I

SYMPLECTIC GEOMETRY

<u>Definition</u>: Let V be a vector space and Ω a skew-symmetric bilinear map, and let $W \subset V$. The <u>symplectic complement</u> or <u>orthogonal</u> is

$$W^{\Omega} := \{ v \in V : (i_v \Omega)_{|W} = 0 = \Omega(v,-)_{|W} \} = \{ v \in V : \Omega(v, W) = 0 \}$$

and the <u>radical</u> of V is $\operatorname{rad} V = \{ v \in V : \Omega(v, V) = 0 \}$.

<u>Theorem</u>. Let Ω be a skew-symmetric bilinear map. There exists a basis $\{ u_1, \dots, u_k, e_1, \dots, e_n, f_1 \dots f_n \}$ such that

$$\Omega(u_i, -) = 0 \quad , \quad \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j) = 0 \quad , \quad \Omega(e_i, f_j) = \delta_{ij} ,$$

ie, the matrix of Ω takes the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix}$$

We will say that the invariant $2n$ is the <u>rank</u> of Ω.

Note that ~~if~~ $U = \langle u_1 \dots u_k \rangle$, and $\Omega^\flat : V \to V^*$, $v \mapsto i_v \Omega$ is the polarity, then

$$\operatorname{Ker} \Omega^\flat = U .$$

<u>Definition</u>: A 2-form Ω is <u>symplectic</u> or non-degenerated when $\Omega^\flat$ is isomorphism, ie, $U = 0$.

We say that (V, Ω) is a <u>sympl. vector space</u>. Note, by the theorem, that its dimension must be <u>even</u>.

- $V^\Omega = \operatorname{rad} V$, and $\operatorname{rad} W \subseteq W^\Omega$.

• In $\mathbb{R}^{2n}$, let $\{e_1, \dots e_n, v_1 \dots v_n\}$ a basis and $\{\omega_1, \dots \omega_n, \xi_1, \dots \xi_n\}$ its dual basis. Then $\Omega_0 = \sum \omega_i \wedge \xi_i$ is a symplectic form.

<u>Definition</u>: A ~~sb~~ vector subspace W is <u>symplectic</u> when $(W, \Omega|_W)$ is a symplectic vector space.

<u>Definition</u>: A vector subspace W is <u>isotropic</u> when $\Omega|_W = 0$, ie, when rad $W = W$.

<u>Lemma</u>: Let (V, Ω) be a space with a 2-form, and $\Omega^\flat : V \to V^*$ the polarity

1) $\operatorname{Ker} \Omega^\flat|_W = W \cap W^\Omega$.

5) $(W^\Omega)^\Omega = W$. *

2) W isotropic $\Rightarrow W \subseteq W^\Omega$

3) W symplectic $\Rightarrow W \cap W^\Omega = 0$. $\Leftrightarrow V = W \oplus W^\Omega$

4) $\boxed{\dim V = \dim W + \dim W^\Omega}$ *

* needs (V, Ω) symplectic or W: symplectic

<u>Definition</u>: A <u>symplectomorphism</u> or linear symplectic map if a linear isomorphism $\varphi : (V, \Omega) \to (V', \Omega')$ s.t. $\varphi^* \Omega' = \Omega$, i, $\Omega'(\varphi(u), \varphi(v)) = \Omega(u, v)$.

<u>SYMPLECTIC MANIFOLDS</u>

<u>Definition</u>: A <u>symplectic manifold</u> is a pair (M, ω) where $\omega \in \Omega^2(M)$ satisfies

i) $d\omega = 0$

ii) ω_x is symplectic $\forall x \in M$, ie, ω is non-deg pointwise

It follows that $\dim M = \underline{\underline{\text{even}}}$.

<u>E.g.</u>: 1) $(\mathbb{R}^{2n}, \omega_0 = \sum dx_i \wedge dy_i)$; 2) $(S^2, \omega_p(u, v) := p \cdot (u \times v))$

3) $(\mathbb{C}^n, \omega = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k)$, where $z = x + iy \Rightarrow \boxed{dz := dx + i\,dy}$

<u>Definition</u>: A <u>symplectomorphism</u> is a diffeomorphism $\varphi : (M, \omega) \to (M', \omega')$ such that $\varphi^* \omega' = \omega$.

Lemma: Let $(U; q_1 \dots q_n)$ and $(U'; q_1' \dots q_n')$ be charts in a nbhd of $x \in M$, and let $p_i = \omega \cdot (\partial_{q_i})$, $p_i' = \omega(\partial_{q_i'})$ the ~~set~~ of coordinates for T^*M. Then the change of coordinates is:

$$\boxed{p_i = \sum_{j=1} p_j' \frac{\partial q_j'}{\partial q_i}.}$$

• (Beginning of symp. geometry): If M is a mnfd, T^*M has a canonical symplectic structure, given by

$$\omega_{can} := -d\theta_L ,$$

where θ_L is a canonical 1-form, ~~in~~ T^*M, called the Liouville 1-form or tautological 1-form, defined by

$$\left.\begin{array}{l} \pi : T^*M \to M \\ \pi_* : T_{\omega_p}(T^*M) \to T_pM \\ \pi^* : T_p^*M \to T_{\omega_p}^*(T^*M) \end{array}\right\} \quad \boxed{(\theta_L)_{\omega_p} := \pi^* \omega_p}$$

and if $(T^*U; q_1 \dots q_n, p_1 \dots p_n)$ are the rel. coordinates of that chart, then

$$\boxed{\begin{aligned} \theta_L &= \sum p_i \, dq_i \\ \omega_{can} &= \sum dq_i \wedge dp_i . \end{aligned}}$$

Definition: Let M_1, M_2 manifolds, and $f: M_1 \to M_2$ a diffeomorphism. The lift of f is the diffeomorphism

$$f_\# : T^*M_1 \to T^*M_2 ,$$

$f_\# |_{T_p^*M_1} = f_{*,p} \equiv (f_p^*)^{-1}$ (inverse of the cotangent linear map). Moreover, the following diagram is commutative:

$$\begin{array}{ccc} T^*M_1 & \xrightarrow{f_\#} & T^*M_2 \\ \pi \downarrow & & \downarrow \pi \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

Proposition (Naturality of the Liouville 1-form): The lift of a diffeomorphism $f: M \to \bar{M}$ pulls the Liouville 1-form on $T^*\bar{M}$ back to the Liouville 1-form on T^*M, i.e.,

$$f_\#^* \bar{\theta}_L = \theta_L .$$

Corollary: If $f: M \longrightarrow \overline{M}$ is a diffeomorphism $\Rightarrow$ $f_{\#}: (T^*M, \omega_{can}) \longrightarrow (T^*\overline{M}, \overline{\omega}_{can})$ is a symplectomorphism.

II

• Let (M,ω) be symplectic. Since ω is non-deg, we have an isomorphism
$$\mathfrak{X}(M) \xrightarrow{\ \ \sim\ \ } \Omega^1(M)$$
$$X \longmapsto i_X\omega\ .$$

<u>Definition</u>: Let $H \in \mathcal{C}^\infty(M)$. We will call <u>Hamiltonian vector field</u> with Hamiltonian function H to the unique v.f. $X_H \in \mathfrak{X}(M)$ that corresponds with dH in the previous isomph., i.e., that $i_{X_H}\omega = dH$.

