and $\tilde{h}^0(S^0)$ is called the coefficients of the cohomology theory

• We can define lots of (co)homology theories for spaces. How do they relate each other?

Theorem (Uniqueness) : All (co)homology theories on pointed CW complexes are naturally isomorphic, both reduced and unreduced.

More precisely, if h and $\tilde{h}$ are (unreduced) cohomology theories, then there are natural isomorphisms

$$h^n(X,A) \simeq H^n(X,A;h^0(*)) \quad , \quad \tilde{h}^n(X) \simeq \tilde{H}^n(X, \tilde{h}^0(S^0)).$$

For the proof one needs a result valid for general homology theories:

Lemma: For any unreduced homology theory, the degree of maps between spheres is additive, i.e.,

$$\mathbb{Z} = \pi_n(S^n) \longrightarrow \mathrm{Hom}(h_n(S^n), h_n(S^n))$$
$$[f] \longmapsto f_*$$

is a group homomorphism.

EILENBERG-MACLANE SPACES AS CLASSIFYING SPACES

Lemma: $\Sigma X = S^1 \wedge X$.

Theorem (Suspension-loop adjunction): Let X, Y be pointed spaces, with X Hausdorff and loc. compact. There is a bijection

$$[\Sigma X, Y]_* = [X, \Omega Y]_*$$

which is natural on X and Y. That is, "$\Sigma -$ is left adjoint to $\Omega -$".

- Explicitly, the inverse is given by $(f : X \to \Omega Y) \longmapsto (\hat{f} : \Sigma X \to Y,\ \hat{f}(x,t) := f(x)(t))$.
- In ΩY we have an obvious operation given by concatenation of loops, but it fails to be associative. Nevertheless, it induces a multiplication on $[X, \Omega Y]_*$ which is a group. Even more, $[X, \Omega^2 Y]_*$ is abelian, by similar arguments to the ones which claim that π_2 is.

Definition: An infinite loop space is a space X such that $X \simeq \Omega^k X_k$ for some space X_k for all $k \geq$

Lemma: $K(A,n)$ is an infinite loop space:

$$K(A,n) \simeq \Omega K(A, n+1) \simeq \Omega^2 K(A, n+2) \simeq \cdots$$

Corollary: The groups $[X, K(A,n)]_* = [X, \Omega^2(A, n+2)]_*$ are abelian, $\forall n \geq 0$.

<u>Theorem</u>: The functors $\{[-, K(A,n)]_* : n \geqslant 0\}$ define a reduced cohomology theory on pointed CW complexes.

* <u>Corollary</u>: "<u>Eilenberg-MacLane spaces represent cohomology</u>".

Precisely, for a pointed CW-complex X, there is a natural isomorphism

$$[X, K(A,n)]_* = \widetilde{H}^n(X; A).$$

XIV : POSTNIKOV TOWERS

Theorem (Postnikov Towers): Let X be a connected, pointed space. There are CW pairs (P_nX, X), $n \geq 1$, unique up to (weak) htpy equivalence, together with maps $i_n : X \to P_nX$ satisfying

1) Relative cells have dim $\geq n+2$
2) $\pi_i(X) \simeq \pi_i(P_nX) \quad \forall i \leq n$
3) $\pi_i(P_nX) = 0 \quad \forall i > n$.

Moreover, there exist maps $P_{n+1}X \xrightarrow{\varphi_{n+1}} P_nX$, unique up to homotopy, fitting in the following diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_nX \arrow[d, "\varphi_n"] \\
 & \vdots \arrow[d] \\
 & P_2X \arrow[d, "\varphi_2"] \\
X \arrow[r, "i_1"] \arrow[ur, "i_2"] \arrow[uuur, "i_n"] & P_1X
\end{tikzcd}$$

(Postnikov tower)

For this there is a useful

Lemma: Let $m \leq n$ and let $f : X \to Y_m$, $g : X \to Y_n$ satisfying 1) -3) of the theorem. Then there is a map $\varphi : Y_n \to Y_m$, unique up to homotopy, such that $\varphi \circ f = g$.

Lemma: The homotopy fiber of $\varphi_n : P_nX \to P_{n-1}X$ is a $K(\pi_n(X), n)$.

Up to what extent does the Postnikov tower of X encode information of X?

Theorem: Let $\cdots \to X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$ a sequence of Hurewicz fibrations and let $i \geq 1$. There is natural map

$$\Phi : \pi_i \left(\varprojlim_n X_n \right) \longrightarrow \varprojlim_n \pi_i (X_n)$$

is surjective, and an isomorphism if $\pi_{i+1}(X_n) \xrightarrow{f_{n,*}} \pi_{i+1}(X_{n-1})$ is surjective for $n >> 0$.

Corollary: If all maps $\varphi_n : P_n X \to P_{n-1} X$ are Serre fibrations, then the natural map

$$X \longrightarrow \varprojlim_n P_n X$$

is a weak htpy equivalence.

- There is a dual notion of the Postnikov towers:

Theorem (Whitehead towers): Let X be a connected, pointed space. There are spaces $W_n X$, $n \geq 1$, together with maps $p_n : W_n X \to X$ satisfying

1) $\pi_i(W_n X) \simeq \pi_i(X) \quad \forall\, i > n$
2) $\pi_i(W_n X) = 0 \quad \forall\, i \leq n$.

Moreover, there are morphisms $\phi_{n+1} : W_{n+1} X \to W_n X$ fitting in the following diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & \\
W_n X \arrow[d, "\phi_n"'] \arrow[ddr, "p_n"] & \\
\vdots \arrow[d] & \\
W_2 X \arrow[d, "\phi_2"'] \arrow[r, "p_2"] & X \\
W_1 X \arrow[ur, "p_1"'] &
\end{tikzcd}$$

(Whitehead tower)

- In particular, $W_1 X$ is weak htpy equivalent to the universal covering of X (if it exists)

• (Action of $\pi_1(X)$ on the higher homotopy groups) Let X be a pointed space. For every $n\geq 1$, $\pi_1(X)$ acts on $\pi_n(X)$ as follows: given $[\gamma]\in\pi_1(X)$ and $[\alpha]\in\pi_n(X)$ represented by $\gamma: I\to X$ and $\alpha: S^n\to X$, we have a map $\alpha\cup\gamma: S^n\times\{0\}\cup_{\{s_0\}\times\{0\}}\{s_0\}\times I\longrightarrow X$ coming from the u. property of the pushout. Now the HEP of the pair $(S^n,\{s_0\})$ precisely says that H exists in the below diagram:

$$\begin{tikzcd} S^n\times\{0\}\cup_{\{s_0\}\times\{0\}}\{s_0\}\times I \arrow[r, "\alpha\cup\gamma"] \arrow[d, hook] & X \\ S^n\times I \arrow[ur, dashed, "H"] & \end{tikzcd}$$

Now setting $[\gamma]*[\alpha]:=[H_1]$ defines an action $\pi_1(X)\times\pi_n(X)\longrightarrow\pi_n(X)$.

