I : SINGULAR HOMOLOGY

- We want to associate algebraic invariants for top. spaces, analogous to the fundamental group.

__Definition__: Let $(A,+)$ an abelian group and let M be a set. The __A-linearization__ of M is

$$A[M] := \{ f : M \to A \; : \; f^{-1}(A - 0) \text{ is finite} \}$$
$$= \{ f : M \to A : \; f(m) = 0 \text{ up to a finite quantity} \}.$$

- $A[M]$ inherits an abelian group structure from A: $(f+g)(x) = f(x) + g(x)$, $\text{cont}_0 : M \to A, m \mapsto$ is the neutral element and $(-f)(x) = -f(x)$ is the inverse of f.

- We will denote as $a \cdot x$ for the element $\begin{array}{l} f : M \to A \\ \quad x \mapsto a \\ \quad \text{else} \mapsto 0 \end{array}$. Since $f(m) \neq 0$ for a finite quantity, it is clear that every $f \in A[M]$ is $a_1 x_1 + \dots + a_r x_r$. Also it holds: $a \cdot x + a' x = (a + a') x$, and $0 \cdot x$ is omitted.

__Definition__: A map of set $\gamma : M \to N$ induces a morphism between the linearizated spaces:

$$\begin{array}{l} \gamma_* : A[M] \longrightarrow A[N] \\ \quad f \longmapsto \gamma_* f : N \longrightarrow A \\ \qquad\qquad\qquad y \longmapsto (\gamma_* f)(y) := \sum_{x \in \gamma^{-1}(y)} f(x) \end{array}$$

It holds that $\gamma_* \left(\sum a_i x_i \right) = \sum a_i \gamma(x_i)$, so it says that the construction of $A[M]$ is functorial.

- "Motivating example":  and , $\frac{\operatorname{Ker} \partial_1}{\operatorname{Im} \partial_2}$ have different dimensions!

Definition: Let $n \in \mathbb{N}_0$. We'll call **n-simplex** to

$$\Delta^n := \{ (t_0 \dots t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \ \sum t_i = 1 \} \subset \mathbb{R}^{n+1}$$

it is the equation of a hyperplane in $\mathbb{R}^{n+1}$).

Δ^0 = point in $\mathbb{R}$; Δ^1 = segment in $\mathbb{R}^2$ (from $(1,0)$ to $(0,1)$) ; Δ^2 = piece of plane in $\mathbb{R}^3$.

Definition: Let $\alpha : \{0, \dots, m\} \longrightarrow \{0, \dots, n\}$. It induces a continuous map

$$\alpha_* : \Delta^m \longrightarrow \Delta^n \quad , \quad \alpha_*(t_0, \dots, t_m) = \left(\sum_{i \in \alpha^{-1}(0)} t_i \ , \dots , \sum_{i \in \alpha^{-1}(n)} t_i \right)$$

Note that $\alpha_*(e_i) = e_{\alpha(i)}$.

Definition: For $n \geq 0$ and $i = 0, \dots, n$ let $\delta_i : \{0, \dots, n-1\} \longrightarrow \{0, \dots, n\}$ the unique order preserving injection not hitting i, ie, $0 \longmapsto 0$, $1 \longmapsto 1$, $\dots$, $i-1 \longmapsto i-1$, $i \longmapsto i+1$, $\dots$, $n-1 \longmapsto n$.
If we write $\delta_i : \Delta^{n-1} \longrightarrow \Delta^n$ for $(\delta_i)_*$ it does

$$\delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, \overset{i)}{0}, t_i, \dots, t_{n-1})$$

Lemma: Let $n \geq 2$ and $0 \leq j < i \leq n$. It holds $\delta_i \circ \delta_j = \delta_j \circ \delta_{i-1}$.

Definition: Let X be a top space. A **singular n-simplex** is a continuous map $\sigma : \Delta^n \longrightarrow X$. We denote as $S(X)_n$ the set of all singular n-simplices.

0-simplex = point , 1-simplex = general edge  , ...

Definition: Let A be an abelian group and let X be a top space. The group of singular n-chains (with coefficients in A) is $C_n(X;A) := A[\mathcal{S}(X)_n]$

If $f: X \to X$ is continuous, it induces a morphism $\begin{array}{ccc} \mathcal{S}(X)_n & \longrightarrow & \mathcal{S}(X)_n \\ \sigma & \longmapsto & f\circ\sigma \end{array}$, and a group homomorphism $f_*: C_n(X;A) \to C_n(X;A)$. Taking "$*$" is indeed a functor.

Definition: Let X top-space and A abelian. The map $\mathcal{S}(X)_n \to \mathcal{S}(X)_{n-1}$, $\sigma \longmapsto \sigma\circ\delta_i$ induces (taking $*$) a group homomorphism $d_i: C_n(X;A) \to C_{n-1}(X;A)$; and the singular boundary operator is the group homomorphism

$$\partial_n := \sum_{i=0}^{n} (-1)^i d_i : C_n(X;A) \longrightarrow C_{n-1}(X;A)$$

Lemma: $\partial_{n-1}\circ\partial_n = 0$

Definition: A chain complex C is a sequence of abelian groups C_n, $n\geq 0$, together with group homomorphisms $\partial_n: C_n \to C_{n-1}$ s.t. $\partial_{n-1}\circ\partial_n = 0$.

The previous lemma shows that $C_n(X;A)$ form a chain complex.

Definition: Let C be a chain complex. We'll call

a) differential to ∂_n.

b) n-chains to the elements of C_n

c) n-cycles to the elements of $\operatorname{Ker}\partial_n$

d) n-boundaries to the elements of $\operatorname{Im}\partial_{n+1}$

Since $\partial_{n-1}\circ\partial_n = 0 \Rightarrow \operatorname{Im}\partial_{n+1} \subseteq \operatorname{Ker}\partial_n$,

<u>Definition</u>: Let C be a chain complex. The <u>n-th homology group</u> is the abelian group

$$H_n(C) := \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

- For $n=0$, we define $H_0(C) := \frac{C_0}{\operatorname{Im} \partial_1}$.

<u>Definition</u>: Let X be a top. space and A an abelian group. The <u>n-th singular homology group</u> of X with coefficients in A is

$$H_n(X;A) := H_n(C(X;A)).$$

<u>Example</u>: Let $X = *$ point space. It holds

$$H_0(*;A) = A \quad ; \quad H_n(*;A) = 0, \quad n \geq 1.$$

<u>Definition</u>: Let C, D be chain complexes. A sequence of group homomorphisms $g_n : C_n \to D_n$ is a <u>chain map</u>, and we will write $g : C \to D$, if the following diagram commutes:

$$\begin{tikzcd} C_n \arrow[r, "g_n"] \arrow[d, "\partial_n"'] & D_n \arrow[d, "\partial_n"] \\ C_{n-1} \arrow[r, "g_{n-1}"] & D_{n-1} \end{tikzcd} \quad , \quad \partial_n \circ g_n = g_{n-1} \circ \partial_n.$$

<u>Lemma</u>: A chain map $g : C \to D$ induces a group homomorphism

$$\begin{aligned} g_* : H_n(C) &\longrightarrow H_n(D) \\ [x] &\longmapsto [g_n(x)] \end{aligned}$$

Lemma: Let $f : X \to Y$ be a continuous map, and A ab. The induced group homomorphism

$$f_* : C_n(X;A) \to C_n(Y;A)$$

form a chain map $f_* \; C(X;A) \to C(Y;A)$

Corollary: Any continuous map $f: X \to Y$ induces group homomorphisms

$$f_* : H_n(X;A) \to H_n(Y;A) \quad , \; n \geq 1$$

Note that the induced maps on homology are functorial:

$$(\mathrm{Id}_X)_* = \mathrm{Id}_{H_n(X)} \quad ; \quad (f \circ g)_* = f_* \circ g_* \quad ,$$

e,

$$H_n(-, A) : \mathbf{Top} \longrightarrow \mathbf{AbGrp}$$

is a functor.

Definition: Let $(C^i)_{i \in I}$ a family of chain complexes. We can define a chain complex $\bigoplus_{i \in I} C^i$ by

$$\left(\bigoplus C^i\right)_n := \bigoplus C^i_n \quad ; \quad \partial_n (a_i)_{i \in I} := \left(\partial_n(a_i)\right)_{i \in I}$$

and the inclusions $i_{(C^i)_n} : C^i_n \hookrightarrow \bigoplus (C^i)_n$ form a chain map $i_{C^i} : C^i \longrightarrow \bigoplus C^i$.

Proposition: Given a chain complex D and a family of chain maps $f_i : C^i \to D$, there exists a unique chain map

$$f : \bigoplus_{i \in I} C^i \longrightarrow D$$

with $f_i = f \circ i_{C^i}$.

Proposition: There's an isomorphism

$$\bigoplus_{i \in I} H_n(C^i) \xrightarrow{\sim} H_n\left(\bigoplus_{i \in I} C^i\right)$$

whose composite with $i_{H_n(C^i)}$ is the map $H_n(C^i) \to H_n(\bigoplus C^i)$ induced by i_{C^i}.

II : SINGULAR HOMOLOGY IN DEGREES 0 AND 1

Lemma: Let $(X_j)_{j\in J}$ be the path components of X. The inclusions $X_j \hookrightarrow X$ induce an isomorphism

$$\bigoplus_{j\in J} H_n(X_j;A) \xrightarrow{\sim} H_n(X;A)$$

Corollary: Let Z be a set viewed as a discrete space (i.e, with the discrete topology). Then $H_n(Z;A)=0\ \forall n\geq 1$, and $H_0(Z;A)=\bigoplus_Z A$.

- Let $\pi_0(X)$ the set of path components and let $\pi_X : X \longrightarrow \pi_0(X)$, $x \longmapsto [x]$.
- Let $\tilde{x} : \Delta^0 \longrightarrow X$, $1 \longmapsto x$ be the singular 0-simplex with ~~value~~ x. Obviously there is a bijection $X \xrightarrow{\sim} S(X)_0$, $x \longmapsto \tilde{x}$ and therefore an isomorphism $A[X] \to C_0(X;A)$.

Set

$$\begin{tikzcd} A[X] \arrow[r, "\sim"] \arrow[rrr, bend right, "\phi_X"'] & A[S(X)_0] = C_0(X;A) \arrow[r] & \dfrac{C_0(X;A)}{\operatorname{Im}\partial_1} & = H_0(X;A) \end{tikzcd}$$

Lemma: The map ϕ_X factors as

$$\begin{tikzcd} A[X] \arrow[r, "\phi_X"] \arrow[d, "(\pi_X)_*"'] & H_0(X;A) \\ A[\pi_0(X)] \arrow[ur, "\gamma_X"'] & \end{tikzcd} \qquad \phi_X = \gamma_X \circ (\pi_X)_* .$$

- If $*$ is the one-point space, the unique map $X \longrightarrow *$ and the isomorphism $H_0(*;A)\simeq A$ induce an homomorphism $\varepsilon : H_0(X;A) \longrightarrow H_0(*;A) \xrightarrow{\sim} A$ (sometimes called augmentation)

Theorem: The homomorphism γ_X from before is an isomorphism, and if X is path-connected, the augmentation ε is also an isomorphism, $H_0(X;A)\simeq A$.

• For the rest of this chapter fix $A = \mathbb{Z}$ and set $H_n(X) = H_n(X;\mathbb{Z})$.

• Take the canonical maps $S(X)_n \longrightarrow C_n(X;\mathbb{Z})$, $\sigma \longmapsto 1\cdot\sigma$ and the homeomorphism $\iota : \Delta^1 \longrightarrow I$, $(1-t,t) \longmapsto t$.