• In $(\mathbb{R}^{2n}, \omega_0 = \sum dq_i \wedge dp_i)$, we have $X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i}\partial_{q_i} - \frac{\partial H}{\partial q_i}\partial_{p_i}\right)$. Moreover, a curve $\sigma(t) = (q_1(t), \dots, q_n(t), p_1(t), \dots, p_n(t))$ is an integral curve for X_H $\Leftrightarrow$

$$\begin{cases} q_i' = \dfrac{\partial H}{\partial p_i}(q_1 \dots p_n) & , i = 1\dots n \\[2ex] p_i' = -\dfrac{\partial H}{\partial q_i}(q_1, \dots, p_n) & , i = 1\dots n \end{cases}$$

(<u>Hamilton equations</u>)

• (<u>Classical mechanics</u>): Let $(\mathbb{R}^3, q_1 q_2 q_3)$. A particle of mass m moving along $\sigma(t) = (q_1(t), q_2(t), q_3(t))$ under a conservative force with potential $V \in \mathcal{C}^\infty(\mathbb{R}^3)$, according to Newton's 2^{nd} law, satisfies

$$m(q_1'', q_2'', q_3'') = -\operatorname{grad} V$$

("F = m a")

Introduce the <u>momenta</u> $p_i(t) := m q_i'(t)$, and let $T^*\mathbb{R}^3 = \mathbb{R}^6$ be the <u>phase space</u>, with coordinates $(q_1 q_2 q_3, p_1 p_2 p_3)$. If we define the energy function as $H(q,p) := \frac{1}{2m}|p|^2 + V(q)$, then Newton's law is equivalent to the Hamilton equations in $\mathbb{R}^6$:

$$\begin{cases} q_i' = \dfrac{\partial H}{\partial p_i}(q_1 \cdots p_3) & , i = 1 2 3 \\[2ex] p_i' = -\dfrac{\partial H}{\partial q_i}(q_1 \dots p_3) & , i = 1 2 3 \end{cases}$$

Since H is a first integral of X_H (see now), it turns out that $H \equiv$ the energy is constant along the integral curves of X_H, i.e., the physical motions.

Lemma: H is a first integral of X_H.

$$\left[(\varphi_X^t)^* X\right]_x \\ \| $$

Take $X \in \mathfrak{X}(M)$ and its flow φ_X^t. We know that $\frac{d}{dt}\varphi_X^t(x) = X(\varphi_X^t(x))$ (in particular in $t=0$ we have the well-known $\frac{d}{dt}\big|_{t=0}\varphi_X^t(x) = X_x$, ie $\frac{d}{dt}\big|_{t=0}\varphi_X^t = X$.

Denote by $\varphi_H^t \equiv \varphi_{X_H}^t$.

Proposition: $\varphi_H^t : M \longrightarrow M$ is a symplectomorphism, $\forall t$.

Definition. A v.f $X \in \mathfrak{X}(M)$ is said to be <u>symplectic</u> when $i_X\omega$ is closed, and <u>hamiltonian</u> when it is exact (it coincides with the previous def, $i_X\omega = dH$).

- Obviously hamiltonian $\Rightarrow$ symplectic; and "$\Leftarrow$" locally (Poincaré), then symplectic v.f. are also called <u>locally hamiltonian</u>

Proposition: $f:(M,\omega) \longrightarrow (N,\omega')$ symplectomorphism $\Longleftrightarrow$ $X_H = f_* X_{f^*H}$ $\forall H \in \mathcal{C}^\infty(N)$

BRACKETS

Definition: A <u>Lie algebra</u> is a real v.s. V endowed with a map

$$[\cdot,\cdot] : V \times V \longrightarrow V$$

satisfying

i) Bilinear

ii) Anti-symmetric

iii) Jacobi.

Lemma. $X, Y \in \mathfrak{X}(M)$ symplectic $\Rightarrow$ $[X,Y]$ Hamiltonian with Ham. function $H = -\omega(X,Y)$.

Definition: Let (M,ω) symplectic. The <u>Poisson bracket</u> of $f,g \in \mathcal{C}^\infty(M)$ is the function

$$\{f,g\} := \omega(X_f, X_g)$$

LET OP! Opposite than in "Poisson Geometry". Everything will be with different sign

- The v.f (derivation) $\{-,g\} = X_g$.

Corollary: $X_{\{f,g\}} = -[X_f, X_g]$, ie, $(\mathcal{C}^\infty(M), \{\cdot,\cdot\}) \longrightarrow (\mathfrak{X}(M), [\cdot,\cdot])$, $f \mapsto X_f$ is an anti-homomorphism of Lie algebras.

- For (M,ω), if $(U; q_1 \dots q_n, p_1 \dots p_n)$ and $\omega = \sum dq_i \wedge dp_i$, then

$$\{F,G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

and in particular $\{q_i, p_j\} = \delta_{ij}$, $\{q_i, q_j\} = \{p_i, p_j\} = 0$.

Theorem: Let $\varphi : (M,\omega) \to (N,\omega')$ be diffeomorphism between sympl. manifolds. The following are equiv.

1) φ is symplectomorphism

2) $X_H = \varphi_* X_{\varphi^* H} \quad \forall H \in \mathcal{C}^\infty(M)$

3) $\varphi^*\{f,g\}_N = \{\varphi^* f, \varphi^* g\}_M \quad \forall f,g \in \mathcal{C}^\infty(N)$ (φ is a Poisson map)

- In $\mathbb{R}^{2n}$, the canonical sympl. form is $\omega_0 = \sum dx_i \wedge dy_i$. More exmpls? Locally not:

Theorem (Darboux): Let (M,ω) be a symplectic manifold, and $x \in M$. There exists a chart $(U; q_1 \dots q_n, p_1 \dots p_n)$ centered at $x \in M$ such that

$$\omega = \sum dq_i \wedge dp_i .$$

Remarks concerning Diff. Geometry: With Navarro notation too.

1) $\varphi : X \to Y$, $\sigma : I \to X$, $\sigma(0) = p$, $T_0 = v$ $\Rightarrow$ if $\bar{\sigma} := \varphi \circ \sigma$ $\Rightarrow$ $\bar{T}_0 = \varphi_*(v)$.

2) What is $\frac{d}{dt}\big|_{t=0} \sigma$? As vector, $\sigma_*((\partial_t)_0)$; as derivation, $(\partial_t)_0 \sigma^*$, ie, $\left(\frac{d\sigma}{dt}\big|_{t=0} f\right) = \frac{\partial f \circ \sigma}{\partial t}$

3) τ uniparametric group with inf generator D. Then $D_p = \frac{d}{dt}\big|_{t=0} \tau_t(p)$, or more gen, $D_{\tau(t_0,p)} = \frac{d}{dt}\big|_{t=t_0} \tau_t(p)$.

4) $D^L \omega = \frac{d}{dt}\big|_{t=0} \tau_t^* \omega$, and in general $\frac{d}{dt} \tau_t^* \omega = \tau_t^*(D^L \omega)$.