PRINCIPAL FIBRATIONS

Definition: Let $F\hookrightarrow E\to B$ be a fibration sequence. We say that it is principal if there is a space Z and a map $B\to Z$ st $E\to B\to Z$ is a fibration sequence.

For the fibration $K(\pi_n(X),n)\to P_nX\to P_{n-1}X$, the only possibility is $P_{n-1}X\to K(\pi_n(X),n+1)$. The homotopy class of this map corresponds to a cohomology class $k_n\in H^{n+1}(P_{n-1}X;\pi_n(X))$; called the n-th k-invariant.

Definition: A space X is called simple if $\pi_1(X)$ acts trivially on $\pi_n(X)$ $\forall n\geq 1$.

Lemma: A fibration $K(A,n)\to X\to Y$ is principal $\Leftrightarrow$ the action of $\pi_1(X)$ on $\pi_n(K(A,n))=A$ is trivial

Theorem: X has a Postnikov tower of principal fibrations $\Leftrightarrow$ X is simple.

OBSTRUCTION THEORY

- let (Y, A) be a CW pair and suppose we have a map $A \to X$ that we want to extend to X,

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ Y \arrow[ur, dashed, "\exists ??"'] & \end{tikzcd}$$

is that possible ?

The point is, if we are able to extend Y to the Postnikov tower of X, then we can extend it to X,

__Lemma__. To give a lift

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & P_n X \arrow[d] & \\ Y \arrow[ur, dashed] \arrow[r] & P_{n-1} \arrow[r] & K(\pi_n(X), n+1) \end{tikzcd}$$

is the same as to give a nullhty of the lower row, and the same as to give a nullhty $Y/A \to K(\pi_n(X), n+1)$.

__Definition__. The lifty class $\omega_n \in H^{n+1}(Y, A; \pi_n(X))$ corresponding with the map $Y/A \to K(\pi_n(X), n+1)$ is called the __n-th obstruction__ to extend $A \to X$ to $Y \to X$.

__Corollary__. If $\omega_n = 0$ $\forall n$, then the extension $Y \to X$ exists.

__Corollary__: If $H^{n+1}(Y, A; \pi_n(X)) = 0$ $\forall n$, then the extension $Y \to X$ exists.

XV : BLAKERS - MASSEY EXCISION THEOREM

<u>Theorem</u> (Blakers - Massey) : let X be a connected space and let U_1, U_2 two connected open subsets which form an open cover of X, and suppose that $U_0 = U_1 \cap U_2$ is also connected. If the pairs (U_1, U_0) and (U_2, U_0) are p-connected and q-connected, resp., then the <u>excision map</u>

$$i_* : \pi_n(U_1, U_0) \longrightarrow \pi_n(X, U_2)$$

induced by the inclusion is an isomorphism for all $n < p+q$ and surjective for $n = p+q$ (here the basepoint is taken in U_0).

<u>Corollary</u> (Freudenthal suspension) :

1) $\pi_i(S^n) \longrightarrow \pi_i(S^{n+1})$ is isomorphism $\forall\, i < 2n-1$, and surjective for $i = 2n-1$
2) If X is n-connected, then $\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$ is isomorphism $\forall\, i < 2n-1$ and surj for $i = 2n-$

<u>Corollary</u> : $\pi_n(S^n) = \mathbb{Z}$.

The groups $\pi_{n+k}(S^n)$ with $k < n-1 \Leftrightarrow n > k+1$ are called the <u>stable range</u> because by 2) there are isos $\pi_{n+k}(S^n) \xrightarrow{\simeq} \pi_{n+k+1}(S^{n+1})$, and therefore

<u>Definition</u> : The <u>k-th stable homotopy group of the spheres</u> is

$$\pi_k^{st} := \varinjlim_n \pi_{n+k}(S^n)$$

<u>Ex</u> : $\pi_0^{st} = \mathbb{Z}$, $\pi_1^{st} = \mathbb{Z}_2$, $\pi_2^{st} = \mathbb{Z}_2$, $\pi_3^{st} = \mathbb{Z}_{24}$

XVI : THE SERRE SPECTRAL SEQUENCE

- Groups $\pi_k(S^n)$ very hard to compute. The first serious tool for that was the Serre ss (1951).
- For a cofibre sequence $A \xrightarrow{f} X \to X/A$, we can relate (co)homology of them via the les

$$\dots \to H^n(X/A) \to H^n(X) \to H^n(A) \to H^{n+1}(X/A) \to H^{n+1}(X) \to \dots$$

and also the homotopy groups (Blakers–Massey long exact sequence)

$$\pi_{r+s}(A) \to \pi_{r+s}(X) \to \pi_{r+s}(X/A) \to \pi_{r+s-1}(A) \to \quad \dots \quad \to \pi_2(X/A) \to 0,$$

when A is r-connected and X/A s-connected.

- For a fibre sequence $F \to E \to B$, we have a les of htpy groups

$$\dots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \dots$$

but there is nothing yet relating (co)homology of a fibre sequence! (If $E \to B$ is a fibre bundle, Leray–Hirsch).
The Serre ss will be a gadget to relate these.

- (<u>Overview</u>): The Serre ss will consist of a sequence of pages E_r (as a book); where each page has a first quadrant grid. Each entry has a group $H^i(B; H^j(F; M))$ for a fibration $F \to E \to B$, and there are diagonal differentials $d_r : (i,j) \to (i-r, j+r-1)$. The page E_{r+1} arises by taking the cohomology of these differentials, and it will have new differentials. At some point there will be a stable value E_∞, and one can read off the cohomology of E out of this, eg when all groups of the E_∞ page are free, $H^n(E)$ is just direct sum of the $i+j=n$ diagonal.