We will again write $\tilde{x}$ for the singular 0-simplex with value x.

If $f: I \longrightarrow X$ is a loop in X ($f(0) = f(1) = x_0$), we have a 1-simplex $\Delta^1 \xrightarrow{f\circ\iota} X$, and $\partial(f\circ\iota) = 0$. (Watch out! Here $f\circ\iota \equiv 1\cdot(f\circ\iota) \in C_1(X;\mathbb{Z})$).

<u>Lemma</u>: the map

$$\phi : \pi_1(X;x_0) \longrightarrow H_1(X)$$
$$[f] \longmapsto [f\circ\iota]$$

is well-defined

homotopy class of loops

homology class

<u>Lemma</u>: Let $f,g: I \longrightarrow X$ be paths in X with $f(1) = g(0)$. The 1-chain

$$(g\circ\iota) + (f\circ\iota) - ((f*g)\circ\iota)$$

is a boundary.

<u>Corollary</u>: The map $\phi : \pi_1(X,x_0) \longrightarrow H_1(X)$ is a group homomorphism.

<u>Corollary</u>: Let $\omega \in S(X)_1$, and let $\overline{\omega}(t,1-t) := \omega(1-t,t)$ the "reverse 1-simplex". Then $\omega + \overline{\omega}$ is a boundary.

• Is ϕ an isomorphism? In general no, for 2 reasons: 1) $\pi_1(X,x_0)$ only "looks" at the path component of x_0, and 2) $H_1(X)$ is always abelian, but in general $\pi_1(X,x_0)$ is not. We are going to see that basically these are the only that exist.

Definition: Let G be a group. The abelianization of G is the abelian group $G^{ab} := \dfrac{G}{\langle ghg^{-1}h^{-1} \rangle_{g,h \in G}}$

Theorem (Universal property of the abelianized group): Every group homomorphism $\varphi : G \to A$, where A is abelian, factors in a unique way through G^{ab},

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & A \\ G^{ab} \arrow[ur, dashed, "\varphi^{ab}"'] & \end{tikzcd} \quad , \text{ i.e., } \quad \mathrm{Hom}_{gr}(G, A) = \mathrm{Hom}_{gr}(G^{ab}, A)$$

• Then we have a homomorphism $\phi^{ab} : \pi_1(X, x_0)^{ab} \longrightarrow H_1(X)$

Theorem: Let X be path-connected. Then

$$\phi^{ab} : \pi_1(X; x_0)^{ab} \xrightarrow{\sim} H_1(X)$$

is an isomorphism.

III : RELATIVE HOMOLOGY GROUPS

Recall:

$A \xrightarrow{f} B \to 0$ exact $\iff f$ surjective

$0 \to A \xrightarrow{f} B$ " $\iff f$ injective

$0 \to A \xrightarrow{f} B \to 0$ $\iff f$ isomorphism

Definition. A sequence of chain complexes and chain maps

$$0 \to C' \to C \to \overline{C} \to 0$$

is a <u>short exact sequence</u> if $0 \to C_n' \to C_n \to \overline{C}_n \to 0$ is exact $\forall n \geq 0$.

• If $i : C' \to C$ is an injective chain map, and $\overline{C}_n := C_n / C_n'$, then $0 \to C' \xrightarrow{i} C \xrightarrow{\pi} \overline{C} \to 0$ is a short exact sequence.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_n' \arrow[r, "i_n"] \arrow[d] & C_n \arrow[r, "\pi_n"] \arrow[d, "\partial_n"] & \overline{C}_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}' \arrow[r, "i_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "\pi_{n-1}"] \arrow[d] & \overline{C}_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Given $[\overline{x}] \in H_n(\overline{C})$, one can obtain an element $[x'] \in H_{n-1}(C')$ through the previous arrows:

$$\delta : H_n(\overline{C}) \longrightarrow H_{n-1}(C')$$
$$[\overline{x}] \longmapsto [x']$$

Lemma: This is a well-defined group homomorphism, called the <u>connecting map</u> or <u>homomorphism</u>

Theorem (Exactness of the LES): Let $0 \to C' \xrightarrow{i} C \xrightarrow{\pi} \bar{C} \to 0$ be a short exact sequence of chain complexes. Then the following sequence is exact:

$$\begin{tikzcd}
\arrow[r, hook] & H_n(C') \arrow[r, "i_*"] & H_n(C) \arrow[r, "\pi_*"] & H_n(\bar{C}) \arrow[dll, "\delta"] \\
& H_{n-1}(C') \arrow[r, "i_*"] & H_{n-1}(C) \arrow[r, "\pi_*"] & H_{n-1}(\bar{C}) \arrow[dll, "\delta"] \\
& \dots & & \dots \arrow[dll, "\delta"] \\
& H_0(C') \arrow[r, "i_*"] & H_0(C) \arrow[r, "\pi_*"] & H_0(\bar{C}) \arrow[dll] \\
& 0 & &
\end{tikzcd}$$

RELATIVE HOMOLOGY GROUPS

Definition: A pair of spaces (X, X') is a top space X together with a subspace $X' \subset X$.
A morphism of pairs $f : (X, X') \to (Y, Y')$ is a continuous map $f : X \to Y$ s.t. $f(X') \subset Y'$.

- Given (X, X'); we have a canonical injection $\mathcal{S}(X')_n \hookrightarrow \mathcal{S}(X)_n$, $\sigma \mapsto i \circ \sigma$, and therefore an injective morphism $C_n(X'; A) \hookrightarrow C_n(X; A)$, and a chain map $C(X'; A) \to C(X; A)$*

Definition: We will call relative chain complex to the quotient complex

$$C(X, X'; A) := \frac{C(X; A)}{C(X'; A)}$$

and relative homology groups to

$$H_n(X, X'; A) := H_n(C(X, X'; A))$$

- Note that $C_n(X, X'; A) \xrightarrow{\bar{\partial}_n} C_{n-1}(X, X'; A)$ is given by $\bar{\partial}_n([x]) := [\partial_n(x)]$.

*: Note that $C(X'; A) \to C(X; A)$ being a chain map means that
$$\begin{tikzcd}
C_n(X') \arrow[r, hook] \arrow[d, "\partial_n'"] \arrow[dr, phantom, "//"] & C_n(X) \arrow[d, "\partial_n"] \\
C_{n-1}(X') \arrow[r, hook] & C_{n-1}(X)
\end{tikzcd}$$
, i.e., $\partial_n|_{C_n(X')} = \partial_n'$

• Since we have the SES $0 \to C(X';A) \to C(X;A) \to C(X,X';A) \to 0$, we have by the Thm

<u>Corollary</u>: Let (X,X') be a pair of top spaces. Then the following sequence is exact.

$$\hookrightarrow H_n(X';A) \to H_n(X;A) \to H_n(X,X';A) \to$$
$$\hookrightarrow H_{n-1}(X';A) \to H_{n-1}(X;A) \to H_{n-1}(X,X';A) \to \cdots$$

<u>Corollary</u>: $H_n(X,X';A)=0 \ \forall n \iff i: X' \hookrightarrow X$ induces isomorph $H_n(X';A) \xrightarrow{\sim} H_n(X;A)$ $\forall n$

• A map of pairs $f:(X,X') \to (Y,Y')$ induces $f_*: C(X) \to C(Y)$ and $(f|_{X'})_*: C(X') \to C(Y')$. It is satisfied that $f_*|_{C(X')} = (f|_{X'})_*$.

Because of the previous thing, there's a well-defined chain map $C(X,X';A) \to C(Y,Y';A)$, with

$$\begin{aligned} C_n(X,X';A) &\to C_n(Y,Y';A) \\ [x] &\longmapsto [f_*(x)] \end{aligned}$$

<u>Lemma</u>: Let $f:(X,X') \to (Y,Y')$ be a map of pairs. Consider

$$\begin{tikzcd} \cdots \arrow[r, "\delta"] & H_n(X';A) \arrow[r] \arrow[d] & H_n(X;A) \arrow[r] \arrow[d] & H_n(X,X';A) \arrow[r, "\delta"] \arrow[d] & \cdots \\ \cdots \arrow[r, "\delta"] & H_n(Y';A) \arrow[r] & H_n(Y;A) \arrow[r] & H_n(Y,Y';A) \arrow[r, "\delta"] & \cdots \end{tikzcd}$$

(LES for (X,X'))

(LES for (Y,Y'))

Then if 2 out of 3 vertical maps are isomorphisms for all $n \geq 0$, then so is the third.

• The two main results so far are (to be proved soon):

<u>Theorem</u> (Excision): Let (X, X') be a pair of spaces and let $Y \subset X'$, with $\overline{Y} \subset \mathring{X}'$. Then the inclusion $(X-Y, X'-Y) \hookrightarrow (X, X')$ induces isomorphisms of relative homology groups:

$$H_n(X-Y, X'-Y; A) \xrightarrow{\sim} H_n(X, X'; A).$$

<u>Theorem</u> (Homotopy invariance of singular homology): If $f, f' : X \to Y$ are homotopic, then $f_* = f'_* : H_n(X; A) \to H_n(Y; A)$.

<u>Corollary</u>: Homotopic spaces have isomorphic homology groups, ie, homotopy equivalences induce isomorphisms on homology groups:

$$X \simeq Y \implies H_n(X; A) = H_n(Y; A).$$

<u>Corollary</u>: Let $f : (X, X') \to (Y, Y')$ be a pairs map. If $f : X \to Y$ and $f|_{X'} : X' \to Y'$ are homotopy equivalences (ie, $X \simeq Y$, $X' \simeq Y'$) then f induces isomorphisms

$$H_n(X, X'; A) = H_n(Y, Y'; A).$$

<u>Example</u>: $$H_n(S^m; A) = H_n(D^m, S^{m-1}; A) = \begin{cases} A \oplus A, & \text{if } m = n = 0 \\ A, & \text{if } m = n > 0 \\ 0, & \text{else.} \end{cases}$$

IV: HOMOTOPY INVARIANCE OF S. HOMOLOGY

∘ Recall our construction of singular homology:

$$\mathrm{Top} \xrightarrow[\mathcal{S}(-)]{\text{singular complex}} \text{"discrete data"} \xrightarrow[A[-]]{A\text{-lin.}} \mathrm{Ch} \xrightarrow[H(-)]{\text{homology}} \mathrm{Ab}$$

· We will denote as Δ the category whose objects are ordered sets $[n] = \{0, \dots, n\}$, $n \geq 0$, and where arrows are order preserving maps $\alpha : [m] \to [n]$. Recall

- $\delta_i : [n-1] \to [n]$ order preserving injection not hitting i
- $\sigma_i : [n] \to [n+1]$ " " surjection hitting i twice.

<u>Definition</u>: A <u>simplicial set</u> is a contravariant functor

$$K : \Delta \longrightarrow \mathrm{Set}$$
$$[n] \longmapsto K_n$$
$$(\alpha : [m] \to [n]) \longmapsto (\alpha^* : K_n \to K_m),$$

where the α^*'s are called <u>the structure maps</u>. I.o.w., a simplicial set is a sequence of sets K_n, $n \geq 0$ with a maps $\alpha^* : K_n \to K_m$ for every arrow in Δ (and $(\mathrm{Id}_{[m]})^* = \mathrm{Id}_{K_m}$, $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$)

· The singular n-simplexes $\mathcal{S}(X)_n$ form a simplicial set: for every $\alpha : [m] \to [n]$ we have

$$\alpha^* : \mathcal{S}(X)_n \longrightarrow \mathcal{S}(X)_m$$
$$\sigma \longmapsto \sigma \circ \alpha_* .$$

Definition: A morphism of simplicial sets $h: K \to L$ is a collection of maps $h_n : K_n \to L_n$ s.t.

$$\begin{tikzcd} K_n \arrow[r, "h_n"] \arrow[d, "\alpha^*"'] & L_n \arrow[d, "\alpha^*"] \\ K_m \arrow[r, "h_m"] & L_m \end{tikzcd}$$

commutes $\forall \alpha : [m] \to [n]$. Ie, a morphism of simplicial sets is a natural transformation of contravariant functors $\Delta \to Set$.