III

<u>Definition</u>: A <u>Hamiltonian system</u> is a triple (M, ω, H), where (M,ω) is a symplectic mnfld & $H \in \mathcal{C}^\infty$
is a function called <u>Hamiltonian function</u>. If $\dim M = 2n$, we say that the system has n <u>degrees of freedom</u> (DOF).

<u>Definition</u>: Given (M, ω, H), the <u>orbit</u> of X_H through $x \in M$ is $\mathcal{O}_H(x) = \{ \varphi_H^t(x) : t \in \mathbb{R} \} = \{ \sigma_x(t) : t \in \mathbb{R} \}$ where σ_x is the integral curve of X_H passing through $x \in M$.

Note that, since H is first integral of X_H, motion takes place on the level sets (preimages, fibres) of H. Ie, each level set can be decomposed into the union of orbits of X_H (with the same value of H).

LIOUVILLE INTEGRABLE HAMILTONIAN SYSTEMS

<u>Definition</u>: We will say that a Hamiltonian system (M, ω, H) with n DOF is <u>Liouville integrable</u> if there exists n functions $F_1, F_2, \dots, F_n \in \mathcal{C}^\infty(M)$ s.t. (typically $F_1 = H$)

i) $\{F_i, F_j\} = 0$ (<u>involution</u>)

ii) $dF_1 \wedge \dots \wedge dF_n \neq 0$ almost everywhere (<u>independence</u>)

- Note that since $\{-, F_1\} = X_H$, every F_i is an first integral (= integral of motion) of X_H. (if $F_1 = H$)
- Can we obtain more that n independent functions in involution? No:

<u>Lemma</u>: If $F_1, \dots, F_m \in \mathcal{C}^\infty(M)$ satisfy i) & ii) $\Rightarrow m \leq n$.

Let H be a Hamiltonian.

$$\begin{cases} x \text{ regular point} \Leftrightarrow d_x H \neq 0 \Leftrightarrow (X_H)_x \neq 0 \\ x \text{ critical point} \Leftrightarrow d_x H = 0 \Leftrightarrow (X_H)_x = 0 \end{cases}$$

$$\begin{cases} h \in \mathbb{R} \text{ regular value if every } x \in H^{-1}(h) \text{ is regular} \\ h \in \mathbb{R} \text{ critical } \text{ " } \quad \dots\dots \quad \text{critical} \end{cases}$$

<u>Theorem</u> (Liouville – Arnold). Let (M, ω, H) be a Liouville integrable system and let $F = (F_1 \dots F_n) : M \to \mathbb{R}^n$ be the so called <u>integral map</u>. If $y \in \mathbb{R}^n$ is a regular value of F, then each connected component of $F^{-1}(y)$ is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$. In particular, if $F^{-1}(y)$ is compact, then every connected component must be diffeomorphic to $\mathbb{T}^n$.

Moreover, in this last case, if $T \simeq \mathbb{T}^n$ is the compact connect. compont, there exists a ngbh V of T on M, an open set $U \subset \mathbb{R}^n$ and a diffeomorphism

$$V \xrightarrow{\sim} U \times \mathbb{T}^n$$

$$x \longmapsto (\underbrace{I_1 \dots I_n}_{\text{actions}}, \underbrace{\varphi_1 \dots \varphi_n}_{\text{angles}})$$

The coordinates I_k are called <u>actions</u> and they are smooth functions of $F_1 \dots F_n$ only, and the coordinats φ_k are the <u>angles</u>.

Also, in these action-angles coordinates, one has $\omega = \sum d\varphi_i \wedge dI_i$; and the Hamiltonian function is only function of $I_1 \dots I_n$, ie, $H = H(I_1, \dots, I_n)$. Therefore, the Hamilton equations take the form

$$\begin{cases} \dot{\varphi}_i = \dfrac{\partial H}{\partial I_i} = \omega_i(I_1, \dots, I_n) \\[2ex] \dot{I}_i = -\dfrac{\partial H}{\partial \varphi_i} = 0 \end{cases}$$

(φ_i's play the role of the position; and I_i's of the momenta).

In practice, $I_i = \dfrac{1}{2\pi} \displaystyle\int_{C_i} \theta_L$, where C_i are loops running through S_1 in $\mathbb{T}^n = S_1 \times \dots \times S_1$

IV

LIE GROUPS

Definition: A *Lie group* G is a manifold equipped with a group structure, where the operations $(g,g')\mapsto g\cdot g'$ & $g\to g^{-1}$ are smooth.

$(\mathbb{R},+)$, $S_1 = SO(2) = \mathcal{U}(1)$, $S\mathcal{U}(n)$, $O(n)$, ...

Definition. A *representation* of a Lie group G on a v.s. V is a group homomorphism $G\to GL(V)$.

Definition: An *action* of a Lie group G on a manifold M is a group homomorphism

$$\begin{aligned}\psi: G &\longrightarrow \mathrm{Diff}(M)\\ g &\longmapsto \psi_g\end{aligned} \quad , \quad \text{ie}, \quad \psi_{gh} = \psi_g\circ\psi_h \ \& \ \psi_{g^{-1}} = \psi_g^{-1} .$$

and its *evaluation map* is

$$\begin{aligned}ev_\psi: G\times M &\longrightarrow M\\ (g,x) &\longmapsto \psi_g(x) \equiv g * x\end{aligned}$$

We will also say that the action is *smooth* when ev_ψ is smooth.

The flow of a complete v.f is an $\mathbb{R}$-action (of course, it is a uniparametric group).

We write $\varphi^t = \exp(tX)$ for $X\in\mathfrak{X}(M)$ and φ^t the flow of X. (X complete v.f.)

Definition: An action ψ is *symplectic* when ψ_g is a symplectomorphism $\forall g\in G$, so $\psi: G\to \mathrm{Sympl}(M)$

Definition: A $\mathbb{R}$-action or S_1-action ψ is *Hamiltonian* if the vector field generated by ψ (for $\mathbb{R}$-actions = uniparametric groups, it is clear; for S_1-actions, take it as an $\mathbb{R}$-action repeating values every 2π) is Hamiltonian, ie, if $\exists H\in\mathcal{C}^\infty(M)$: $dH = i_X\omega$, $\psi = \exp tX$.

ADJOINT & COADJOINT REPRESENTATIONS

<u>Definition</u>: Let G be a Lie group. The <u>left-multiplication</u> is the action $L_g : G \to G$, $g' \mapsto g \cdot g'$. We also say that $X \in \mathfrak{X}(G)$ is <u>left-invariant</u> when $(L_g)_* X = X$ $\forall g \in G$, i.e., $(L_g)_{*,a} = X_{ga}$.

<u>Proposition</u>: $\mathfrak{g} := \mathfrak{X}^{inv}(G) = T_e G$ is the <u>Lie algebra</u> of the Lie group G

- Consider the conjugation $\psi_g(a) := gag^{-1}$. It's derivative at the identity is

$$(\psi_g)_{*,e} := Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g}$$

<u>Definition</u>: The <u>adjoint representation</u> of G on $\mathfrak{g}$ is $\begin{aligned} Ad : G &\longrightarrow GL(\mathfrak{g}) \\ g &\longmapsto Ad_g \end{aligned}$,

and the <u>coadjoint representation</u> is $\begin{aligned} Ad^* : G &\longrightarrow GL(\mathfrak{g}^*) \\ g &\longmapsto Ad_g^* \end{aligned}$, where Ad_g^* is the dual of the adjoint action (up to sign), i.e, $Ad_g^*(\xi) := \xi \circ Ad_{g^{-1}}$

<u>Lemma</u>: 1) $Ad_{gh} = Ad_g \circ Ad_h$, $Ad_g^{-1} = Ad_{g^{-1}}$.