<u>Definition</u>: Let N be a R-module. A <u>filtration</u> of N of length n is a sequence of submodules

$$F^0N \subseteq F^1N \subseteq \dots \subseteq F^nN = N.$$

The <u>associated graded module</u> of F^*N is the graded abelian group gr^*N, where $gr^iN := F^iN/F^{i-1}N$.

* *Definition*: A (cohomological, first quadrant) *spectral sequence* of R-modules is a pair (E_r, d_r), $r \geq 1$ where

a) $E_r = \bigoplus_{i,j \geq 0} E_r^{ij}$ is a bigraded R-module,

b) $d_r : E_r \to E_r$ is a bigraded R-module homomorph of bidegree $(r, 1-r)$, with $d_r^2 = 0$.

c) $E_{r+1} = H(E_r, d_r)$, ie,
$$E_{r+1}^{ij} := \frac{\mathrm{Ker}\,(d_r : E_r^{ij} \to E_r^{i+r, j-r+1})}{\mathrm{Im}\,(d_r : E_r^{i-r, j+r-1} \to E_r^{ij})}.$$

Let $N^\bullet = \bigoplus_{n \geq 0} N^n$ be a graded R-module, where each N^n has a filtration $F^* N^n$ of length n. We say that a spectral sequence (E_r, d_r) converges to $(N^\bullet, F^* N^\bullet)$, and we write

$$E_2^{ij} \Rightarrow N^{i+j}$$

if there are isomorphisms $E_\infty^{n-i,i} \xrightarrow{\cong} gr^i N^n$.

∘ (Action of $\pi_1(B)$ on $H^\bullet(F; M)$): let $F \to E \xrightarrow{p} B$ be a fibration sequence. If $\alpha : I \to B$ is a path on B, $\alpha(0) = x_0$, $\alpha(1) = x_1$, $E_{x_i} = p^{-1}(x_i)$, we can consider the lifting problem to the right, where H is the composite $E_{x_0} \times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$. Since p is a fibration, the dashed arrow exists and by commutativity at $t=1$ defines a map

$$\begin{tikzcd} E_{x_0} \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ E_{x_0} \times I \arrow[r, "H"] \arrow[ur, dashed] & B \end{tikzcd}$$

$$\tilde{H}(-,1) : E_{x_0} \to E_{x_1}$$

This is a htpy eq, so it induces iso in cohomology. Specializing to loops at the basepoint y, $p^{-1}(y) = F$, gives the required action. If $F \to E \to B$ is a Serre fib only, argue with HLP at the level of n-simplices.

* Theorem (Cohomological Serre spectral sequence): let $F \to E \to B$ be a Serre fibration, and assume that $\pi_1(B)$ acts trivially on $H^\bullet(F; M)$ (eg when B is simply connected). Then there is a cohomological, first quadrant spectral sequence

$$E_2^{ij} = H^i(B; H^j(F; M)) \Rightarrow H^{i+j}(E; M).$$

Remark: If the action is nontrivial, then one has to use homology with coeff in a local system = loc. constant sheaf.

• $H^{\bullet}(X;R)$ has a (graded) ring structure coming from $\cup$. This interacts nicely with the Serre ss:

Proposition: The $E_r = \bigoplus_{i,j} E_r^{i,j}$ page of the cohomological Serre ss forms a bigraded ring, i.e., if $x \in E_r^{i,j}$ and $y \in E_r^{p,q}$ then $xy \in E_r^{i+p,\, j+q}$. Moreover the differentials d_r satisfy the Leibniz rule,

$$d_r(xy) = d_r(x)\cdot y + (-1)^{i+j}\, x \cdot d_r(y) .$$

• In a completely analogous way, we can define a homological spectral sequence (E^r, d^r), $E^r = \bigoplus E^r_{ij}$ and $d^r : E^r \to E^r$ R-mod map of bidegree $(-r, r-1)$ (now diff goes up left). Under the same hypothesis, the homological Serre ss states that there is a ss $E^2_{ij} = H_i(B; H_j(F;M)) \Longrightarrow H_{i+j}(E;M)$.

Example: From the Serre fib $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ one gets a ss with E_2 page

$$E_2 = \begin{array}{c|ccccccc}
1 & H^0\cdot a & & H^2\cdot a & & H^4\cdot a & \cdots & H^{2n}\cdot a \\
0 & H^0 & & H^2 & & H^4 & \cdots & H^{2}a \\
\hline
 & 0 & 1 & 2 & 3 & 4 & \cdots & 2n
\end{array}$$

(a gen of $\mathbb{Z} = \pi_1(S^1)$), $H^i = H^i(\mathbb{C}P^n)$, $H^i\cdot a = H^i(\mathbb{C}P^n)\otimes H^1(S^1) \overset{uc}{\cong} H^i(\mathbb{C}P^n; H^1(S^1))$

Since there is no room for non-trivial diff in the E_3 page, $E_3 = E_\infty$ and all diff in the E_2 page are isos.

From the Leibniz rule one also gets that if $x := d_2 a$ is the gen of H^2, then x^i is the gen of H^{2i}. So one also gets the ring str $H^{\bullet}(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^n)$, $|x| = 2$.

Example: From the Serre fib $\Omega S^n \to PS^n \to S^n$, $n \geq 2$, one gets that the first diff that can be non-zero is d_n. Since PS^n is contractible, only the $\mathbb{Z}$ at $(0,0)$ can survive. Thus the homological Serre ss yields

$$E_n = \begin{array}{c|ccc}
\vdots & & & \\
3(n-1) & H_{3n-3} & & H_{3n-3} \\
\vdots & & & \\
2(n-1) & H_{2n-2} & & H_{2n-2} \\
\vdots & & & \\
n-1 & H_{n-1} & & H_{n-1} \\
\vdots & & & \\
0 & \mathbb{Z} & & \mathbb{Z} \\
\hline
 & 0 & \cdots & n
\end{array}$$

$$H_i(\Omega S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i \in (n-1)\mathbb{Z} \\ 0, & \text{else} \end{cases}$$

Using univ. coeff (or the cohom. Serre ss) one gets $H^i(\Omega S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i \in (n-1)\mathbb{Z} \\ 0, & \text{else} \end{cases}$, and the coh Serre ss even says that

$$H^{\bullet}(\Omega S^{2n+1}; \mathbb{Z}) \cong \Gamma[z] \quad , \quad H^{\bullet}(\Omega S^{2n}; \mathbb{Z}) \cong \Gamma[y] \otimes \mathbb{Z}[x]/(x^2) ,$$

where $|z| = 2n$, $|x| = 2n-1$ and $|y| = 2(n-1)$, and $\Gamma[z] \subset \mathbb{Q}[z]$ the subring spanned by the elements $\{ \frac{x^i}{i!} \}_{i \geq 0}$.