- Any continuous map $f: X \to Y$ induces a morphism of simplicial sets $f_* : S(X) \to S(Y)$, with $(f_*)_n : S(X)_n \to S(Y)_n$, $\sigma \mapsto f \circ \sigma$. Ie,

$$S : Top \to sSet$$

is a covariant functor from top. spaces to simplicial sets.

Definition: For $k \geq 0$, let $\underline{\Delta}^k$ be the simplicial set $(\Delta^k)_n := Hom_\Delta([n], [k])$ (order preserv.

- $\underline{\Delta}^0$ plays the role of the point, $\underline{\Delta}^1$ of the interval, ...

- Denote $j_i : [0] \to [1]$, $0 \mapsto i$, $i = 0,1$. They induce map $j_{i*} : \underline{\Delta}^0 \to \underline{\Delta}^1$, that are "the inclusions of end points".

$$(\alpha : [m] \to [0]) \mapsto ([m] \xrightarrow{\alpha} [0] \to [1])$$

Definition: For simplicial sets, K and L, their product $K \times L$ is a simplicial set with $(K \times L)_m = K_m \times L_m$ and structure maps $\alpha^* = \alpha^* \times \alpha^* : K_m \times L_m \to K_n \times L_n$.

- Since K_m and $K_m \times \{*\}$ is bij, there's a canonical isom. $K \xrightarrow{\simeq} K \times \underline{\Delta}^0$.

Definition: Let $h_0, h_1 : K \to L$ be morphisms of simplicial sets. A simplicial homotopy from h_0 to h_1 is a map of simplicial sets

$$H : K \times \underline{\Delta}^1 \to L$$

such that h_i equals

$$K \xrightarrow{\sim} K \times \underline{\Delta}^0 \xrightarrow{Id_K \times j_i} K \times \underline{\Delta}^1 \xrightarrow{H} L$$

Proposition: $f_0 \simeq f_1 : X \to Y$ are homotopic cont. maps $\Rightarrow (f_0)_* \simeq (f_1)_* : S(X) \to S(Y)$ are simplicial homotopic maps

In other words,

$$S : Top \to sSet$$

is a homotopy preserving functor.

Now we'd like to have the notion of homotopy in chain maps:

Definition: Let $f_0, f_1 : C \to D$ be chain maps. A chain homotopy from f_0 to f_1 is a sequence of groups homomorphism $P_n : C_n \to D_{n+1}$ s.t.

$$\begin{tikzcd}
C_{n+1} \arrow[r] \arrow[d, "\partial^C_{n+1}"'] & D_{n+1} \arrow[d, "\partial^D_{n+1}"] \\
C_n \arrow[r, "(f_1)_n - (f_0)_n"'] \arrow[ur, "P_n"] \arrow[d, "\partial^C_n"'] & D_n \arrow[d, "\partial^D_n"] \\
C_{n-1} \arrow[r] \arrow[ur, "P_{n-1}"] & D_{n-1}
\end{tikzcd}$$

$$\underbrace{\partial^D_{n+1} \circ P_n} + \underbrace{P_{n-1} \circ \partial^C_n} = \underbrace{(f_1)_n - (f_0)_n}$$

<u>Proposition</u>: $f_0 \simeq f_1 : C \to D$ homotopic as chain maps $\Rightarrow (f_0)_* = (f_1)_* : H_n(C) \to H_n(D)$

- The construction $\mathcal{S}(X) \xrightarrow{A[-]} C(X;A)$ generalizes to a functor

$$C(-;A) : sSet \to Ch \,,$$

by setting $C_n(K;A) = A[K_n]$; and $\delta_i : [n-1] \to [n]$ induces $\delta_i^* : K_n \to K_{n-1}$ and thus one $\delta_i^* : C_n(K;A) = A[K_n] \to A[K_{n-1}] = C_{n-1}(K;A)$, and define $\partial_n := \sum (-1)^i \delta_i^*$.

<u>Proposition</u>: $h_0 \simeq h_1 : K \to L$ homotopic morphisms of simplicial sets $\Rightarrow (h_0)_* \simeq (h_1)_* : C(K;A) \to C(L;A)$ are homotopic as chain maps.

In other words, the functor

$$C(-;A) : sSet \to Ch$$

preserves homotopy.

<u>Theorem</u> (Homotopy invariance of singular homology): $f_0 \simeq f_1 : X \to Y$ homotopic continuous maps $\Rightarrow (f_0)_* = (f_1)_* : H_n(X;A) \to H_n(Y;A)$.

V : THE MAPPING DEGREE

· A sometimes cleaner version of homology is:

<u>Definition</u>: Let X be a top. space. The <u>reduced homology</u> of X is

$$\widetilde{H}_n(X;A) := \begin{cases} \ker\left(H_0(X;A) \to H_0(*;A)\right) & , n=0 \\ H_n(X;A) & , n \geq 1 \end{cases},$$

Equivalently, $\widetilde{H}_n(X;A)$ is the homology of the chain complex (with $n \geq -1$)

$$\underset{n}{\to C_n(X;A)} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} \underset{1}{C_1(X;A)} \xrightarrow{\partial_1} \underset{0}{C_0(X;A)} \xrightarrow{\partial_0} \underset{-1}{A = C_0(*;A)}$$

where ∂_0 is the map induced by $X \to *$.

<u>Lemma</u>: Both def coincide. In general, $\boxed{H_0(X;A) = A \oplus \widetilde{H}_0(X;A)}$

· For a pair (X,X'), we set $\widetilde{H}_n(X,X';A) := H_n(X,X';A)$, and again we have SES with the (-1)-th term of chain complexes and LES of reduced homology groups.

· Reduced homology allows us to write cleaner homologies:

- $\widetilde{H}_n(*;A) = 0 \quad \forall n \geq 0$ (in general if X is contractible $\widetilde{H}_n(X;A) = 0 \ \forall n$)
- $\widetilde{H}_n(S^m;A) = \begin{cases} A & , m=n \geq 0 \\ 0 & , \text{else} \end{cases}$.

· In the rest of the chapter we will write $\widetilde{H}_n(X) \overset{\text{not}}{=} \widetilde{H}_n(X;\mathbb{Z})$.

· Recall that every group homomorphism $\mathbb{Z} \to \mathbb{Z}$ is $k\cdot$, $k \in \mathbb{Z}$. (k is determined by the image of 1).

Definition: Let $f: S^n \to S^n$ a continuous map. The mapping degree of f is the unique integer $\deg(f) \in \mathbb{Z}$ s.t.

$$f_*(a) = \deg(f) \cdot a \, ,$$

where $f_*: \widetilde{H}_n(S^n) = \mathbb{Z} \longrightarrow \mathbb{Z} = \widetilde{H}_n(S^n)$.

Proposition (Properties):

1) $\deg(g \circ f) = \deg(g) \cdot \deg(f)$

2) $f \simeq f' : S^n \to S^m$ homotopic $\Rightarrow \deg f = \deg f'$.

3) $f \simeq \mathrm{Id}_{S^n}$ (homotopic equivalence) $\Rightarrow \deg f = \pm 1$.

Proposition: Consider the reflexion $f_m : S^m \to S^m$, $(x_0, \dots, x_m) \longmapsto (-x_0, \dots, x_m)$. Then $\deg f_m = -1$.

Corollary: The reflexion $S^m \to S^m$, $(x_0 \dots x_m) \longmapsto (x_0, \dots, -x_i, \dots, x_m)$ has degree -1

Corollary: The antipodal map $-1 : S^m \to S^m$, $x \longmapsto -x$ has degree $(-1)^{m+1}$

Corollary: If m is even, the antipodal map $-1 : S^m \to S^m$ cannot be homotopic to Id.

Lemma: Let $f : S^m \to S^m$ be a continuous map, with m even. There exists an $x \in S^m$ such that either $f(x) = x$ or $f(x) = -x$.

Definition: A vector field on S^m is a continuous map $F : S^m \to \mathbb{R}^{m+1}$ s.t. $F(x) \perp x \ \forall x \in S^m$.

Theorem (Hairy ball): If m is even, every vector field on S^m vanishes at some point.

THE EXCISION THEOREM

Definition: Let $\mathcal{O} = (O_i)_{i\in I}$ be a cover of a top space X. We say that $\mathcal{O}$ is <u>admissible</u> if
$X = \bigcup_{i\in I} \mathring{O}_i$.

Definition: A singular n-simplex $\sigma : \Delta^n \to X$ is <u>$\mathcal{O}$-small</u> if $\sigma(\Delta^n) \subset O_i$ for some i.
We will denote as $\mathcal{S}_{\mathcal{O}}(X)_n$ the set of $\mathcal{O}$-small singular n-simplices.

- Since, if $\sigma \in \mathcal{S}_{\mathcal{O}}(X)_n$, and $\alpha : [m] \to [n]$, $\sigma \circ \alpha_* \in \mathcal{S}_{\mathcal{O}}(X)_m$, then $\mathcal{S}_{\mathcal{O}}(X)$ forms a sub-simplicial set of $\mathcal{S}(X)$.

Theorem (Small simplices): Let $\mathcal{O}$ be an admissible cover of a top space X. The inclusion of $\mathcal{S}_{\mathcal{O}}(X)$ in $\mathcal{S}(X)$ induces an isomorphism

$$H_n(C(\mathcal{S}_{\mathcal{O}}(X);A)) \xrightarrow{\sim} H_n(C(\mathcal{S}(X);A)) = H_n(X;A)$$

(To be proved)

Theorem (Excision): Let (X,X') be a pair of spaces and $Y \subset X' \subset X$, with $\overline{Y} \subset \mathring{X}'$. Then the inclusion $X - Y \hookrightarrow X$ induces isomorphisms of relative homology groups

$$H_n(X-Y, X'-Y; A) \xrightarrow{\sim} H_n(X,X';A)$$

VI : PROOF OF THE "SMALL SIMPLICIES" THEOREM

· The whole lecture is technical without useful results, only technical lemmas to prove the theorem.
It is skipped.

VII: CELL ATTACHMENTS

REVIEW & NOTATION

<u>Definition</u>: Let X be a top space. We will say that X is <u>quasi-compact</u> if every open cover of X admits a finite subcover; and we will say that X is <u>compact</u> if it is quasi-compact and Hausdorff.

<u>Proposition</u> (Properties):

1) U quasi-compact in X Hausdorff $\Rightarrow$ U closed

2) $f: X \to Y$ cont, X quasi-compact $\Rightarrow$ $f(X)$ quasi-compact

3) U closed in X (quasi-) compact $\Rightarrow$ U (quasi-) compact

4) $\underset{\text{quasi-compact}}{f: X} \longrightarrow \underset{\text{Hausdorff}}{Y}$ continuous + bijective $\Rightarrow$ f homeomorphism.

<u>Definition</u>: Consider the following top spaces and continuous maps

$$B \xleftarrow{i} A \xrightarrow{f} X.$$

The <u>pushout</u> of this diagram is the quotient space

$$X \cup_A B := \frac{X \amalg B}{\sim}$$

where $\sim$ identifies $i(a) \sim f(a)$ $\forall a \in A$.