2) $Ad_{gh}^* = Ad_g^* \circ Ad_h^*$, $(Ad_g^*)^{-1} = Ad_{g^{-1}}^*$.

<u>Definition</u>: Let (M, ω) symplectic; G a Lie group and consider a symplectic action $\psi : G \longrightarrow Sympl(M, \omega)$. We say that the action is <u>Hamiltonian</u>, or that M is a <u>G-Hamiltonian space</u>, if there is a map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

called the <u>momentum map</u>, such that

i) $\boxed{i_{a(v)}\,\omega = d\mu_v}$, where

a Lie algebra ~~anti-isomorphism~~

- $a : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ is the <u>infinitesimal action</u> , and $a(v)$ is the <u>infinitesimal generator induced by v</u> , the vector field whose flow is given by $\phi^t_{a(v)}(x) = \exp(tv) * x$

(and $\exp(tv) = \phi^t_{v^L}(e)$, in general $\exp : \mathfrak{g} \to G$, $\exp(\alpha) := \phi^1_{\alpha^L}(e)$)

in other words, $a(v)_x = \frac{d}{dt}\big|_{t=0} \exp(tv) * x$

- $\mu_v \in \mathcal{C}^\infty(M)$ as $\mu_v(x) := \mu(x)(v)$

ii) μ is <u>G-equivariant</u>, ie, a G-sets morphism, ie, $\mu(g * x) = g * \mu(x)$, ie,

G-action on M $\qquad$ Ad^* action

$$\boxed{\mu \circ \psi_g = Ad^*_g \circ \mu} \qquad \forall g \in G$$

<u>Important remark</u> : If G is abelian, then $Ad^*_g = Id : \mathfrak{g}^* \to \mathfrak{g}^*$ $\forall g$, and the G-equivariance is equivalent to the <u>invariance</u> of μ, ie, $\mu \circ \psi_g = \mu$, ie, $\mu(g * x) = \mu(x)$.

<u>Corollary</u>: For a $\mathbb{R}$-action or S_1-action , both conditions become

i) $i_{a(v)}\,\omega = d\mu$ $\qquad$ & $\qquad$ ii) $\mu(t * x) = \mu(x)$

with the identification $\mu : M \to \mathfrak{g}^* = \mathbb{R}$.

<u>Theorem</u> (Noether, Hamiltonian version) : $f \in \mathcal{C}^\infty(M)$ is G-invariant $\iff$ μ is constant along the trajectories of X_f .

<u>ORBIT SPACES</u>

· Let $\psi : G \longrightarrow \mathrm{Diff}(M)$ be an action on a Lie group on a manifold M; $\psi_g(x) \equiv g * x$.

<u>Definition</u> : The <u>orbit</u> of $x \in M$ is $G * x \overset{\text{not}}{=} \mathrm{Orb}(x) := \{ g * x : g \in G \}$, and the <u>isotropy group</u> (or stabilizer) at $x \in M$ is $I_x \overset{\text{not}}{=} G_x := \{ g \in G : g * x = x \}$

Definition: We say that the G-action on M is

(a) <u>transitive</u> if there is just one orbit, $Orb(x) = M \ \forall x \in M$

(b) <u>free</u> if $G_x = \{e\} \ \forall x \in M$ (equivel., $g * x = h * x$ for some $x \Rightarrow g = h$)

(c) <u>locally free</u> if G_x is a discrete group $\forall x \in M$

- Recall that two orbits either are the same or are disjoint, and define the following equiv relation:

$$x \sim y \Leftrightarrow x \in Orb(y)$$

The quotient M/G, ie, the space of orbits of G, is called the <u>reduced space</u> of the G-action on M endowed with the projection to the quotient $\rho: M \to M/G$, called the <u>reduction map</u>. We consider on M/G the quotient topology: $U \subset M/G$ open $\Leftrightarrow \rho^{-1}(U)$ open.

* <u>Theorem</u> (Marsden – Weinstein – Meyer reduction): Let (M, ω, G, μ) be a G-Hamiltonian space for a compact Lie group G. Assume that G acts freely on $\mu^{-1}(0)$, and write $i: \mu^{-1}(0) \hookrightarrow M$. Then

1) The orbit space $M_r = \mu^{-1}(0)/G$ is a manifold

2) $\rho: \mu^{-1}(0) \to M_r$ is a principal G-bundle

3) M_r is a symplectic manifold with symplectic form ω_r satisfying $\rho^* \omega_r = i^* \omega$.

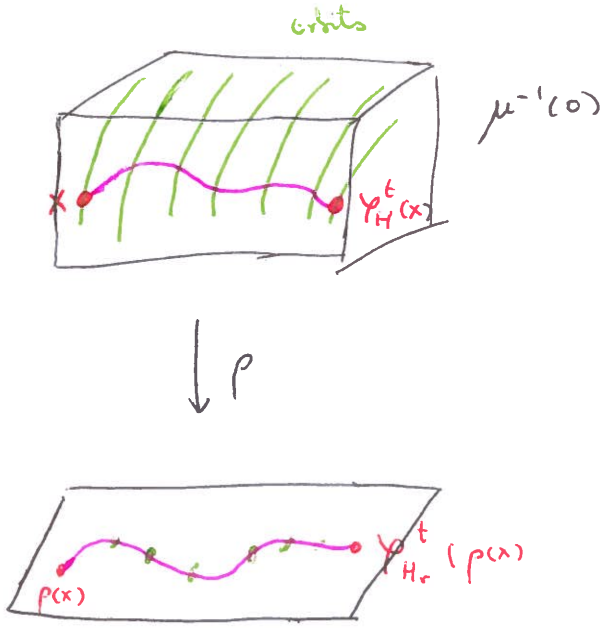
$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "\rho"'] & M \\ M_r & \end{tikzcd}$$

Corollary: Let $H \in \mathcal{C}^\infty(M)$ G-invariant, and $H_r \in \mathcal{C}^\infty(M_r)$: $\rho^* H_r = i^* H$. Let X_H and X_{H_r} be the Ham. v.f. of H and H_r. For $x \in \mu^{-1}(0)$ we have

$$\rho_{*,x}\left(X_H\right)_{i(x)} = \left(X_{H_r}\right)_{\rho(x)}$$

and therefore

$$\boxed{\rho\left(\varphi_H^t(x)\right) = \varphi_{H_r}^t(\rho(x))}$$



V

SINGULAR REDUCTION USING ALGEBRAIC INVARIANTS

- We can determinate the reduced spaces $M_m = \mu^{-1}(m)/\mathbb{T}^r$ using algebraic invariants. By sake of simplicity we will fixed a S_1-action on $\mathbb{C}^2 \underset{\Gamma}{\simeq} \mathbb{R}^4$

$$e^{it} * (z_1, z_2) := (e^{it} z_1, e^{it} z_2).$$

- Momentum map: One identifies $\mathfrak{g}^* \simeq \mathbb{R}$ and $\mu : \mathbb{R}^4 \to \mathbb{R}$. Computing $a : \mathbb{R} \to \mathcal{H}(\mathbb{R}^4)$ one determinate μ by the condition $i_{a(v)} \omega = d\mu$.