CONSTRUCTION OF THE SERRE SS

Definition: An exact couple is a pair of R-modules (A, E) together with an exact triangle

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & {\scriptstyle k}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & E & \end{array}$$

• Given an exact triangle, we can define a differential $d_1 := j \circ k$ on E. Set $E_2 := H(E)$, $A_2 := \operatorname{Im} i$, and maps

$$i_2 : A_2 \to A_2, \quad a \mapsto i(a), \qquad j_2 : A_2 \to E_2, \quad a = i(a') \mapsto [j(a')], \qquad k_2 : E_2 \to A_2, \quad [e] \mapsto k(e)$$

Lemma: The previous R-modules (A_2, E_2) and module maps i_2, j_2, k_2 form a exact couple

• We can iterate this and obtain $E_2, E_3, \dots$ with differentials $d_2, d_3, \dots$. Ultimately this will be our ss.

Definition: An unrolled exact couple is a diagram

$$\begin{array}{ccccccccc} \cdots \to & A^{p+1} & \xrightarrow{i} & A^p & \xrightarrow{i} & A^{p-1} & \xrightarrow{i} & A^{p-2} & \to \cdots \\ \swarrow_j & \nwarrow_k & \swarrow_j & \nwarrow_k & \swarrow_j & \nwarrow_k & \swarrow_j & \nwarrow & \\ & & E^p & & E^{p-1} & & E^{p-2} & & \end{array}$$

such that every triangle is exact.

• We can get an exact couple out of a unrolled one by setting $A := \bigoplus A^p$, $E := \bigoplus E^p$, so now our modules are graded. In particular the differential $d_1 : E \to E$ has degree one, $d_1 : E^p \to E^{p+1}$. In particular, $E_2 = H(E) = H(\bigoplus E^p) = \bigoplus H^p(E)$. In general, $E_r = \bigoplus H^p(E_{r-1}) = \bigoplus E^p_{r-1}$.

Lemma: $d_r : E_r \to E_r$ has degree r, ie, if $[e] \in E^p_r$, then $d_r([e]) \in E^{p+r}_r$.

• We will now construct a unrolled exact couple from a chain complex.

<u>Definition</u>: Let C^* be a cochain complex of R-modules. A <u>filtration</u> of C^* is a sequence of subcomplexes

$$\cdots \subseteq F^2C^* \subseteq F^1C^* \subseteq F^0C^* = C^*.$$

The <u>associated graded complex</u> consists of cochain complexes gr^pC^* given by $gr^pC^* := F^pC^*/F^{p+1}C^*$, with differential induced by the one of C^*.

• There is a ses of cochain complexes $0 \to F^{p+1}C^* \to F^pC^* \to gr^pC^* \to 0$.

The les's of cohomology fits into a unrolled exact couple, varying p,

$$\begin{tikzcd}
\cdots \arrow[r] & H^\bullet(F^{p+1}C^*) \arrow[rr, "i"] \arrow[dl] & & H^\bullet(F^pC^*) \arrow[rr, "i"] \arrow[dl, "j"] & & H^\bullet(F^{p-1}C^*) \arrow[r] \arrow[dl, "j"] & \cdots \\
\cdots & & H^\bullet(gr^pC^*) \arrow[ul, "k"] & & H^\bullet(gr^{p-1}C^*) \arrow[ul, "k"] & & \cdots
\end{tikzcd}$$

where $H^\bullet(\ ) = \bigoplus_q H^q(\ )$ as usual. As before, we can get a exact couple setting

$$A := \bigoplus_{p,q} H^q(F^pC^*) \quad , \quad E := \bigoplus_{p,q} H^q(gr^pC^*)$$

Observe also that the differential $d_1 : E \to E$ has bidegree $(1,1)$ because k is induced by the connecting. If we set $E_1^{p,q} = H^q(gr^pC^*)$, d_r will have bidegree $(r,1)$. To be able to read off the column of E in the Serre ss we consider the <u>Serre grading</u>

$$E_1^{p,q} := H^{p+q}(gr^pC^*)$$

so that d_r has bidegree $(r, 1-r)$.

• Lastly, observe that given a filtered cochain complex C^*, its filtration induces a filtration on $H^q(C^*)$, setting $F^pH^q(C^*) := \operatorname{Im}\left(H^q(F^pC^*) \xrightarrow{i^p} H^q(F^0C^*) \right)$.

Theorem: let C^* be a filtered cochain complex, inducing a filtration on its cohomology as above. Suppose that the filtration F^*C^q of any C^q has finite length and that the cochain complex F^pC^* is concentrated in degrees $\geq p$. Then the spectral sequence constructed above converges to the coh. of C^*,

$$E_1^{p,q} = H^{p+q}(gr^p C^*) \Rightarrow H^{p+q}(C^*) .$$

• The next step is to study how a double complex gives rise to a filtered cochain complex and therefore to a SS:

Definition: A double (cochain) complex of R-modules is a collection of R-modules $C^{n,m}$, $n,m \in \mathbb{Z}$, equipped with differentials $\delta : C^{n,m} \to C^{n+1,m}$ and $\bar\delta : C^{n,m} \to C^{n,m+1}$ st $\delta^2=0$, $\bar\delta^2=0$, $\delta\bar\delta = \bar\delta\delta$.

$$C^{*,*} = \begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C^{n-1,m+1} \arrow[r] \arrow[u] & C^{n,m+1} \arrow[r] \arrow[u] & C^{n+1,m+1} \arrow[r] \arrow[u] & {} \\
\cdots \arrow[r] & C^{n-1,m} \arrow[r, "\delta"] \arrow[u] & C^{n,m} \arrow[r, "\delta"] \arrow[u] & C^{n+1,m} \arrow[r, "\delta"] \arrow[u] & {} \\
\cdots \arrow[r] & C^{n-1,m-1} \arrow[r] \arrow[u, "\bar\delta"] & C^{n,m-1} \arrow[r] \arrow[u, "\bar\delta"] & C^{n+1,m-1} \arrow[r] \arrow[u, "\bar\delta"] & {} \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

• Of course, $C^{n,*}$ is a cochain complex (wrt $\bar\delta$), so can take $H^m_{\bar\delta}(C^{n,*})$. But δ induces a new differential $\delta : H^m_{\bar\delta}(C^{n,*}) \to H^m_{\bar\delta}(C^{n+1,*})$. We denote by $H^n_\delta H^m_{\bar\delta}(C^{*,*})$ the resulting cohomology groups.