• It comes with canonical maps (projections to the quotient) $X \to X \cup_A B$ and $B \to X \cup_A B$.

__Theorem__ (Universal Property of the pushout). For every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

there exists a unique map $X \cup_A B \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & X \cup_A B \arrow[dr, dashed, "\exists"] & \\ & & Y \end{tikzcd}$$

The first square is called a pushout when such map $X \cup_A B$ is homeomorphism.

__Example__: Let $(X, x_0), (Y, y_0)$ pointed spaces. The __wedge sum__ $X \vee Y := \dfrac{X \amalg Y}{x_0 = y_0}$ is the pushout $X \cup_* Y$, with $f : \{*\} \to X$, $* \mapsto x_0$; and $i : \{*\} \to Y$, $* \mapsto y_0$.

__Definition__. We say that __Y arises from X by attaching an n-cell along $f : \partial D_n \to X$__ if there is a pushout diagram

$$\begin{tikzcd} \partial D_n \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d] \\ D_n \arrow[r] & Y \end{tikzcd}$$

i.e. $Y \simeq X \cup_{\partial D_n} D_n$ = gluing X and D_n along ∂D_n and $\operatorname{Im} f$. We refer to f as the __attaching map__.

Definition: We say that Y arises by attaching n-cells indexed by a set J if there exists a continuous map $f : J \times \partial\mathbb{D}_n \longrightarrow X$ (where in J one consider the discrete top, ie, f cont $\iff f(j,-)$ cont $\forall j \in J$) and a pushout square

$$\begin{tikzcd} J \times \partial\mathbb{D}_n \arrow[r] \arrow[d, hook] & X \arrow[d] \\ J \times \mathbb{D}_n \arrow[r] & Y \end{tikzcd}$$

• $S_n \simeq S_{n-1} \cup_{\{a,b\} \times \partial\mathbb{D}_n} \{a,b\} \times \mathbb{D}_n$.

CELL ATTACHMENTS AND HOMOTOPIES

• Consider

$$\begin{tikzcd} A \times [0,1] \arrow[r, "f \times Id"] \arrow[d] & X \times [0,1] \arrow[d, "H"] \\ B \times [0,1] \arrow[r, "G"] & Z \end{tikzcd}$$

(not product diagram)

commutative with H and G homotopies. One wonders: do H and G induce a homotopy in the pushout? Ie, a homotopy $(X \cup_A B) \times [0,1] \longrightarrow Z$?

Definition: A top space is said to be *locally compact* if every point has a compact neighbourhoods basis ie, if $\forall x \in X$ $\forall U \in \mathcal{U}(x)$ $\exists V \subset U$ compact.

Proposition: Let $f : X \to Y$ be a quotient map, and let K be a loc. compact space. Then $f \times Id_K : X \times K \longrightarrow Y \times K$ is a quotient map.

Theorem: If

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

is a pushout square, and K is a loc. compact top space, then the resulting square

$$\begin{tikzcd} A \times K \arrow[r] \arrow[d] & X \times K \arrow[d] \\ B \times K \arrow[r] & Y \times K \end{tikzcd}$$

obtained by replacing all spaces with their product with K and all maps with their product with Id is again a product, ie, $X \times K \cup_{A \times K} B \times K \xrightarrow[\sim]{p} (X \cup_A B) \times K$.

In our particular case, given

$$\begin{tikzcd} J \times \partial \mathbb{D}_n \times [0,1) \arrow[r] \arrow[d, hook] & X \times [0,1) \arrow[d, "F"] \\ J \times \mathbb{D}_n \times [0,1) \arrow[r, "G"] & Z \end{tikzcd}$$

it says that p is homeomorphism in the following square, and therefore the homotopy H exists:

$$\begin{tikzcd} X \times [0,1) \cup_{J \times \partial \mathbb{D}_n \times [0,1)} J \times \mathbb{D}_n \times [0,1) \arrow[r] \arrow[d, "p", "\wr"'] & Z \\ (X \cup_{J \times \partial \mathbb{D}_n} J \times \mathbb{D}_n) \times [0,1) \arrow[ur, dashed, "H"'] & \end{tikzcd}$$

Theorem: Two homotopies $F : X \times [0,1) \to Z$ and $G : J \times \mathbb{D}_n \times [0,1) \to Z$ can be glued into ..
$(X \cup_{J \times \partial \mathbb{D}_n} J \times \mathbb{D}_n) \times [0,1) \to Z$

ANALYSIS OF CELL ATTACHMENTS

Lemma: Consider the quotient map $\pi : X \amalg J \times \mathbb{D}_n \to X \cup_{J \times \partial \mathbb{D}_n} J \times \mathbb{D}_n$, and let $\mathring{\mathbb{D}}_n$ the interior of the disk, an open n-cell.

1) The subspace $\pi(X) \subset X \cup_{J \times \partial \mathbb{D}_n} J \times \mathbb{D}_n$ is closed, and induces a homeomorphism $X \xrightarrow[\sim]{\pi} \pi(X)$

2) The subspace $\pi(J \times \mathring{\mathbb{D}}_n) \subset X \cup_{J \times \partial \mathbb{D}_n} J \times \mathbb{D}_n$ is open, and induces a homeomorphism $J \times \mathring{\mathbb{D}}_n \xrightarrow[\sim]{\pi} \pi(J \times \mathring{\mathbb{D}}_n)$

• 2) can be generalized to obtain more examples of open subsets of the product:

Definition: We say that $U \subseteq X \amalg J \times \mathbb{D}_n$ is <u>saturated</u> if $\pi^{-1}(\pi(U)) = U$.

Lemma: Let $U \subseteq X \amalg J \times \mathbb{D}_n$ be a <u>saturated</u> subset.

1) U open $\Rightarrow$ $\pi(U)$ open

2) U closed $\Rightarrow$ $\pi(U)$ closed

Corollary: For every $j \in J$, let $V_j \subseteq \{j\} \times \mathbb{D}_n$ be open with $\{j\} \times \partial\mathbb{D}_n \subseteq V_j$. Then $V := X \cup \left(\bigcup_{j \in J} \pi(V_j) \right)$ is open in $X \cup_{J \times \partial\mathbb{D}_n} J \times \mathbb{D}_n$.

• Observe that any point in $X \cup_{J\times\partial\mathbb{D}_n} J \times \mathbb{D}_n$ is either in $\pi(X)$ or $\pi(J \times \mathring{\mathbb{D}}_n)$ (the "identified" points lie on $\pi(X)$), so $\pi(X) \amalg \pi(J \times \mathring{\mathbb{D}}_n) \doteq X \cup_{J\times\partial\mathbb{D}_n} J \times \mathbb{D}_n$, and with the homeomorphism,

$$X \amalg J \times \mathring{\mathbb{D}}_n \simeq X \cup_{J\times\partial\mathbb{D}_n} J \times \mathbb{D}_n. \qquad (*)$$

To page 14

• What exactly are the characteristic maps? If $\pi : X_{n-1} \amalg J_n \times \mathbb{D}_n \longrightarrow X_n$, then

$\chi_j : \mathbb{D}_n \longrightarrow X_n$ is just $\chi_j := \pi|_{\{j\} \times \mathbb{D}_n}$. Now since by $(*)$

- $X_{n-1} \amalg J_n \times \mathring{\mathbb{D}}_n = X_n \Rightarrow X_n - X_{n-1} = J_n \times \mathring{\mathbb{D}}_n = \coprod_{J_n} \mathring{\mathbb{D}}_n$.
- π identifies $\partial\mathbb{D}_n$ with X_{n-1}

They say that

$$\begin{aligned} \chi_j : \mathbb{D}_n &\longrightarrow X_n \\ \mathring{\mathbb{D}}_n &\longmapsto X_n - X_{n-1} \\ \partial\mathbb{D}_n &\longmapsto X_{n-1} \end{aligned} \qquad \text{(by one inside of)}$$

VIII : CW-COMPLEXES

Some more results about cell attachments:

<u>Lemma</u>. X Hausdorff $\Rightarrow$ $X \cup_{J\times\partial\mathbb{D}_n} J\times\mathbb{D}_n$ Hausdorff

<u>Corollary</u>: X compact, J finite $\Rightarrow$ $X \cup_{J\times\partial\mathbb{D}_n} J\times\mathbb{D}_n$ compact.

<u>Lemma</u>: If $K \subseteq X \cup_{J\times\partial\mathbb{D}_n} J\times\mathbb{D}_n$ is compact, then K intersects with only finitely many open cells.

<u>Corollary</u>: Let X be compact.

$$J \text{ finite} \iff X \cup_{J\times\partial\mathbb{D}_n} J\times\mathbb{D}_n \text{ compact.}$$

CW - COMPLEXES

<u>Definition</u>: Let A be a top space. A <u>CW-complex</u> relative to A is a top space X together with a sequence of subspaces (called the <u>filtration</u>)

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X = \bigcup_{i\geq -1} X_i$$

where every X_n is the <u>n-skeleton</u> of (X,A); such that:

i) For every $n \geq 0$, the space X_n arises from X_{n-1} by attaching n-cells

ii) $\mathcal{O} \subset X$ open $\iff$ $\mathcal{O} \cap X_n$ open $\forall n \geq -1$. (it has the final topology given by the inclusions $X_n \hookrightarrow X$)

The first property means that for every $n \geq 0$ there's a pushout square

$$\begin{tikzcd} J_n \times \partial\mathbb{D}_n \arrow[r, "f"] \arrow[d, hook] & X_{n-1} \arrow[d] \\ J_n \times \mathbb{D}_n \arrow[r] & X_n \end{tikzcd}$$

• In particular, this says that $X_n - X_{n-1}$ is homeomorphic to a union of open n-cells, i.e., $X_n - X_{n-1} \simeq J_n \times \mathring{\mathbb{D}}_n$ (recall the final of lect. VIII). We call open n-cell to every path component of $X_n - X_{n-1}$.

Of course it is allowed that $J_n = \emptyset$ and $X_{n-1} = X_n$.

<u>Definition</u>: For every $j \in J_n$, the <u>characteristic map</u> $\chi_j : \mathbb{D}_n \to X_n$ is defined as the following composite:
$x \longmapsto [(j,x)]$

$$\chi_j : \mathbb{D}_n \simeq \{j\} \times \mathbb{D}_n \hookrightarrow J_n \times \mathbb{D}_n \xrightarrow{\pi \circ i} X_{n-1} \cup_{J_n \times \partial\mathbb{D}_n} J_n \times \mathbb{D}_n .$$

i: inclusion in $\amalg$

hence, it is continuous, and induces a homeomorphism between $\mathring{\mathbb{D}}_n$ and the open n-cell in X_n indexed with j.

<u>Let opl.</u> χ_j is not unique, precomposing with any homeomorphism $\mathbb{D}_n \to \mathbb{D}_n$ will give another one!

• $\chi_j = \pi|_{\{j\} \times \mathbb{D}_n}$, thus $\chi_j(\mathring{\mathbb{D}}_n) \simeq \mathring{\mathbb{D}}_n$.

<u>Corollary</u>: Let (X,A) a CW-complex, and Y a top space.

$$f : X \to Y \text{ cont} \implies f|_{X_n} : X_n \to Y \text{ cont } \forall n \geq -1.$$

<u>Definition</u>: Let (X,A) a CW-complex. We will say that it is...

a) <u>absolute</u> if $A = \emptyset$.

b) <u>finite dimensional</u> if $X = X_n$ for some $n \geq 0$.

c) <u>finite</u> if it is finite dim and $\# J_n < \infty$ $\forall n$.