- Singular reduction: The key is to use algebraic invariants of the S_1-action. A theorem of Schwarz states that every S_1-invariant function on $\mathbb{R}^4 \simeq \mathbb{C}^2$ can be written as smooth function of the generators of the algebraic invariants of the S_1-action.

To determine such alg invts, consider a general monomial in the z_1, z_2 variables, $p(z_1, z_2) = z_1^{a_1} \bar{z}_1^{a_2} z_2^{b_1} \bar{z}_2^{b_2}$, for $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. For $p(z_1, z_2)$ to be S_1-invt, it must hold

$$p(z_1, z_2) = p(e^{it} * (z_1, z_2)) = p(e^{it} z_1, e^{it} z_2)$$

what means that $z_1^{a_1} \bar{z}_1^{a_2} z_2^{b_1} \bar{z}_2^{b_2} = (e^{it} z_1)^{a_1} \cdots (e^{-it} \bar{z}_2)^{b_2} = e^{it(a_1 - a_2 + b_1 - b_2)} z_1^{a_1} \bar{z}_1^{a_2} z_2^{b_1} \bar{z}_2^{b_2}$, i.e., $a_1 + b_1 = a_2 + b_2$. Consider the following diagram, where both rows have the same length because of the latter condition:

$$\begin{array}{l} \overbrace{z_1\, z_1\, z_1 \cdots z_1\, z_1\, z_1}^{a_1}\ \overbrace{z_2\, z_2 \cdots z_2\, z_2}^{b_1} \\ \underbrace{\bar{z}_1\, \bar{z}_1\, \bar{z}_1 \cdots \bar{z}_1\, \bar{z}_1}_{a_2}\ \underbrace{\bar{z}_2\, \bar{z}_2\, \bar{z}_2 \cdots \bar{z}_2\, \bar{z}_2}_{b_2} \end{array}$$

This pairing says that every invariant monomial can be written as a product of monomials of the form

$$\rho_1 := z_1 \bar{z}_1, \quad \rho_2 := z_2 \bar{z}_2, \quad \chi + i\psi := z_1 \bar{z}_2, \quad \chi - i\psi := \bar{z}_1 z_2$$

and thus as a function of $\rho_1, \rho_2, \chi, \psi$.

Observe that these invariants are not independent, $\rho_1\rho_2 = z_1\bar{z}_1 z_2\bar{z}_2 = (\chi+i\psi)(\chi-i\psi) = \chi^2+\psi^2$.

Since $\mu = m = \text{const}$ on $\mu^{-1}(m)$, it is convenient to use, instead of ρ_1, ρ_2, the invariants

$$\mu = \frac{1}{2}(\rho_1+\rho_2) \qquad , \qquad \vartheta = \frac{1}{2}(\rho_1-\rho_2)$$

and we obtain $\mu^2-\vartheta^2 = \rho_1\rho_2 = \chi^2+\psi^2$ and since $\mu^2 = m^2$ the reduced space is given by

$$\boxed{\chi^2+\psi^2+\vartheta^2 = m^2} \quad , \text{ i.e., } M_m \simeq S^2_m .$$

Important: Depending on the action, there might be some conditions that are not explicit in the above expression of the sort $\vartheta \le |m|$, ... which arise after manipulating $\mu+\vartheta = \rho_1 = |z_1|^2 \ge 0 \Rightarrow m=\mu \ge -\vartheta \Rightarrow \underline{\underline{\vartheta \ge -m}}$;

and $\mu-\vartheta = \rho_2 = |z_2|^2 \ge 0 \Rightarrow m = \mu \ge \vartheta \Rightarrow \underline{\underline{\vartheta \le m}} \Rightarrow \boxed{|\vartheta| \le |m|}$

Dynamics: If $H \in \mathcal{C}^\infty(\mathbb{R}^4)$ is S_1-invariant, $\exists\, H_r \in \mathcal{C}^\infty(M_m)$: $H = \rho^* H_r$, that can be expressed as a function in $\mu, \vartheta, \chi, \psi$. If we define $\dot{f} := X_{H_r} f = \{f, H_r\}$ for $f \in \mathcal{C}^\infty(M_r)$, then the dynamics is given by

$$\left.\begin{aligned}
\dot{\chi} &= \{\chi, H_r\} = \{\chi,\psi\}\frac{\partial H_r}{\partial \psi} + \{\chi,\vartheta\}\frac{\partial H_r}{\partial \vartheta} \\
\dot{\psi} &= \dots \\
\dot{\vartheta} &= \dots \\
\dot{\mu} &= 0 \qquad \left(\text{because } F\ S_1\text{-invariant} \overset{\text{Noether}}{\Longleftrightarrow} \underset{\{\mu, F\}}{\underline{\underline{X_F\,\mu = 0}}} \right)
\end{aligned}\right\}$$

recall

$$\boxed{\{F,G\} = \sum_{i,j=1}^{n} \frac{\partial F}{\partial u_i}\frac{\partial G}{\partial u_j}\{u_i,u_j\}}$$

Poisson structure: Recall (*). We can express $\mu, \vartheta, \chi, \psi$ in $\mathbb{R}^4 \simeq \mathbb{C}^2$ in terms of q_1, p_1, q_2, p_2. Since the sympl. str. in $\mathbb{R}^4$ is $\omega = dq_1\wedge dp_1 + dq_2 \wedge dp_2$, in $\mathbb{R}^4$ the Poisson bracket is given by

$$\{F,G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

So we just need to compute the derivatives

$$\{X, Y\} = 2\partial \quad ; \quad \{X, \partial\} = -2Y, \quad \{Y, \partial\} = 2X, \quad \{\mu, -\} = 0.$$

• Reduced symplectic form

Take $m \neq 0$, so M_m is a 2-dim manifold. Take "cilindrical" coordinates $\begin{cases} h := \partial \\ \theta := \mathrm{Arg}(X + iY) \end{cases}$.

The bivector associated to the Poisson str. is $\pi_r = \{\theta, h\} \partial_\theta \wedge \partial_h$, thus the reduced sym. form is

$$\omega_r = \frac{1}{\{\theta, h\}} \, d\theta \wedge dh$$

• Symplectic volume: $\mathrm{Vol}(M_m) = \int_{M_m} \omega_r$,

VI

HAMILTONIAN TORUS ACTION

• Consider a Hamiltonian S_1-action $\psi : S_1 \to \mathrm{Sympl}(M,\omega)$ with moment map $\mu : M \to \mathbb{R}$, $i_{a(s)}\omega = d\mu$ and $\psi_s^* \mu = \mu$ $\forall e^{is} \in S_1$. Assume that μ is proper ($\mu^{-1}(m)$ compact) and S_1 acts freely on $\mu^{-1}(0)$, $M_0 = \mu^{-1}(0)/S_1$ is a sympl. mfld whose sympl form is determined by $\rho_0^* \omega_0 = i_0^* \omega$ ($\rho_0 : \mu^{-1}(0) \to M_0$, $i_0 : \mu^{-1}(0) \hookrightarrow M$).