Definition: Given a double complex $C^{*,*}$, its total complex is the cochain complex $tot(C)^*$ with $tot(C)^q := \bigoplus_{n+m=q} C^{n,m}$ and differential $\delta^{tot} := \delta + (-1)^n \bar\delta$.

• A double complex $C^{*,*}$ can be filtered forgetting everything to the left, $F^p(C)^{n,m} = \begin{cases} C^{n,m}, & \text{if } n \geq p \\ 0, & \text{else} \end{cases}$, which in turn induces a filtration in $tot(C)$, $F^p tot(C)^* := tot(F^pC)^*$. Applying this to the previous thm we get

Theorem: Let $C^{*,*}$ be a double complex with only non-zero elements in degrees $n, m \geq 0$. Then the above construction yields a ss with pages

$$E_1^{p,q} = H_{\bar{\delta}}^{q}(C^{p,*}) \quad , \quad E_2^{p,q} = H_{\delta}^{p} H_{\bar{\delta}}^{q}(C^{*,*})$$

converging to the cohomology of $tot(C)^*$,

$$E_2^{p,q} \Rightarrow H^{p+q}(tot(C)^*).$$

• We finally go back to topology. The Serre ss will arise from a double complex constructed as follows:

Definition: Let $p: E \to B$ be a Serre fibration. A singular (n, m)-simplex is a commutative diagram

$$\begin{tikzcd} \Delta^n \times \Delta^m \arrow[r] \arrow[d, "pr_1"] & E \arrow[d, "p"] \\ \Delta^n \arrow[r] & B \end{tikzcd}$$

• Denote $S_{n,m}(p)$ the set of singular (n,m)-simplices and $C_{n,m}(p) := R[S_{n,m}(p)]$ its R-linearization. Taking alternating sum of faces we get differentials $\partial: C_{n,m}(p) \to C_{n-1,m}(p)$ and $\bar{\partial}: C_{n,m}(p) \to C_{n,m-1}(p)$. Setting $C^{n,m}(p;M) = Hom_R(C_{n,m}(p); M)$ for a R-module M gives a double complex with diff $\delta := \partial^*$ and $\bar{\delta} := \bar{\partial}^*$.

• The multiplication and Leibniz rule of the Serre ss can be stated in more general terms:

Definition: A double complex $C^{*,*}$ is multiplicative if there is a multiplication map

$$\mu: C^{n,m} \otimes C^{s,t} \to C^{n+s, m+t}$$

such that

i) $tot(C)^\bullet = \bigoplus_{n,m} C^{n,m}$ is a graded ring

ii) $\delta^{tot} = \delta + (-1)^{n}\bar{\delta}$ satisfies the graded Leibniz rule

$$\delta^{tot}(xy) = \delta^{tot}(x) \cdot y + (-1)^{n+m} x \cdot \delta^{tot}(y)$$

Proposition: If (E_r, d_r) is a ss associated to a multiplicative double complex, then E_r inherits a graded product structure from E_{r-1} and the differential d_r satisfies the graded Leibniz rule.

Remark: When the action $\pi_1(B) \curvearrowright H^0(F; n)$ is not trivial, one still have a ss but the action is encoded in a system of local coefficients. Generally, if B is a path-connected CW-complex and h_* is an additive homology theory, there is a ss

$$E^2_{p,q} = H_p(B; h_q(F)) \Rightarrow h_{p+q}(E)$$

where $F \hookrightarrow E \to B$ is a fibration space and $H_p(B; h_q(F))$ is homology with local coeff, as any loop σ gives rise to a htpy eq $\tilde{\sigma}: F \to F$ and therefore an iso $h_n(F) \to h_n(F)$, so $h_n(F)$ is a $\mathbb{Z}[\pi_1 B]$-module.

If h^n is an additive cohomology theory and B is finite-dimensional, there is a cohomological ss

$$E_2^{p,q} = H^p(B; h^q(F)) \Rightarrow h^{p+q}(E).$$

Example: One can use the previous ss to compute $\tilde{K}(\mathbb{CP}^n)$.

XVII : APPLICATIONS OF THE SERRE SS

__Theorem__ (Serre exact sequence) : Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration sequence with B, F path-connected and with $\pi_1(B) \curvearrowright H_*(F;R)$ trivially. Suppose that $\tilde{H}_i(B;R) = 0$ for $i < m$ and $\tilde{H}_i(F;R) = 0$ for $i < n$. Then there is an exact sequence

$$H_{m+n-1}(F;R) \xrightarrow{i_*} H_{m+n-1}(E;R) \xrightarrow{p_*} H_{m+n-1}(B;R) \to H_{m+n-2}(F;R) \to \dots \to H_1(B;R) \to 0 .$$

If $\pi_1(B) \curvearrowright H^*(F;R)$ trivially, ~~and~~ the cohomology in the given range vanishes, then there is an exact seque[nce] as the previous with arrows reversed in cohomology.

__Theorem__ (Gysin sequence) : Let $S^n \hookrightarrow E \xrightarrow{p} B$ be a fibration, and suppose $\pi_1(B) \curvearrowright H_*(S^n;R)$ trivially. B path conn. Then there is a exact sequence

$$\dots \to H_k(E;R) \xrightarrow{p_*} H_k(B;R) \to H_{k-n-1}(B;R) \to H_{k-1}(E;R) \to \dots .$$

For cohomology, there is a similar one reversing arrows.