• Is a CW-complex be finite... how, if the J_n's are <u>not</u> part of the data? Well,... the pushout shows that there's a bijection $J_n \to \pi_0(X_n - X_{n-1})$, hence one can always know $\# J_n$.

~~• (Relation with Sandro's definition). Sandro's definition can be recovered from this one, in particular, the one there refers to a absolute, finite~~

<u>Examples</u>: 1) $S_n = \mathbb{D}_0 \cup_{\partial\mathbb{D}_n} \mathbb{D}_n$, with attaching map $\partial\mathbb{D}_n \to \mathbb{D}_0$ (the unique); thus S_n can be viewed as a CW-complex with 1 0-cell and 1 n-cell.

2) $\mathbb{P}_n(\mathbb{R})$ is a CW-complex with one i-cell $\forall i = 0, \dots, n$.

Definition: Let X be a absolute, finite CW-complex, and let c_i the number of i-cells. The Euler characteristic of X is the alternated sum

$$\chi(X) := c_0 - c_1 + c_2 - c_3 + \dots + (-1)^n c_n \qquad , X = X_n .$$

Theorem (Topological invariance of the characteristic): Every CW-complex decomposition of a topological space has the same Euler characteristic.

Theorem (Euler): For any convex polyhedron with V vertices, E edges and F faces it holds

$$\boxed{V - E + F = 2.}$$

COMPACTNESS AND CW-COMPLEXES

Lemma: Let (X,A) be a finite CW-complex, with A compact. Then X is compact.

Lemma: Let (X,A) be a CW-complex, with A Hausdorff. Then X is Hausdorff.

Corollary: Every finite, absolute CW-complex is compact (quasi-compact + Hausdorff).

Definition: Let (X,A) be a relative CW-complex. A relative CW-complex (Y,A) is a subcomplex of (X,A) if $Y \subseteq X$ such that $Y_n = X_n \cap A$ and $Y_n - Y_{n-1}$ is union of some of the open n-cells in $X_n - X_{n-1}$ $\forall n \geq 0$.

Lemma: Let (X,A) be a relative CW-complex, and let $K \subset X$ be a compact subset s.t. $K \subseteq X_n$ for some n. Then K is contained in a finite subcomplex of (X,A)

Corollary: The closure of every n-cell in (X,A) is contained in a finite subcomplex.

Theorem: Let (X,A) be a relative CW-complex and let $K \subset X$ be compact. Then K is also contained in a finite subcomplex of (X,A) (no $K \subset X_n$ needed!)

• (Relation with Sancho's definition): Last year's Sancho definition of CW-complex can be recovered from this one: he refered, with this terminology, to a absolute, finite CW-complex. The Hausdorff and quasi-compact properties in that def are automatic by the Corollary. Condition ii) now says nothing bc $X = X_n$ for some n; the interesting is the i):

- $X_n - X_{n-1} \simeq J_n \times \mathring{\mathbb{D}}_n \equiv \pi(J_n \times \mathbb{D}_n)$, ie, $X_n - X_{n-1} = \coprod_{j \in J_n} Z_{n_j}$, with $Z_{n_j} \simeq \mathring{\mathbb{D}}_n$.

- Such homeomorphism is given by $\chi_j |_{\mathring{\mathbb{D}}_n} : \mathring{\mathbb{D}}_n \xrightarrow{\sim} Z_{n_j}$; thus it extends (well, it comes from) the map $\chi_j : \mathbb{D}_n \to X_n$, but restricting the target to $\chi_j(\mathbb{D}_n) = \chi_j(\overline{\mathring{\mathbb{D}}_n}) \overset{\text{ex.8.1}}{=\!=} \overline{Z_{n_j}}$.

• Why the letters "CW"?

- Closure-finiteness: The closure of every cell meets only finitely many other cells.

- Weak topology: $V \subset X$ closed $\Leftrightarrow$ $V \cap \overline{Z_{n_j}}$ closed $\forall\, n_j$.

J.H.C. Whitehead gave the original definition with these two properties playing a more central role.

IX: CELLULAR HOMOLOGY

- It's time to mix both topics: CW complexes + homology:

<u>Corollary</u>: Let X be an absolute CW-complex, A an abelian group and take $c \in C_n(X;A)$. Then there exists $p \geq 0$ such that $c \in \operatorname{Im}\left(C_n(X_p;A) \xrightarrow{i_*} C_n(X;A)\right)$.

- Recall the les of a triple from the appendix.

<u>Definition</u>: Let X be an absolute CW-complex and let A be an abelian group. The <u>cellular chain complex</u> of X with coefficients in A is the chain complex $\tilde{C}(X;A)$ that in n-th degree is

$$\tilde{C}_n(X;A) := H_n(X_n, X_{n-1};A)$$

and its differential is $\tilde{\partial}_n : \tilde{C}_n(X;A) = H_n(X_n, X_{n-1};A) \longrightarrow H_{n-1}(X_{n-1}, X_{n-2};A) = \tilde{C}_{n-1}(X;A)$ the connecting homomorphism from the les of the triple $X_{n-2} \subset X_{n-1} \subset X_n$.

- Let X an absolute CW-complex, and choose char. maps $\chi_j : \mathbb{D}_n \longrightarrow X_n$. There's a pushout diagram

$$\begin{tikzcd}
J_n \times \partial\mathbb{D}_n \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
J_n \times \mathbb{D}_n \arrow[r, "\amalg \chi_j"] & X_n
\end{tikzcd}$$

and $\chi_j : (\mathbb{D}_n, \partial\mathbb{D}_n) \to (X_n, X_{n-1})$ can be viewed as a map of pairs (because $\chi_j = \pi|_{\{j\} \times \mathbb{D}_n}$ and π identifies $\partial\mathbb{D}_n$ with X_{n-1} in the pushout).

<u>Theorem</u>: The characteristic maps induce isomorphisms

$$\bigoplus_{J_n} H_n(\mathbb{D}_n, \partial\mathbb{D}_n;A) \cong H_n(X_n, X_{n-1};A)$$

<u>Corollary</u>: For X an absolute CW-complex, $\widetilde{C}_n(X;A) \simeq \bigoplus_{I_n} A = A[I_n]$.

• Let's head to prove the big thm from here which will say "$H_n(X;A) \simeq H_n(\widetilde{C}(X;A))$", as has as corollary the ~~theorem~~ f invariance of the Euler char.

• We will fix X an absolute CW-complex.

<u>Lemma</u>: $n > m \geq -1$, and $k > n$ or $k \leq m$ $\Rightarrow H_k(X_n, X_m; A) = 0$.

<u>Corollary</u>: $H_n(X_{n+1};A) \simeq H_n(X_{n+2};A) \simeq H_n(X_{n+3};A) \simeq \dots$

<u>Proposition</u>: $H_n(X_{n+1};A) \simeq H_n(X;A)$.

<u>Lemma</u>: $H_n(X_{n+1};A) \simeq H_n(X_{n+1}, X_{-1};A) \simeq H_n(X_{n+1}, X_0;A) \simeq \dots$

<u>Corollary</u>: $H_n(X;A) \simeq H_n(X_{n+1};A) \simeq H_n(X_{n+1}, X_{n+2};A)$

* <u>Theorem</u>: Let X be an absolute CW-complex, and A an abelian group. There is an isomorphism

$$H_n(X;A) \simeq H_n(\widetilde{C}(X;A))$$

identifying the homology groups of the cellular chain complex with the singular homology groups of X.

<u>Corollary</u>: Let X be a finite CW-complex and k a field. Then the homology groups $H_n(X;k)$ are finite-dimensional v.s, $H_n(X;k)$ is only non-trivial for finitely many and

$$\chi(X) = \sum_{n \geq 0} (-1)^n \dim_k H_n(X;k)$$

• <u>Theorem</u> (Topological invariance of the Euler characteristic): Homotopic spaces have the same Euler characteristic.

Example : $\mathbb{P}_n(\mathbb{C}) := \dfrac{\mathbb{C}^{n+1} - 0}{\sim}$. There's a push-out diagram

$$\begin{tikzcd} \partial \mathbb{D}_{2n} \arrow[r, "\chi_{2n}|\partial \mathbb{D}_{2n}"] \arrow[d, hook] & \mathbb{P}_{n-1}(\mathbb{C}) \arrow[d] \\ \mathbb{D}_{2n} \arrow[r, "\chi_{2n}"] & \mathbb{P}_n(\mathbb{C}) \end{tikzcd}$$

ie, $\mathbb{P}_n(\mathbb{C})$ arises from $\mathbb{P}_{n-1}(\mathbb{C})$ by attaching a $2n$-cell.

- $H_n(\mathbb{P}_k(\mathbb{C}); A) = \begin{cases} A, & n \text{ even } \& \ 0 \le n \le 2k \\ 0, & \text{else} \end{cases}$.

- Note : The corollary from before "$*$Theorem" says that for a finite CW-complex (of dim s),

$\underline{H_p(X;A)} \cong H_p(X_{s+1}, X_{s+2}; A) = H_p(X_s, X_s; A) = \underline{0}$. for $\underline{p > s}$.

X : COMPUTATIONS IN CELLULAR HOMOLOGY

• We fix $A = \mathbb{Z}$, characteristic maps $\chi_\alpha : \mathbb{D}_n \to X_n$ for any n-cell $\alpha \in J_n$. Moreover, choose $1_n \in \mathbb{Z} = H_n(\mathbb{D}_n, \partial \mathbb{D}_n)$. Since $\bigoplus_{J_n} H_n(\mathbb{D}_n, \partial\mathbb{D}_n) = H_n(X_n, X_{n-1})$, we have

<u>Lemma</u>: The elements $e^n_\alpha := (\chi_\alpha)_* 1_n \in H_n(X_n, X_{n-1})$ form a $\mathbb{Z}$-basis of $H_n(X_n, X_{n-1})$

• Therefore, to determine $\widetilde{\partial}_n : \widetilde{C}_n(X) \to \widetilde{C}_{n-1}(X)$, it is enough to compute the coefficients $d_{\alpha\beta}$

$$\widetilde{\partial}_n(e^n_\alpha) = \sum_{\beta \in J_{n-1}} d_{\alpha\beta}\, e^{n-1}_\beta$$

<u>Aim</u>: Compute $d_{\alpha\beta}$ using the topological structure of the CW-complex X.

<u>Lemma</u>: $\widetilde{\partial}_n(e^n_\alpha)$ = Image of $\widetilde{\delta}_n(1_n)$ under the lower composite:

$$\begin{tikzcd}
 & H_n(X_n, X_{n-1}) \arrow[rr, bend left, "\widetilde{\partial}_n"] & & \\
H_n(\mathbb{D}_n, \partial\mathbb{D}_n) \arrow[ur, bend left, "(\chi_\alpha)_*"] \arrow[dr] & & & H_{n-1}(X_{n-1}, X_{n-2}) = \widetilde{C}_{n-1}(X) \\
 & \widetilde{H}_n(\partial\mathbb{D}_n) \arrow[r] & \widetilde{H}_{n-1}(X_{n-1}) \arrow[ur] &
\end{tikzcd}$$

... see the table ...