We can repeat the same for a level at $\mu^{-1}(t)$ with t close to 0, $M_t = \mu^{-1}(t)/S_1$, and $i_t^* \omega = \rho_t^* \omega_t$. Is the family of spaces $\{M_t\}$ related? And their sympl. forms $\{\omega_t\}$?

<u>Definition</u>: Let M be a manifold and G a Lie group. A <u>principal G-bundle</u> over M is a manifold P together with

i) An action of G on P

ii) A surjective map $\rho : P \to M$ which is invariant (i.e. $\rho(g*p) = \rho(p)$)

such that the following local triviality condition holds: every point $x_0 \in M$ has an open nbhd U such that

$$\varphi : \rho^{-1}(U) \xrightarrow{\sim} U \times G$$

satisfying $\varphi(\rho^{-1}(x)) \subseteq \{x\} \times G$ and φ is G-invariant (with $g*(x,\tilde{g}) := (x, g\tilde{g})$).

<u>Definition</u>: Consider the principal S_1-bundle $\rho_0 : \mu^{-1}(0) \to M_0$. A <u>connection form</u> is a 1-form $\alpha \in \Omega(\mu^{-1}(0))$ such that $\mathcal{L}_{X_\mu}\alpha = 0$ and $i_{X_\mu}\alpha = \alpha(X_\mu) = 1$. The <u>curvature form</u> is the 2-form $\beta \in \Omega^2(M_0)$ determined by the condition $d\alpha = \rho_0^* \beta$.

<u>Theorem</u> (Duistermaat–Heckman): The symplectic manifold (M_t, ω_t) is symplectomorphic to the symplectic manifold $(M_0, \omega_0 - t\beta)$.

The value 0 is not special, we can take t_0 so that $(M_t, \omega_t) \overset{\text{sympl}}{\simeq} (M_{t_0}, \omega_{t_0} - (t - t_0)\beta)$.

• Given a sympl. manifold (M, ω), there's a canonical volume form $\omega^n := \frac{1}{n!}\omega \wedge \overset{n)}{\cdots} \wedge \omega$.

<u>Corollary</u>: The volume of M_t $Vol(M_t)$ is a polynomial in t of degree $\leq n-1$.

• For a torus action we also have important results.

<u>Theorem</u> (Atiyah - Guillemin - Sternberg): Let $\psi: \mathbb{T}^m \longrightarrow Sympl(M, \omega)$ be a Hamiltonian action on a compact, connected symplectic manifold (M, ω). If $\mu: M \longrightarrow \mathbb{R}^m$ $(\simeq \mathfrak{g}^*)$ is the momentum map, then

1) $\mu^{-1}(m)$ is connected $\forall m \in \mathbb{R}^m$

2) $\mu(M)$ is convex; in particular it is the convex hull of the images of the fixed points of the $\mathbb{T}^m$-action

The image $\mu(M)$ is called the <u>momentum polytope</u>.

<u>Definition</u>: A G-action on M is said to be <u>effective</u> if all elements of G except the identity move at least one point in M; ie, if $\bigcap_{p \in M} I_p = \{e\}$

<u>Theorem</u>: Let $\psi: \mathbb{T}^m \longrightarrow Sympl(M, \omega)$ be a Hamiltonian action on a compact, connected $2n$-dim sympl. manifold. If the action is effective $\Rightarrow$ the action has at least $m+1$ fixed points and $m \leq n$.

<u>Definition</u>: A <u>toric manifold</u> $(M, \omega, \mathbb{T}^n, \mu)$ is a compact, connected $2n$-dim sympl. manifold endowed with an effective $\mathbb{T}^n$-action with momentum map $\mu: M \longrightarrow \mathbb{R}^n$.

<u>Proposition</u>: Every toric manifold defines a Liouville integrable system, whose integrals are the components of the momentum map.

<u>DELZANT POLYTOPES</u>

<u>Definition</u>: A <u>polytope</u> in $\mathbb{R}^n$ is the convex hull of finitely many points; ie, it is the intersection of finitely many closed half-spaces which is bounded.

Definition: A Delzant polytope Δ in $\mathbb{R}^n$ is a polytope satisfying the following properties:

i) It is simple: exactly n edges meet at each vertex p.

ii) It is rational: each of the n edges meeting at p can be parametrized as $p+tu$, with $u \in \mathbb{Z}^n$ and $0 \leq t \leq t_{max}$.

iii) It is smooth: the n vectors $u_1, \dots u_n$ used in ii) at every vertex form a basis of $\mathbb{Z}^n$.

Theorem (Delzant): There is a 1-to-1 correspondence

$$\{\text{Toric manifolds}\} \longleftrightarrow \{\text{Delzant polytopes}\}$$
$$M \longmapsto \mu(M)$$

(Delzant construction): How to reconstruct $(M, \omega, \mathbb{T}^n, \mu)$ from Δ? Write

$$\Delta = \{x \in \mathbb{R}^n : x \cdot v_k \leq \lambda_k \text{ for some } v_1, \dots, v_d\}$$

ie, d = number of faces, and $v_k \in \mathbb{Z}^n$ normal to the k-th face). Consider the $\mathbb{T}^d$-action on $\mathbb{C}^d$

$$(e^{2\pi i t_1}, \dots, e^{2\pi i t_d}) * (z_1, \dots, z_d) = (e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_d} z_d)$$

and the canonical sympl. form on $\mathbb{C}^d$ $\omega = \frac{i}{2} \sum_{k=1}^{d} dz_k \wedge d\bar{z}_k$; so the momentum map $\mu : \mathbb{C}^d \to \mathbb{R}^d$ is

$$\mu \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} = -\pi \begin{pmatrix} |z_1|^2 \\ \vdots \\ |z_d|^2 \end{pmatrix} + \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix}$$

Let $L : \mathbb{R}^d \to \mathbb{R}^n$ which sends the standard basis $e_1, \dots, e_d$ to $v_1, \dots, v_d$, $L(e_i) = v_i$. L induces $L_{|\mathbb{Z}^d} : \mathbb{Z}^d \to \mathbb{Z}^n$; and therefore $L : \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \to \mathbb{R}^n / \mathbb{Z}^n = \mathbb{T}^n$. Set $N = \operatorname{Ker} L \simeq \mathbb{T}^{d-n} \subseteq \mathbb{T}^d$. It happens that N acts on $\mathbb{C}^d$ in a Hamiltonian way so we can consider the corresponding momentum map $\nu : M \longrightarrow \mathbb{R}^{d-n}$. If $i : \operatorname{Ker} L \overset{\subset \mathbb{R}}{\hookrightarrow} \mathbb{R}^{n}$ with matrix (a_{ij}), then

$$\nu = \mu \cdot i = (\mu_1 \dots \mu_d) \begin{pmatrix} a_{11} & \dots \\ \vdots & \\ a_{d1} & \dots \end{pmatrix}$$

if $M_\Delta = \nu^{-1}(0)/N$, it turns out that this is a toric manifold, with $\dim M_\Delta = 2n$, and a sympl. form ω_Δ given by the reduction theorem.