__Theorem__ (Wang sequence) : Let $F \hookrightarrow E \to S^n$ be a fibration, with F path connected. Then there is an exact sequ[ence]

$$\to H_k(F;R) \to H_k(E;R) \to H_{k-n}(F;R) \to H_{k-1}(F;R) \to \dots$$

The same yields in cohomology reversing arrows.

__Corollary__ : Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibration, B, F path connected. Then there is a 5 term exact sequence

$$H_2(E;R) \xrightarrow{p_*} H_2(B;R) \to H_0^{tw}(B; H_1(F;R)) \to H_1(E;R) \xrightarrow{p_*} H_1(B;R) \to 0$$

GROUP (CO)HOMOLOGY

Recall: $H^k(G;M) := H^k(K(G,1);M)$, $H_k(G;M) := H_k(K(G,1);M)$, G group.

Proposition: The group (co)homology of $\mathbb{Z}/n$ is given by

$$H_p(\mathbb{Z}/n,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & p=0 \\ \mathbb{Z}/n, & p \text{ odd} \\ 0, & p \text{ even} > 0 \end{cases} \quad , \quad H^p(\mathbb{Z}/n,\mathbb{Z}) = \begin{cases} \mathbb{Z}, & p=0 \\ 0, & p \text{ odd} \\ \mathbb{Z}/n, & p \text{ even} > 0 \end{cases}$$

Moreover, the cohomology ring is $H^*(K(\mathbb{Z}/n,1);\mathbb{Z}) \cong \mathbb{Z}[x]/(nx)$, $|x|=2$.

+**Theorem**: Let $p: X \to S$ be a Galois covering map with Galois group $G = \mathrm{Aut}_S X$. Then there is a cohomological ss

$$E_2^{p,q} = H^p_{tw}(G; H^q(X;R)) \Rightarrow H^{p+q}(S;R)$$

and a homological ss

$$E^2_{p,q} = H_p^{tw}(G, H_q(X;R)) \Rightarrow H_{p+q}(S;R).$$

Here H_p^{tw} and H^p_{tw} mean (co)homology with twisted coeff, where the twisting is induced by the action of $G = \pi_1(K(G,1))$ on X by cov. transformations.

Corollary: There is a five-term exact sequence, for $p: X \to S$ Galois covering map,

$$H_2(S;R) \to H_2(G;R) \to H_0(G; H_1(X;R)) \to H_1(S;R) \to H_1(G;R) \to 0.$$

Corollary: For a path-connected space X, the sequence

$$\pi_2(X) \xrightarrow{h} H_2(X;\mathbb{Z}) \to H_2(\pi_1(X);\mathbb{Z}) \to 0$$

is exact, where h is the Hurewicz map.

Proposición: The cohomology of the Eilenberg-MacLane spaces $K(\mathbb{Z},n)$ with $\mathbb{Q}$ coefficients are

$$H^*(K(\mathbb{Z},2n),\mathbb{Q}) \cong \mathbb{Q}[x_{2n}], \qquad H^*(K(\mathbb{Z},2n+1);\mathbb{Q}) \cong \frac{\mathbb{Q}[x_{2n+1}]}{(x_{2n+1}^2)} \cong \Lambda_{\mathbb{Q}}[x_{2n+1}]$$

where $|x_n| = n$.

Proposition: Let X be a space and M an R-module. There is a homological ss

$$E^2_{p,q} = \mathrm{Tor}_p^R(M, H_q(X;R)) \Rightarrow H_{p+q}(X;M).$$

• This is equivalent to Universal Coeff for Homology.

Theorem: Let G be a group. Then $\quad H_n(G;\mathbb{Z}) \cong \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},\mathbb{Z})$.

SERRE CLASS THEORY

Definition: A class of abelian groups $\mathcal{C}$ is a Serre class if

i) $0 \in \mathcal{C}$

ii) For a ses $0 \to A \to B \to C \to 0$ of ab groups, we have that $A, C \in \mathcal{C} \Leftrightarrow B \in \mathcal{C}$.

iii) If $A, B \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}_1^{\mathbb{Z}}(A,B) \in \mathcal{C}$.

iv) (Group homology axiom) If $A \in \mathcal{C}$, then $H_n(A;\mathbb{Z}) \in \mathcal{C}$ for all $n \geq 1$.

Examples: 1) $\mathcal{FG}$ the class of finite-generated abelian groups

2) For a fixed set of primes $\mathbb{P}$, $\mathcal{T}_{\mathbb{P}}$ torsion abelian groups whose elements have order divisible only by primes of $\mathbb{P}$.

3) $\mathcal{F}_{\mathbb{P}}$, the finite groups of $\mathcal{T}_{\mathbb{P}}$.

4) As a special case for $\mathbb{P}$ all primes, the class $\mathcal{T}$ of all torsion ab gps and the class $\mathcal{F}$ of all finite ab gps.

Lemma: Let $F \hookrightarrow E \to B$ be a fibration sequence of path-connected spaces, with $\pi_1(B) \curvearrowright H_*(F;\mathbb{Z})$ trivially (eg if B simply-connected). If two out of the three $H_n(F;\mathbb{Z})$, $H_n(E;\mathbb{Z})$, $H_n(B;\mathbb{Z})$ are in a Serre class $\mathcal{C}$ for all $n>0$, so is the third.

Theorem: If $A \in \mathcal{C}$, then $H_k(K(A,n);\mathbb{Z}) \in \mathcal{C}$ for all $k, n > 0$.

Definition: A space X is abelian or simple if $\pi_1(X)$ is abelian and it acts trivially on the higher htpy groups of X.

***Theorem** (mod $\mathcal{C}$ Hurewicz): Let X be a path-connected, abelian space (eg if X is simply connected) and let $\mathcal{C}$ be a Serre class. Then

$$\pi_i(X) \in \mathcal{C} \text{ for } 0<i<n \iff H_i(X;\mathbb{Z}) \in \mathcal{C} \text{ for } 0<i<n.$$

Moreover, under these conditions the Hurewicz map $h: \pi_n(X) \to H_n(X;\mathbb{Z})$ is a isomorphism mod $\mathcal{C}$, ie, $\operatorname{Ker} h$, $\operatorname{Coker} h \in \mathcal{C}$.