* <u>Theorem</u>: The coefficient $d_{\alpha\beta}$ in $\widetilde{\partial}_n(e^n_\alpha) = \sum_{\beta\in J_{n-1}} d_{\alpha\beta}\, e^{n-1}_\beta$ is given by the mapping degree of the following composite,

$$\partial\mathbb{D}_n \xrightarrow{\chi_\alpha|\partial\mathbb{D}_n} X_{n-1} \xrightarrow{q} X_{n-1}/X_{n-2} \xrightarrow{q_\beta} \mathbb{D}_{n-1}/\partial\mathbb{D}_{n-1} \xrightarrow[\sim]{h_{n-1}} \partial\mathbb{D}_n$$

where q_β collapses all $(n-1)$-cells except the one indexed with β, i.e.,

$$\boxed{d_{\alpha\beta} = \deg\left(h_{n-1} \circ q_\beta \circ q \circ \chi_{\alpha|\partial D_n}\right)}$$

ok, to compute the coefficients that determine the differential we do as follows: we first take the attaching map $\partial D_n \to X_{n-1}$ of the n-cell α, then collapse the $(n-2)$-skeleton X_{n-2}, then collapse all $(n-1)$-cells except β, and lastly identify the resulting space $D_{n-1}/\partial D_{n-1}$ with ∂D_n. The degree of this map gives you the coefficient $d_{\alpha\beta}$.

- $H_n(T^2;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0,2 \\ \mathbb{Z}\oplus\mathbb{Z}, & n=1 \\ 0, & n\geq 3 \end{cases}$.

<u>Lemma</u>: Let $f: S_n \to S_n$ and take $v \in S_n$: $f^{-1}(v) = \{u_1 \dots u_m\}$ finite. Take $U_1 \dots U_m$ neighborhoods of u_i on S_n, and V nbhd of v : $f(U_i) \subset V$ $\forall i$. There are isos

$$\mathbb{Z} \simeq H_n(S_n;\mathbb{Z}) \simeq H_n(U_i, U_i - u_i;\mathbb{Z}) \quad ; \quad \mathbb{Z} \simeq H_n(S_n;\mathbb{Z}) \simeq H_n(V, V-v;\mathbb{Z})$$

<u>Definition</u>: Let $f: S_n \to S_n$ in the above definition. The <u>local degree</u> of f at u_i is the integer $\deg f_{|u_i}$ such that $f_* : H_n(U_i, U_i - u_i;\mathbb{Z}) = \mathbb{Z} \to \mathbb{Z} = H_n(V, V-v;\mathbb{Z})$ is the morphism "multiply by $\deg f_{|u_i}$".

<u>Proposition</u> $\deg f = \sum_{i=1}^{m} \deg f_{|u_i}$.

- $H_n(\mathbb{P}_k(\mathbb{R});\mathbb{Z}) = \begin{cases} \mathbb{Z}, & (n=0) \text{ or } (n=k \ \& \ k \text{ odd}) \\ \mathbb{Z}/2\mathbb{Z}, & (n \text{ odd } \& \ 0<n<k) \\ 0, & \text{else.} \end{cases}$

XI : THE HOMOTOPY EXTENSION PROPERTY

<u>Definition</u>: We say that a pair of spaces (X,A) has the <u>homotopy extension property</u> (HEP) if the following property holds:

Given a top space Z, a cont map $f: X \to Z$, and a homotopy $F: A\times[0,1] \to Z$ s.t. $F(-,0) = f_{|A}$, $\exists$ a homotopy $H: X\times[0,1] \to Z$: $H(-,0) = f$ and $H_{|A\times[0,1]} = F$.

In words, (X,A) has the HEP if given a space Z, a cont map $f: X\to Z$ and a homotopy $F: A\times[0,1]\to Z$ starting in the restriction of f to A, then exist an <u>extension</u> of F to a homotopy defined on $X\times[0,1]$ starting from f.

<u>Lemma</u>: (X,A) has HEP $\iff$ every cont. map $X \cup_A (A\times[0,1]) \to Z$ extends to $X\times[0,1] \to Z$

$$\begin{tikzcd}
A \arrow[r, "Id\times i_0"] \arrow[d, hook, "i"'] & A\times[0,1] \arrow[d] \arrow[ddr, bend left, "i\times Id"] \arrow[dddr, bend left=60] & \\
X \arrow[r] \arrow[ddrr, bend right=60] \arrow[drr, bend right, "Id\times i_0"'] & X\cup_A(A\times[0,1]) \arrow[dr, dashed, "\exists"] \arrow[ddr, dashed] & \\
 & & X\times[0,1] \arrow[d, dashed, "¿\exists?"] \\
 & & Z
\end{tikzcd}$$

<u>Lemma</u>: Let (Y,B) a pair of spaces.

B retract of Y $\iff$ every continuous map $B\to Z$ extends to a cont. map $Y\to Z$

Lemma: Let $U, V \subset X$ closed. The inclusions induce a homeomorphism

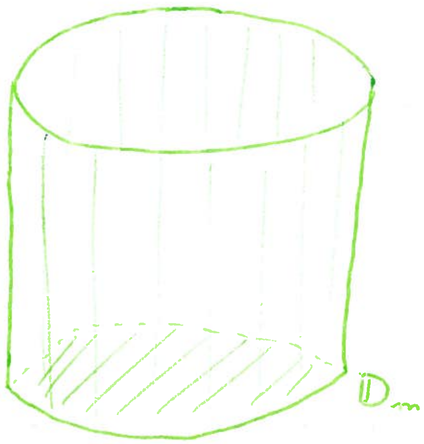
$$U \cup_{U \cap V} V \xrightarrow{\simeq} U \cup V.$$

Corollary: $A \subset X$ closed, then the canonical map $X \cup_A (A \times [0,1]) \longrightarrow X \times \{0\} \cup A \times [0,1]$ is a homeomorphism.

Corollary: Let $A \subset X$ closed.

(X, A) has the HEP $\iff$ $X \times \{0\} \cup A \times [0,1]$ is a retract of $X \times [0,1]$

Proposition: The pair $(\mathbb{D}_n, \partial \mathbb{D}_n)$ has the HEP, $n \geq 0$.



retract of the solid cylinder.

HEP FOR CW-COMPLEXES

Proposition: If X arises from X' by attaching n-cells, the pair (X, X') has the HEP.

Theorem: Every relative CW-complex (X, A) has the HEP.

Definition: Let $f: (X, A) \to (Y, B)$ a map of relative CW-complexes (ie, $f(A) \subset B$). We say that f is <u>cellular</u> if $f(X_n) \subset Y_n$ $\forall n \geq -1$.

Lemma: Let $f:(X,A)\to(Y,B)$ a map of relative CW-complexes with $f(X_i)\subset Y_i$ $\forall i\leq m-1$.
Then $f|X_m$ is homotopic rel. X_{m-1} to a map with image in Y_m.

Corollary: Let $f:(X,A)\to(Y,B)$ a map of relative CW-complexes. Then there exists a sequence of continuous maps $\left(f_m:(X,A)\to(Y,B)\right)_{m\geq -1}$ and homotopies $\left(H_m: X\times[0,1]\to Y\right)_{m\geq 0}$ such that

1) $f_{-1}=f$

2) $f_m(X_i)\subset Y_i$ $\forall i\leq m$

3) H_m is a homotopy from f_{m-1} to f_m rel. X_{m-1}.

* **Theorem** (Cellular Aproximation): Every map of relative CW-complexes $f:(X,A)\to(Y,B)$ is homotopic rel. A to a cellular map. ie, there exists a homotopy $H: X\times[0,1]\to Y$ with ! $H(-,0)=f$ and $H(a,t)=f(a)$ $\forall t$ such that $H(-,1)$ is cellular.

Example: Every continuous map $S^m\to S^n$, $m<n$, is homotopic to the constant map.

XII : CELLULAR APPROXIMATION

- Most of the lecture is focused on technical lemmas to prove the Aprox. Cellular Thm. We skip that part

PRODUCTS OF CW-COMPLEXES

- Let X and X' CW-complexes with sets of n-cells J_n and J_n'. Set $\hat{X} = \coprod_{n\geq 0} J_n \times \mathbb{D}_n$ and $\hat{X}' = \coprod_{n\geq 0} J_n' \times \mathbb{D}_n$. The respectives char maps induces $q : \hat{X} \to X$ and $q' : \hat{X}' \to X'$, which are quotient maps. Their product $q \times q' : \hat{X} \times \hat{X}' \to X \times X'$ is a continuous map, but in general not a quotient map. Why do we want a quotient map? It will ensure that $X \times X'$ is another CW complex.

* In general, the product of two CW-complexes is not a CW-complex

- Consider $X \hat{\times} X'$ the set $X \times X'$ equipped with the quotient topology given by $q \times q'$.

Theorem: The space $X \hat{\times} X'$ inherits a CW-structure from X and X' with set of n-cells $\hat{J}_n = \bigcup_{p+q=n} J_p \times J_q$ and n-squeleton $(X \hat{\times} X')_n = \bigcup_{p+q=n} X_p \times X_q' \subset X \hat{\times} X'$.

*Theorem: The product of two CW-complexes, where one of them is finite, inherits a CW-structure.

- Eg: $[0,1]$ can be viewed as a CW complex with 2 0-cells & 1 1-cell. Therefore, for every CW-complex X, $X \times [0,1]$ has a CW-structure. Every n-cell of X gives rise to two n-cells & 1 $(n+1)$-cells of $X \times [0,1]$.

XIII : HIGHER HOMOTOPY GROUPS

• Denote $[X, Y] = \{$ homotopy classes of cont. maps $X \to Y\} = \dfrac{\mathcal{C}(X, Y)}{\text{homotopy}}$

• Denote $[(X, x_0), (Y, y_0)]_* = \{$ basepoint preserving homotopy classes of pointed space maps $(X, x_0) \to (Y, y_0)\}$

$$= \frac{\mathcal{C}((X, x_0), (Y, y_0))}{\text{basepoint preserving homotopy}}$$

<u>Definition</u> : Let X be a top space with basepoint $x_0 \in X$. The <u>n-th homotopy group</u> of X is

$$\pi_n(X, x_0) := [(S_n, s_0), (X, x_0)]_*$$

where the choice of $s_0 \in S_n$ is arbitrary.

• Unraveling this definition, one finds:

- $\pi_0(X, x_0)$ is the set of path component
- $\pi_1(X, x_0)$ is the fundamental group.

<u>Example</u> : By the example after the Cell. Approx. thm, we already know that $\pi_n(S_m, s_0) = 0$ for $n < m$.

• In contrast, $\pi_n(S_m, s_0)$ is very weird for $n > m$. "Good luck".

• A basepoint preserving cont. map $f : (X, x_0) \to (Y, y_0)$ induces

$$f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$$
$$[\sigma] \longmapsto [f \circ \sigma]$$

- We can consider basepoint preserving maps of pairs eg $(\mathbb{D}_n, S_{n-1} = \partial\mathbb{D}_n, s_0) \xrightarrow{\alpha} (X, A, x_0)$, meaning $\alpha : (\mathbb{D}_n, S_{n-1}) \to (X, A)$ map of pairs with $\alpha(s_0) = x_0$.

<u>Definition</u>: The <u>relative n-th homotopy group</u> of a pointed pair (X, A, x_0) is

$$\pi_n(X, A, x_0) = \frac{\mathcal{C}((\mathbb{D}_n, S_{n-1}, s_0), (X, A, x_0))}{\text{basepoint preserving homotopy rel. } S_{n-1}}$$

(i.e, $H : \mathbb{D}_n \times [0,1] \to X$ satisfies $H(S_{n-1} \times [0,1]) \subset A$ & $H(s_0, -) = x_0$.