VII

(The problem of n bodies) : Given a system of n particles in the space $\mathbb{R}^3$, with a two body interaction due to some force (gravitational, e.g.), where we know the initial positions and momenta, we want to know the trajectories of the n particles. The phase space is $\mathbb{R}^{2\cdot 3\cdot n} = \mathbb{R}^{6n}$.

The Kepler problem is the 2 bodies problem in which they interact by the gravitational force. Setting one of them as center of the system of coordinates, the Hamiltonian in $T^*(\mathbb{R}^3 - 0) = (\mathbb{R}^3 - 0) \times \mathbb{R}^3$ is

$$H(P,Q) = \frac{1}{2} P^2 - \frac{1}{Q} \quad , \quad P = (P_1, P_2, P_3) \; , \quad Q = (Q_1, Q_2, Q_3)$$

We want to deal with collision orbits and near them. When we have a perturbation of the Kepler problem it is convenient to simplify the Hamiltonian. We will use regularization,

KUSTAANHEIMO – STIEFEL REGULARIZATION

VIII

<u>Theorem</u> (Duistermaat): An integrable system admits global action-angle coordinates iff:

1) The monodromy is trivial
2) The fibration admits a global Lagrangian section
3) ω is exact.

DYNAMICS

- From now on we'll always consider standard coordinates $z = (q_1 \dots q_n, p_1 \dots p_n)$ on $\mathbb{R}^{2n}$ such that $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$.

- let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathcal{M}_{2n\times 2n}(\mathbb{R})$, and note that $\boxed{J^{-1} = J^t = -J}$

<u>Definition</u>: A <u>linear Hamiltonian system</u> in $\mathbb{R}^{2n}$ is a Hamiltonian system with $H = \frac{1}{2} z^t S z$, where S is a symmetric matrix $2n \times 2n$, i.e., H is a quadratic form.

- Solutions of X_H (integral curves) are given by $\dot{z} = J \cdot \operatorname{grad} H = J \cdot S z = A z \Rightarrow z = e^{At} z_0$, where $e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n$. Manipulating it happens that $\boxed{A^t J + J A = 0}$.

<u>Definition</u>: A matrix $A \in \mathcal{M}_{2n\times 2n}(\mathbb{R})$ is said to be <u>Hamiltonian</u> or infinitesimally symplectic when $A^t J + J A = 0$

- A Hamiltonian matrix takes the form $A = \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}$, with b, c symmetric.

<u>Lemma</u>: A, B Hamiltonian $\Rightarrow [A,B] = AB - BA$ Hamiltonian, thus Hamiltonian matrices form a Lie algebra.

<u>Definition</u>: A matrix $M \in \mathcal{M}_{2n\times 2n}(\mathbb{R})$ is called <u>symplectic</u> if $\boxed{M^t J M = J}$

<u>Proposition</u>: 1) M symplectic $\Rightarrow M$ is invertible and M^{-1} is symplectic

2) M, N sympl. $\Rightarrow MN$ sympl.

• This last prop. says that sympl. matrices form a group, called the symplectic group, and we will write it as $Sp(2n, \mathbb{R})$.

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2n, \mathbb{R}) \to M^{-1} = \begin{pmatrix} a^t & -b^t \\ -c^t & a^t \end{pmatrix}$.

What relation exists between Hamilt. and sympl matrices?

Theorem: A is Hamiltonian $\iff e^{At}$ is symplectic $\forall t$.

Corollary: The Lie algebra of the Lie group $Sp(2n, \mathbb{R})$ is $sp(2n, \mathbb{R}) := \{$ Hamiltonian matrices $\}$.

Theorem: 1) The characteristic polynomial of a real Hamiltonian matrix is even: if λ is an eigenvalue, so $-\lambda, \bar{\lambda}, -\bar{\lambda}$ are

2) The char. poly. of a real symplectic matrix is reciprocal: if λ is an eigenvalue, so $\bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}}$ are

Lemma: M symplectic $\Rightarrow \det M = +1$.

LINEAR STABILITY

Definition: We say that $x_0 \in \mathbb{R}^{2n}$ is a equilibrium point of X_H when $(X_H)_{x_0} = 0 \iff d_{x_0} H = 0$.

Around a fixed point, $x = x_0 + z$, H happens to be $H(x_0 + z) = H(x_0) + \frac{1}{2} z^t S z + \mathcal{O}(z^3)$, where S is the Hessian of H, a sym. matrix. Ignoring the constant,

Definition: Given a Hamiltonian H and an eq. point x_0, we'll call linearized system to the one defined by $H(z) = \frac{1}{2} z^t S z$, so $\dot{z} = JSz = Az$.

Definition: Let x_0 be an eq. point. We say that x_0 is linearly unstable if A has (at least) an eigenvalue with positive real part, and we say that x_0 is linearly stable if all eigenvalues lie on the imaginary axis (they do not have real part) and A is diagonalizable.

• Linear stability means that orbits of $\dot{z} = Az$ are bounded.

• Linear instability $\Rightarrow$ non-linear instability; but linear stability $\not\Rightarrow$ non-linear stability

Definition: A linear Hamiltonian system is called parametrically stable if it is stable and the system defined by $\dot{z} = (A + \varepsilon A_1) z$, $\forall A_1$ Hamiltonian, is stable for sufficiently small $\varepsilon > 0$.

Proposition: If the quadratic form $H = \frac{1}{2} z^t S z$ is positive (or negative) defined $\longrightarrow$ $\dot{z} = Az$ is parametric. stable

Theorem (Krein-Gelfand): A Ham. linear system $\dot{z} = Az$ is parametrically stable if and only if

1) All eigenvalues of A are purely imaginary
2) A is non-singular (ie, 0 is not eigenvalue)
3) A is diagonalizable
4) The Hamiltonians H_j are positive (or negative) defined $\forall j$, where if $\pm i\beta_1, \dots, \pm i\beta_s$ are the eigenvalues of A, H_j is the restriction of H to the corresponding eigenspace $V_{\pm i\beta_i}$.

IX

- There's a reduced form for a Hamiltonian matrix

Lemma: Let A be a 2×2 Hamiltonian matrix with eigenvalues (A). Then there exists a real 2×2 symplectic matrix S such that $S^{-1}AS =$ (B) (change of basis); and the Hamiltonian in the new coordinates are, therefore, (C).

	A	B	C
Hyperbolic	$\pm\alpha$, $\alpha>0$	$\begin{pmatrix}\alpha & 0\\ 0 & -\alpha\end{pmatrix}$	$H=\alpha q p$
Elliptic	$\pm i\beta$, $\beta>0$	$\begin{pmatrix}0 & \beta\\ -\beta & 0\end{pmatrix}$ or $\begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}$	$H=\frac{\beta}{2}(q^2+p^2)$ or $H=-\frac{\beta}{2}(q^2+p^2)$

Lemma. Let A be a 4×4 Hamiltonian matrix with eigen values $\pm\gamma\pm\delta i$ (the four possibilities, according with the char. pol proposition), $\gamma\neq 0$, $\delta\neq 0$. Then there exists a real 4×4 sympl. matrix S s.t.

$$S^{-1}AS=\left(\begin{array}{cc|cc}\gamma & -\delta & & \\ \delta & \gamma & \multicolumn{2}{c}{0} \\ \hline & & -\gamma & -\delta \\ \multicolumn{2}{c|}{0} & \delta & -\gamma\end{array}\right)=\begin{pmatrix}B^t & 0\\ 0 & -B\end{pmatrix},\qquad B=\begin{pmatrix}\gamma & \delta\\ -\delta & \gamma\end{pmatrix}$$

and the Hamiltonian is given by $H=\gamma(q_1p_1+q_2p_2)+\delta(q_1p_2-q_2p_1)$.