***Corollary**: Let X be a path-connected, abelian space (eg if X is simply-connected). Then the homotopy groups $\pi_i(X)$ are finite-generated for all $i>0$ if and only if the homology groups $H_i(X;\mathbb{Z})$ are finite generated for all $i>0$.

***Corollary**: $\pi_i(S^n)$ are finite-generated for all $i, n>0$.

Corollary: If A is finite (generated), so is $H_i(K(A,n);\mathbb{Z})$ for all $i, n>0$.

WHITEHEAD THEOREMS

***Theorem** (Whitehead II): Let $f: X \to Y$ be a map between simply-connected spaces. The following two conditions are equivalent:

1) f induces an isomorphism on $H_i(-;\mathbb{Z})$ for $i<n$ and a surjection for $i=n$

2) —————— π_i ——————

***Corollary**: A map $f: X \to Y$ between simply-connected spaces is a weak htpy eq $\iff$ it induces isos on singular homology.

In particular, any weak homotopy equivalence between any spaces induce isos on singular homology.

Definition: We say that a Serre class $\mathcal{C}$ satisfies the torsion axiom if given $A \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}_1^{\mathbb{Z}}(A,B) \in \mathcal{C}$ for any abelian group B.

Remark: $\mathcal{F}$ and $\mathcal{FG}$ do not satisfy this axiom.

Theorem (mod $\mathcal{C}$ Whitehead). Let $f: X \to Y$ be a map between simply connected spaces inducing a surjection on π_2, and let $\mathcal{C}$ be a Serre class satisfying the tensor axiom. Then the following are equivalent:

1) f induces a $\mathcal{C}$-iso on $H_i(-;\mathbb{Z})$ for $i < n$ and a $\mathcal{C}$-surjection for $i = n$.

2) ——— π_i ———

(here $\mathcal{C}$-surjection means that $\mathrm{Coker}\, f_* \in \mathcal{C}$).

- If $S \subset \mathbb{Z}$ is a multiplicative subset of $\mathbb{Z}$, we let $\mathcal{T}_S$ be the class of torsion abelian groups whose element have order in S. One can show that $\mathcal{T}_S$ is another Serre class, satisfying the torsion axiom. Moreover, a map $A \to B$ is a $\mathcal{T}_S$-isomorphism $\iff A \otimes_{\mathbb{Z}} \mathbb{Z}_S \to B \otimes_{\mathbb{Z}} \mathbb{Z}_S$ is a isomorphism.

Since $\mathbb{Z}_S$ is a flat $\mathbb{Z}$-module (i.e. $- \otimes_{\mathbb{Z}} \mathbb{Z}_S$ is exact), Tor vanishes, so $H_i(X;\mathbb{Z}_S) \cong H_i(X;\mathbb{Z}) \otimes_S \mathbb{Z}_S$ and we get

Theorem: Let $f: X \to Y$ be a map between simply-connected spaces which induces a surjection on π_2, and let $S \subset \mathbb{Z}$ be a multiplicative subset. Then the following are equivalent:

1) f induces an iso on $H_i(-;\mathbb{Z}_S)$ for $i < n$ and a surjection on $i = n$.

2) ——— $\pi_i(-) \otimes \mathbb{Z}_S$ ———

In particular we get that f is a rational equivalence $\iff f$ induces isos on $H_i(-;\mathbb{Q})$.

XVIII : HOMOTOPY GROUPS OF SPHERES

• The theory of Serre classes told us that the groups $\pi_k(S^n)$ are finite generated for $n, k > 0$. We tackle how to actually compute them.

• The main tool is to use the <u>Whitehead tower</u> of a space X. Recall that $W_n X := \mathrm{hofib}(i_n : X \to P_n X)$, they assemble into a tower

$$\begin{tikzcd}
W_n X \arrow[d, "\phi_n"'] \arrow[ddddr, bend left, "p_n"] & \\
\vdots \arrow[d] & \\
W_2 X \arrow[d, "\phi_2"'] \arrow[ddr, "p_2"] & \\
W_1 X \arrow[r, "p_1"] & X
\end{tikzcd}$$

$$\pi_i(W_n X) \cong \pi_i(X), \quad i > n,$$
$$\pi_i(W_n X) \cong 0, \quad i \leq n.$$

(Killing of the hty gps from below). The space $W_n X$ is also called the <u>n-connected cover</u> of X.

• For our purpose the relevance is that, for any space X,

$$\pi_n(X) \cong \pi_n(W_{n-1}X) \overset{\text{Hurew}}{\cong} H_n(W_{n-1}X; \mathbb{Z})$$

The latter group is potentially computable using the Serre ss.

<u>Proposition</u> : $\pi_4(S^3) \cong \mathbb{Z}/2$ (using $W_3 S^3 \to S^3 \to P_3 S^3 = K(\mathbb{Z}, 3)$, and shifting it we get $K(\mathbb{Z},2) \simeq \mathbb{CP}^\infty \to W_3 S^3 \to S^3$).

<u>Proposition</u> : $\pi_5(S^3) \cong \mathbb{Z}/2$

• Since the higher hty groups of spheres are fg ab groups, they must be $\pi_i(S^n) \cong (\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}) \oplus T$, where T is a torsion finite group. It turns out that the free part can be determined completely:

* **Theorem** (Serre): If n is odd,

$$\pi_i(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & i = n \\ 0, & \text{else} \end{cases}$$

and if n is even,

$$\pi_i(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & i = n, 2n-1 \\ 0, & \text{else} \end{cases}$$

In particular, $\pi_i(S^n)$ is a finite group for $i \neq n$, n odd; and $i \neq n, 2n-1$, n even.

A STEP TOWARDS THE POINCARÉ CONJECTURE

- We start stating the the strong (and now proven) Poincaré conjecture

* **Theorem** (Smale 1961, Freedman 1983, Perelman 2002): Every closed manifold homotopy equivalent to S^n is also homeomorphic to S^n.

- Our goal is to show that for $n=3$, the conjecture can be stated as "every simply-connected 3-manifold is homeomorphic to S^3".

- As consequence of Whitehead II we get

Proposition: Every simply-connected CW-complex which is a $\mathbb{Z}$-homology "sphere (ie, that has the same homology groups as S^n) is homotopy equivalent to S^n.