- Do these higher homotopy groups have a group structure? How? I cannot concentrate!
- By shake of simplicity, observe the bijection $\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$ (note $I^n = [0,1]^n \simeq \mathbb{D}_n$), because $I^n / \partial I^n \simeq S_n$ $(I^n \simeq \mathbb{D}_n)$
- Consider $\alpha, \beta : (I^n, \partial I^n) \to (X, x_0)$. For $i = 1 \dots n$ define

$$(\alpha +_i \beta)(t_1 \dots t_n) := \begin{cases} \alpha(t_1 \dots 2t_i \dots t_n) & , \quad t_i \in [0, \tfrac{1}{2}] \\ \beta(t_1 \dots 2t_i - 1 \dots t_n) & , \quad t_i \in [\tfrac{1}{2}, 1] \end{cases}$$

and analogously $[\alpha] +_i [\beta] := [\alpha +_i \beta]$. One checks that it is well-defined.

- But now we have a lot of operations!!

<u>Lemma</u>: Let M be a set with two operations $*$ and $\circ$ that have the same unit $1 \in M$ and

$$(a \circ b) * (c \circ d) = (a * c) \circ (b * d) \qquad \forall a, b, c, d \in M$$

Then both group structures coincide, and they are abelian.

Corollary: For $n \geq 2$, the group structures $+_i$, $i = 1 \dots n$, on $\pi_n(X, x_0)$ agree and they are abelian, so $(\pi_n(X, x_0), +_i)$ are abelian groups (for any i)

- How to define a group structure on $\pi_n(X, A; x_0)$? Consider $I^n \cong D_n$, $I^{n-1} \equiv \{t_n = 0\}$ in the I^n (one face) & J^{n-1} = union of rest of faces. We have $I^{n-1} \cup J^{n-1} = \partial I^n$ and $I^{n-1} \cap J^{n-1} = \partial I^{n-1}$. So we have homeomorphisms $I^n / J^{n-1} \simeq D_n$ and $\partial I^n / J^{n-1} \simeq S_{n-1}$, and therefore a bijection

$$\pi_n(X, A; x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A; x_0)]$$

- For $n = 2$, $+_1 \dots +_{n-1}$ define group structures on $\pi_n(X, A; x_0)$. But they may be different!!
- For $n \geq 3$, all of them are the same, and $\pi_n(X, A; x_0)$ is abelian.

LES OF HOMOTOPY GROUPS

Lemma: There is a group isomorphism $\pi_n(X, x_0) = \pi_n(X, \{x_0\}, x_0)$ for $n \geq 2$ (for $n=1$ is it just a bijection)

Theorem (Long exact sequence of homotopy groups): there is a les

$$\pi_n(A, x_0) \xrightarrow{i} \pi_n(X, x_0) \xrightarrow{j} \pi_n(X, A, x_0) \xrightarrow{p} \pi_{n-1}(A, x_0) \longrightarrow \pi_{n-1}(X, x_0) \longrightarrow \pi_{n-1}(X, A; x_0) \longrightarrow \dots$$

where

- i sends $[\alpha : (S_n, s_0) \to (A, x_0)]$ to $[i \circ \alpha : (S_n, s_0) \to (A, x_0) \hookrightarrow (X, x_0)]$
- j is the composite $\pi_n(X, x_0) \xrightarrow{\simeq} \pi_n(X, \{x_0\}, x_0) \longrightarrow \pi_n(X, A, x_0)$, $[\alpha] \longmapsto [\alpha]$ (view a map to $\{x_0\}$ as a map to A)

$\rightsquigarrow p$ sends $[\alpha : (\mathbb{D}_n, S_{n-1}, s_0) \to (X, A, x_0)]$ to $[\alpha_{|\partial \mathbb{D}_n = S_{n-1}} : (S_{n-1}, s_0) \to (A, x_0)]$.

• Note that for $n \leq 2$ the maps may not be groups homomorphisms. Then the exactness means

<u>Definition</u>: Let (A_i, a_i), $i = 1,2,3$ pointed sets. We say that a sequence of basepoint preserving maps

$$(A_1, a_1) \xrightarrow{f_1} (A_2, a_2) \xrightarrow{f_2} (A_3, a_3)$$

is <u>exact</u> if $f_1(A_1) = f_2^{-1}(a_3)$.

◦ How does the choice of x_0 depend for $\pi_n(X, x_0)$?

<u>Proposition</u>: Let X be a top space and let $n \geq 1$ an integer. Every path $\sigma : [0,1] \to X$ induces a group isomorphism

$$\sigma_* : \pi_n(X, \sigma(1)) \xrightarrow{\sim} \pi_n(X, \sigma(0))$$

such that

1) $\sigma \simeq \sigma'$ rel. $\{0,1\}$, $\Rightarrow \sigma_* = \sigma'_*$.

2) $\sigma = \text{const} \Rightarrow \sigma_* = \text{Id}$

3) σ and σ' composables, then $(\sigma \sigma')_* = \sigma_* \circ \sigma'_*$.

4) If $f : X \to Y$ is continuous, $f_* \circ \sigma_* = (f \circ \sigma)_* \circ f_*$, ie, the diagram

$$\begin{tikzcd}
\pi_n(X, \sigma(1)) \arrow[r, "f_*"] \arrow[d, "\sigma_*"'] & \pi_n(Y, f(\sigma(1))) \arrow[d, "(f \circ \sigma)_*"] \\
\pi_n(X, \sigma(0)) \arrow[r, "f_*"] & \pi_n(Y, f(\sigma(0)))
\end{tikzcd}$$

commutes.

XIV : THE WHITEHEAD THEOREM

Corollary: Let $f: X \to Y$, $x_0 \in X$. If $f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is bijective (or surjective), then for every x_1 in the same path-component of x_0, $f_*: \pi_n(X, x_1) \to \pi_n(Y, f(x_1))$ is also bijective (surjective).

Theorem: Homotopic maps induce "the same" maps in n-th homotopy.

Precisely, if $f_0, f_1: X \to Y$ are homotopic with homotopy $H: X \times [0,1] \to Y$, we fix $x \in X$, $y_0 = f_0(x)$, $y_1 = f_1(x)$, and we set $\gamma(t) := H(x,t)$ (path from y_0 to y_1), then the following diagram commutes:

$$\begin{tikzcd} & \pi_n(Y, y_0) \arrow[d, "\gamma_*", "\wr"'] \\ \pi_n(X, x) \arrow[ur, "f_{0*}"] \arrow[r, "f_{1*}"'] & \pi_n(Y, y_1) \end{tikzcd}$$

Definition: A continuous map $f: X \to Y$ is a <u>weak homotopy equivalence</u> if it induces bijections

$$f_*: \pi_n(X, x_0) \to \pi_n(Y, y_0) \quad \forall n \geq 0, \forall x_0 \in X \quad (y_0 = f(x_0)).$$

Proposition: Every homotopy equivalence is a weak homotopy equivalence.

Lemma: f is a weak hom. eq, and $f' \simeq f \Rightarrow f'$ is a weak hom. eq.

Lemma: Let (X, A) a pair of spaces with the inclusion $i: A \hookrightarrow X$ inducing a bijection $\pi_0(A) \to \pi_0(X)$.

$$i \text{ is a w.h.e} \iff \pi_n(X, A, x_0) = 0 \quad \forall n \geq 1, \forall x_0 \in A.$$

Definition: Let (X,A) be a pair of spaces. We say that (X,A) is <u>n-connected</u> if for every $m \leq n$ and every map of pairs $g:(\mathbb{D}_m, \partial\mathbb{D}_m) \to (X,A)$ there is a homotopy $H: \mathbb{D}_m \times [0,1] \to X$ rel. $\partial\mathbb{D}_m$ with image in A, i.e., $H(-,0) = g$, $H(-,1) \in A$ and $H|_{\partial\mathbb{D}_m \times [0,1]} = g$.

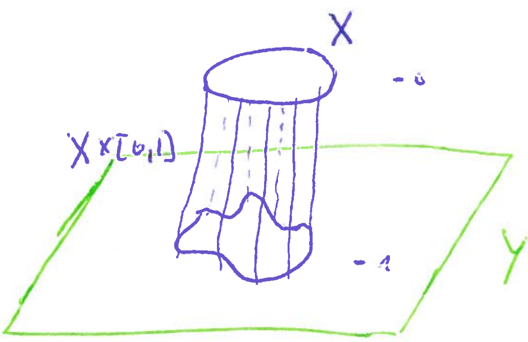
Corollary. Let $A \subset X$. If the inclusion $A \hookrightarrow X$ is a whe, then (X,A) is n-connected $\forall n$.

Lemma. If $A \subset X$ is a deformation retract of X, then (X,A) is n-connected $\forall n$.

Theorem (Whitehead for relative CW-complexes): Let (X,A) be a relative CW-complex. If the inclusion $A \hookrightarrow X$ is a whe, then it is also a homotopy equivalence.

Definition: Let $f: X \to Y$ be a continuous map. The <u>mapping cylinder</u> $M(f)$ is the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\mathrm{ind}_1"'] & Y \arrow[d] \\ X \times [0,1] \arrow[r] & M(f) \end{tikzcd} \quad \text{, i.e., } \quad M(f) = \frac{X \times [0,1] \amalg Y}{(x,1) \sim f(x)}$$



- Write $\pi: X \times [0,1] \amalg Y \to M(f)$, and let $j_X : X \to M(f)$, $x \mapsto \pi(x,0)$. It induces a homeomorphism $X \xrightarrow{\simeq} j_X(X)$.
- By the universal prop. of the pushout, $\exists\, p: M(f) \to Y$ cont. : $p \circ j_X = f$.

Proposition: Every continuous map $f: X \to Y$ factors as a composite

$$X \xrightarrow{j_X} M(f) \xrightarrow{p} Y$$

of a closed inclusion and a homotopy equivalence.

Lemma: $(M(f), X)$ has the HEP.

Theorem (Whitehead): Every weak homotopy equivalence $f: X \to Y$ between CW-complexes is a homotopy equivalence.

Important! It is not enough that two CW-complexes X, Y for which there are bijections between $\pi_n(X, x_0)$ and $\pi_n(Y, y_0)$ are homotopy equivalent. It is an essential assumption that such bijections are induced by continuous maps!

Eg: The spaces $\mathbb{P}_2(\mathbb{R})$ and $S_2 \times \mathbb{P}_\infty(\mathbb{R})$ are not hom. eq. but they have isomorphic homotopy groups in all degrees.

XV : THE HUREWICZ THEOREM

• We have seen that not all top spaces can be equipped with a CW-structure. Ok, but at least can I find a CW which is homotopic to a top space? NO! Eg: the Hawaiian earring. OK... at least can I find a CW-complex st there is a weak homotopic equivalence? Yes:

Theorem (CW-approximation): Every topological space is weak homotopic to a CW complex, i.e., for any top space X $\exists$ a CW-complex $cw(X)$ and a weak hom. eq. $cw(X) \to X$.

• In II we showed that $\phi : \pi_1(X, x_0)^{ab} \xrightarrow{\sim} H_1(X, x_0)$ when X is path-connected. One wonders, is there any analogous iso for $n \geq 2$? $\pi_n(X, x_0)$, $n \geq 2$ is already abelian... . The most naive try fails! In general, even for path-connected spaces, $\pi_n(X, x_0) \not\simeq H_n(X, \mathbb{Z})$.

• Take a generator $1 \in H_n(\mathbb{D}_n, \partial \mathbb{D}_n, \mathbb{Z}) \simeq \mathbb{Z}$ (recall that it depends on the iso $\simeq \mathbb{Z}$). Let (X, A) be a pair of spaces and take $x_0 \in A$.