Proposition: Every closed manifold M is a retract of a finite CW-complex. In particular, the homology groups $H_k(M;\mathbb{Z})$ are finite-generated abelian groups.

Corollary: Every closed manifold is homotopy equivalent to a CW-complex.

Actually, for $n \neq 4$, every closed manifold is homeomorphic to a CW-complex (Kirby-Siebenmann).

Theorem: If M is a simply-connected closed n-manifold with $H_i(M;\mathbb{Z}) \cong 0$ for $1 < i \leq \frac{n}{2}$, then M is homotopy equivalent to S^n.

• Corollary: Every simply-connected 3-manifold is homotopy equivalent to S^3.

• Something more about spheres:

Theorem (Serre): For any prime p, the groups $\pi_i(S^n)$ do not have p-torsion for $i < n+2p-3$, and $(\pi_{n+2p-3} S^n)_{(p)} \cong \mathbb{Z}/p$ if $n \geq 3$, where the group stands for the localization of the lity gp by the prime ideal gen by p.

• Recall: The Freudenthal theorem said, for spheres, that $\pi_{n+i}(S^n) \xrightarrow{\Sigma} \pi_{n+i+1}(S^{n+1})$ is isomorphism as soon as $n > i+1$. By our earlier computation of the rational homotopy groups of spheres we get

Corollary: The stable homotopy groups of spheres $\pi_i^{st} = \pi_i^{st}(S^0) = \operatorname{colim}_n \pi_{n+i}(S^n)$ are finite for all $i > 0$.

• In general, $\pi_i^{st}(X) := \operatorname{colim}_n \pi_{n+i}(\Sigma^n X)$ and we have

Theorem: π_*^{st} forms a reduced homology theory. Moreover, $\pi_i^{st}(X) \otimes \mathbb{Q} \cong \widetilde{H}_i(X;\mathbb{Q})$ for any CW-complex X.

• One last consequence of Freudenthal theorem is

Corollary: 1) $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ if $n \geq 3$.

2) $\pi_{n+2}(S^n) \cong \mathbb{Z}/2$ if $n \geq 3$.

APPENDIX

CELLULAR COHOMOLOGY

<u>Definition</u>: Let X be an absolute CW-complex. The <u>cellular cochain complex</u> (with coefficients in $\mathbb{Z}$) is the cochain complex $\widetilde{C}^*(X)$ given in degree n by $\widetilde{C}^n(X) := H^n(X_n, X_{n-1})$, and whose differential $\widetilde{\delta}^n : \widetilde{C}^n(X) \to \widetilde{C}^{n+1}(X)$ is the composite

$$H^n(X_n, X_{n-1}) \to H^n(X_n) \to H^{n+1}(X_{n+1}, X_n)$$

whose arrows come from the les of the pairs (X_n, X_{n-1}) and (X_{n+1}, X_n).

<u>Theorem</u>: $H^n(\widetilde{C}^*(X)) \cong H^n(X)$.

<u>Proposition</u>: The cellular cochain complex $\widetilde{C}^*(X)$ is isomorphic to $\mathrm{Hom}(\widetilde{C}_*(X); \mathbb{Z})$, where $\widetilde{C}_*$ is the cellular chain complex.

<u>Theorem</u>: Let X be a CW-complex.

1) $H^k(X_n, X_{n-1}) = 0 \quad \forall k \neq n$, and $H^n(X_n, X_{n-1}) = \mathrm{Hom}\left(\bigoplus_{J_n} \mathbb{Z}, \mathbb{Z}\right) \simeq \bigoplus_{J_n} \mathbb{Z}$

2) $H^k(X_n) = 0 \quad \forall k > n$. In particular $H^k(X) = 0 \quad \forall k > \dim X$ (if it is fin. dim)

3) The inclusion $X_n \hookrightarrow X$ induces isomorphisms $H^k(X) \to H^k(X_n) \quad \forall k < n$, and for $k = n$ $H^n(X) \to H^n(X_n)$ is injective.

V.- CONEXION

<u>Lemma</u>: Let X be a space. The following are equivalent:

1) Every map $S^i \to X$ is homotopic to a constant map

2) Every map $S^i \to X$ extends to a map $\mathbb{D}^{i+1} \to X$

3) $\pi_i(X, x_0) = 0 \quad \forall x_0 \in X$.

<u>Lemma</u>: Let (X,A) be a pair of spaces. The following are equivalent:

1') Every map $(\mathbb{D}^i, \partial\mathbb{D}^i) \to (X,A)$ is homotopic to a map $\mathbb{D}^i \to A$ rel. $\partial\mathbb{D}^i$.

2') Every map $(\mathbb{D}^i, \partial\mathbb{D}^i) \to (X,A)$ is homotopic through maps of pairs to a map $\mathbb{D}^i \to A$.

3') Every map $(\mathbb{D}^i, \partial\mathbb{D}^i) \to (X,A)$ is homotopic through maps of pairs to a constant map $\mathbb{D}^i \to A$.

4') $\pi_i(X, A, x_0) = 0 \quad \forall x_0 \in A$

<u>Definition</u>: Let $n \geq 0$.

a) A space X is <u>n-connected</u> if 1) – 3) hold for all $i \leq n$.

b) A pair of spaces (X,A) is <u>n-connected</u> if 1') – 4') hold for $0 < i \leq n$ and 1') – 3') for i

This last condition for $i=0$ means that each path-component of X contains points of A.

HOMOTOPY GROUPS OF A CW-COMPLEX

- The following is a consequence of cellular approximation:

Theorem: If a CW pair (X,A) has relative cells of dimension $> n$, then (X,A) is n-connected ie, $\pi_i(X,A) = 0 \ \forall i \leq n$. In particular for a CW complex X,

1) $\pi_i(X, X_n) = 0 \quad \forall i \leq n$.

2) The inclusion $X_n \hookrightarrow X$ induces isomorphisms $\pi_i(X_n) \xrightarrow{\sim} \pi_i(X) \quad \forall i < n$, and for $i = n$ $\quad \pi_n(X_n) \longrightarrow \pi_n(X)$ is surjective. Ie,

$$\pi_n(X_{n+1}) \xrightarrow{\sim} \pi_n(X_{n+2}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \pi_n(X)$$