Definition: The (relative) Hurewicz map is

$$h_n : \pi_n(X, A, x_0) \longrightarrow H_n(X, A, x_0)$$
$$[\alpha] \longmapsto h_n[\alpha] := \alpha_*(1)$$

where we view $\alpha : (\mathbb{D}_n, \partial \mathbb{D}_n, s_0) \to (X, A, x_0)$ and $\alpha_* : H_n(\mathbb{D}_n, \partial\mathbb{D}_n; \mathbb{Z}) \to H_n(X, A, x_0)$.

• There is an absolute version:

Definition: The Hurewicz map is

$$h_n : \pi_n(X, x_0) \longrightarrow H_n(X, x_0)$$
$$[\alpha] \longmapsto h_n[\alpha] := \alpha_*(1)$$

where $\alpha : (S_n, s_0) \to (X, x_0)$, $1 \in H_n(S_n, \mathbb{Z}) \simeq \mathbb{Z}$ and $\alpha_* : H_n(S_n, \mathbb{Z}) \to H_n(X, \mathbb{Z})$.

Def: A topological space is simply connected if it is path-connected and the fundamental group $\pi_1(X, x_0) = 0$ $\forall x_0 \in X$ (well, only enough for one point).

Theorem (Hurewicz, relative version): Let (X,A) be a pair of spaces, and let $n \geq 2$. Suppose

1) A, X simply connected
2) $\pi_i(X, A, x_0) = 0$ $\forall i = 2, \dots, n$, $\forall x_0 \in A$ (some)

Then $H_i(X, A; \mathbb{Z}) = 0$ $\forall i < n$, and the Hurewicz map

$$h_n : \pi_n(X, A, x_0) \xrightarrow{\sim} H_n(X, A; \mathbb{Z})$$

is isomorphism of groups.

Theorem (Hurewicz): Let X be a top space, and let $n \geq 2$. Suppose

1) X simply connected
2) $\pi_i(X, x_0) = 0$ $\forall i < n$

Then the Hurewicz map

$$h_n : \pi_n(X, x_0) \xrightarrow{\sim} H_n(X; \mathbb{Z})$$

is an isomorphism of groups. Ie, for a simply connected space, the first non-trivial homotopy group coincides with the first non-trivial homology group with $\mathbb{Z}$-coefficients.

Corollary: $\pi_n(S^n, s_0) \simeq \mathbb{Z}$, $n \geq 2$.

Theorem (Whitehead): Let $f: X \to Y$ be continuous between simply connected CW complexes. If $f_* : H_n(X; \mathbb{Z}) \xrightarrow{\sim} H_n(Y; \mathbb{Z})$ are isomorphisms $\forall n \geq 2$ $\Rightarrow$ f is a homotopy equivalence.

Proposition: The Hurewicz map $h_n : \pi_n(X, x_0) \to \tilde{H}_n(X; \mathbb{Z})$ is a group homomorphism.

APPENDIX: REMARKS AND RESULTS FROM EXERCISES

- One point space: $H_n(*;A) = 0$, $n \geq 1$; $H_0(*;A) = A$

- $(X_j)_{j\in J}$ path components: $H_n(X;A) = \bigoplus_{j\in J} H_n(X_j;A)$

- Always: $H_0(X;A) \overset{\gamma_X}{\underset{\sim}{\longleftarrow}} A[\pi_0(X)]$

- X path-connected: $H_0(X;A) = A$.

- Z discrete space, $H_n(Z;A) = 0$ $(n \geq 1)$; $H_0(Z;A) = \bigoplus_Z A = A[Z]$

- $H_n(S_m;A) = \begin{cases} A \oplus A, & n = m = 0 \\ A, & n = m > 0 \\ 0, & \text{else} \end{cases}$

- Two homotopies $F: X \times [0,1] \longrightarrow Z$ and $G: J \times \mathbb{D}_n \times [0,1] \longrightarrow Z$ can be glued into $\left(X \cup_{J \times \partial \mathbb{D}_n} J \times \mathbb{D}_n\right) \times [0,1] \longrightarrow Z$.

__Lemma__ (The five lemma): Consider the following commutative diagram of abelian groups and group homomorphisms in which both rows are exacts

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ f_1\downarrow & & f_2\downarrow & & f_3\downarrow & & f_4\downarrow & & f_5\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

1) f_2, f_4 injective; f_1 surjective $\Rightarrow$ f_3 injective
2) f_2, f_4 surjective; f_5 injective $\Rightarrow$ f_3 surjective
3) f_2, f_4 iso; f_1 surj; f_5 inj $\Rightarrow$ f_3 iso.

Lemma (Splitting): Consider the following short exact sequence of abelian groups.

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "i"] & A \arrow[r, "\pi"] \arrow[l, bend left, "r"] & \bar{A} \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

The following are equivalent:

1) π admits a section, ie, $\exists\ s : \bar{A} \to A \ : \pi \circ s = Id_{\bar{A}}$

2) i admits a retraction, ie, $\exists\ r : A \to A' \ : r \circ i = Id_{A'}$

3) There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, "\wr\ Id"'] & A \arrow[r, "\pi"] \arrow[d, "\wr\ f"'] & \bar{A} \arrow[r] \arrow[d, "\wr\ Id"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "j"] & A' \oplus \bar{A} \arrow[r, "p"] & \bar{A} \arrow[r] & 0 \end{tikzcd}$$

$a' \longmapsto (a', 0)$

$(a', \bar{a}) \longmapsto \bar{a}$

ie, $A \simeq A' \oplus \bar{A}$ (A splits).

• **Note**: In vector spaces, ses always split!! In particular, if $V \subset E$, then $E = V \oplus E/V$.

Corollary: Let (X, X') a pair of spaces and suppose that there exists a retraction $r : X \to X'$. Then exists an isomorphism

$$H_n(X;A) = H_n(X';A) \oplus H_n(X,X';A) \qquad n \geq 0.$$

Theorem (Long exact sequence of a triple): Consider $X'' \subset X' \subset X$ subspaces of X, and consider the maps

- $H_n(X', X''; A) \to H_n(X, X'; A)$ induced by $(X', X'') \xrightarrow{i} (X, X'')$
- $H_n(X, X''; A) \to H_n(X, X'; A)$ induced by $(X, X'') \xrightarrow{Id} (X, X')$
- The composite $H_n(X, X'; A) \xrightarrow[\text{LES of}]{} H_{n-1}(X'; A) \xrightarrow[\text{LES of}]{} H_{n-1}(X', X''; A)$

There is a long exact sequence

$$\begin{tikzcd}
& \cdots \arrow[dl, bend right] & \\
H_n(X',X'';A) \arrow[r] & H_n(X,X'';A) \arrow[r] & H_n(X,X';A) \arrow[dll, bend left] \\
H_{n-1}(X',X'';A) \arrow[r] & H_{n-1}(X,X'';A) \arrow[r] & H_{n-1}(X,X';A) \arrow[dll, bend left] \\
\cdots & &
\end{tikzcd}$$

<u>Theorem</u> (Mayer-Vietoris exact sequence): Let X be a top. space and A an abelian group. Let $U, V \subset X$ be two subsets satisfying $X = \mathring{U} \cup \mathring{V}$, and consider the natural inclusions

$$\begin{tikzcd}
& U \arrow[dr, "j^U"] & \\
U \cap V \arrow[ur, "i^U"] \arrow[dr, "i^V"'] & & X \\
& V \arrow[ur, "j^V"'] &
\end{tikzcd}$$

There's a long exact sequence

$$\begin{tikzcd}
& \cdots \arrow[dl, bend right] & \\
H_n(U\cap V;A) \arrow[r] & H_n(U;A) \oplus H_n(V;A) \arrow[r] & H_n(X;A) \arrow[dll, bend left] \\
H_{n-1}(U\cap V;A) \arrow[r] & H_{n-1}(U;A) \oplus H_{n-1}(V;A) \arrow[r] & H_{n-1}(X;A) \arrow[dll, bend left] \\
H_{n-2}(U\cap V;A) \arrow[r] & \cdots &
\end{tikzcd}$$

called the <u>Mayer-Vietoris exact sequence</u>, where at every row the first morphism is (i^U_*, i^V_*) and the second is $\begin{pmatrix} j^U_* \\ -j^V_* \end{pmatrix}$.

It is!

Very important!!! : For a absolute CW-complex,

$$\underline{\tilde{C}_n(X;A)} = H_n(X_n, X_{n-1};A) = \underline{\bigoplus_{J_n} A}$$

This is very easy to compute. Now just remember that $H(X;A) = H(\tilde{C}(X;A))$, i.e., the chain complex $C(X;A)$ and $\tilde{C}(X;A)$ may be different, but when taking $\frac{\ker}{\mathrm{im}}$ they are the same! <u>So we can use the chain complex $\tilde{C}(X;A)$ to compute the singular homology of X</u>. But watch out! I don't really know $\tilde{\partial}_n$. At this point I can only do this when $\tilde{\partial}_n$ is trivial (because of the form of the chain). If it is not I don't have anything to do at this point.

• <u>Homologies...? I only know two!</u>

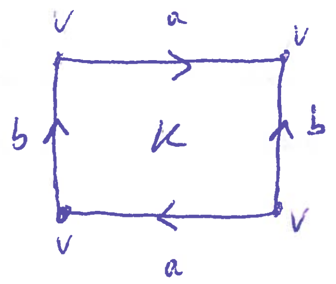
- $H_n(*;A) = 0$, $n \geq 1$; $H_0(*;A) = A$
- $H_n(\text{some points};A) = 0$, $n \geq 1$; $H_0(\text{some points};A) = \bigoplus_{\text{some pts}} A$
- $H_0(\text{path-connected};A) = A$; and in general , $H_0(X;A) = \bigoplus_{\#\pi_0(X)} A$
- To compute $H_n(X, X';A)$:
  - Long exact sequence!!
  - $H_n(X, X';A) = 0 \iff H_n(X;A) = H_n(X';A)$
  - Excision & homotopy invariance
  - $H_n(D_n, \partial D_n;A) = A$

- Naturality:
  - $(f|_X)_* = f_*|_{H_n(X;A)}$
  - $\partial' = \partial|_{C_n(X';A)}$
  - 2 out of 3 diagram

- $\boxed{H_0(X;A) = \tilde{H}_0(X;A) \oplus A}$

- For CW-complexes:
  - $\tilde{C}_n(X;A) = H_n(X_n, X_{n-1};A) = \bigoplus_{J_n} A$

• Computing cellular homology differentials: with the theorem: awful. Better: informal way:



$$\mathbb{Z}[K] \xrightarrow{\tilde{\partial}_2} \mathbb{Z}[a,b] \xrightarrow{\tilde{\partial}_1} \mathbb{Z}[v]$$

$$K \longmapsto \underbrace{b+a-b+a}_{=2a}$$

$$a \longmapsto v-v=0$$

$$b \longmapsto v-v=0$$

Just look to the boundaries (the symbol)

$$H_1(X,\mathbb{Z}) = \frac{\operatorname{Ker}\tilde{\partial}_1}{\operatorname{Im}\tilde{\partial}_2} = \frac{\mathbb{Z}[a,b]}{\mathbb{Z}[2a]} = \frac{a\mathbb{Z}\oplus b\mathbb{Z}}{2a\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

• For speed with exact sequences:

- $0 \to A \xrightarrow[\sim]{f} B \to 0$ exact $\Rightarrow$ f iso

- $A \xrightarrow{\sim} B \to C \to D \xrightarrow{\sim} E$ exact $\Rightarrow$ $C = 0$.

$$\begin{array}{c} C \\ \| \\ \Rightarrow \; 0 \end{array}$$