NONLINEAR ELLIPTIC EQUILIBRIUM

- Consider a Ham. syst. with a point which is an <u>elliptic equilibrium</u>, ie, such that the linearized Ham. sys. takes the form

$$H_0(q,p)=\frac{1}{2}\sum_{k=1}^{n}\omega_k\,(p_k^2+q_k^2)$$

Using symplectic polar coordinates (I,θ) (which are action-angle coord.) we write it as $H_0=\sum_{k=1}^{n}\omega_k I_k$.

Then in this coord. $\omega=\sum d\theta_k\wedge dI_k$ and the integral curves of X_H are given by $\dot{I}_k=0$, $\dot{\theta}_k=\omega_k$, ie,

$I_k(t) = I_k(0)$; $\theta_k(t) = \omega_k t + \theta_k(0)$

<u>Definition</u>: Let $\omega = (\omega_1, \dots, \omega_n)$. We call ω <u>resonant</u> if there exists $l \in \mathbb{Z}^n - 0$ such that $\langle l, \omega \rangle = \sum_{i=1}^{n} l_i \omega_i = 0$. If there doesn't exist such l then ω is called <u>non-resonant</u>

- When ω is non-resonant, the orbit $\theta(t) = (\theta_1(t), \dots, \theta_n(t))$, with $\theta_k(t) = \omega_k t + \theta_k(0)$ is dense on $\mathbb{T}^n$.

- Let $H(q,p) = H_0(q,p) + H_1(q,p) + H_2(q,p) + \dots$ be the Taylor expansion of H around z_0, with $H_s(q,p)$ an homogeneous polynomial of degree $s+2$. We want to construct a (formal) integral of the form $\phi = \phi_0 + \phi_1 + \phi_2 + \dots$ with $\phi_0(q,p) = \frac{1}{2}(q_k^2 + p_k^2) = I_k$; and ϕ_s are determined recursively.

* F pol. of degree r, G pol of degree s $\Rightarrow$ $\{F, G\}$ pol of degree $r+s-2$.

For $\{H, \phi\}$ to be $=0$, we arrange the $\{\cdot,\cdot\}$ terms depending on the degree. Call $\mathcal{L} := \{H_0, -\} = X_{H_0}$ (as derivation). $\mathcal{L}(P_r = \text{pol. of degree } r+2) \subseteq P_r$.

Change coordinates $q = \frac{1}{\sqrt{2}}(z + i w)$, $p = \frac{i}{\sqrt{2}}(z - i w)$, so

<u>Lemma</u>: $\boxed{H_0 = i \sum_{k=1}^{n} \omega_k z_k w_k}$, $\boxed{\mathcal{L}(z^l w^m) = i \langle m - l, \omega \rangle z^l w^m}$

$z^l = (z_1^{l_1}, \dots, z_n^{l_n})$; $w^m = (w_1^{m_1}, \dots, w_n^{m_n})$

<u>Proposition</u>: Let $R_r := \operatorname{Im} \mathcal{L}|_{P_r}$, $N_r := \operatorname{Ker} \mathcal{L}|_{P_r}$. Then $P_r = R_r \oplus N_r$.

If w is non-resonant, $N_r = \operatorname{span}\{ z^l w^m : m = l, \text{ with } \underbrace{l_1 + \dots + l_n + m_1 + \dots + m_n}_{2(l_1 + \dots + l_n)} = r+2 \}$. If r is odd $\Rightarrow$ <u>$N_r = 0$</u>

<u>Proposition</u>: The equation $\mathcal{L}\phi_s = -B_s$ can be solved provided that $B_s \in R_s$. In particular, if $B_s = \sum_{l \neq m} b_{lm} z^l w^m$, then

$$\boxed{\phi_s = \sum_{l \neq m} \frac{i\, b_{lm}}{\langle m - l, \omega \rangle} z^l w^m}$$

Proposition: Let $H = H_0 + H_1 + \dots$ with $H_0 = \langle \omega, I \rangle$ and ω non-resonant, and assume that H is even in the momenta ($H(q,p) = H(q,-p)$). Then there exist n independent formal integrals $\phi^{(1)}, \dots, \phi^{(n)}$ of the form $\phi^{(k)} = \phi_0^{(k)} + \phi_1^{(k)} + \dots$ with $\phi_0^{(k)} = I_k$. This integrals are also in the momenta and they are in involution, $\{\phi^{(k)}, \phi^{(l)}\} = 0$.

Lemma: The Poisson bracket between even/odd functions of the momenta is even/odd as follows:

$\{\cdot,\cdot\}$	even	odd
even	odd	even
odd	even	odd

(Poincaré surface of sections). Consider a system with 2 dof. We define a 2-dim surface. We are interesting in considering trajectories starting at this surface and see when they hit again the surface. In Hamiltonian dyn. the $(k+1)$th piercing only depends on the k-th.

This is construct as follows: fix $H(q_1 p_1 q_2 p_2) = E = \text{const}$. Then the motion is reduced to a 3-dim mnfd (a connected compact M_E of $H^{-1}(E)$). We can now delete p_1, for instance, because we can express it by the remaining three (p_1, q_2, p_2). Now we can fix $q_1 = \text{const}$, obtaining a 2-dim mnfd Σ, called Poincaré surface of sections.

We now define the following map $\phi : \Sigma \to \Sigma$: let $T : \Sigma \to \mathbb{R}$, $T(x) :=$ first positive time such that $\sigma_x(T(x)) = \varphi_H^{T(x)} \in \Sigma$, ie, s.t. $\varphi_H^{T(x)} \in \Sigma$ but $\varphi_H^{t} \notin \Sigma$ for $0 < t < T(x)$. Then $\phi(x)$ is the point of Σ in which the orbit of x hit first in positive time, $\phi(x) := \varphi_H^{T(x)}(x)$. ϕ is called the Poincaré map.

Proposition: The Poincaré map is symplectic, $\phi^* \rho = \rho$, with $\rho = \omega|_\Sigma$, ω standard sympl. form of $\mathbb{R}^{2n}$

X

Setup: Consider a Hamiltonian $H = H_0 + H_1$ even in the momenta and construct $\phi = I_k + \phi_1 + \phi_2 + \dots$ a formal integral. We assume the non-resonance of ω: $\forall s \geq 1$, $|\langle k, \omega\rangle| \geq \alpha_s$ for $k \in \mathbb{Z}^n$, with $0 < |k| \leq s+2$ $\quad (|k| = |k_1| + \dots + |k_n|)$.

Definition: Let $f = \sum_{j+k=s} f_{jk}\, x^j y^k$, $x^j = x_1^{j_1} \cdots x_n^{j_n}$, $y^k = y_1^{k_1} \cdots y_n^{k_n}$, be a homogeneous polynomial of degree s. The **norm** of f is $\|f\| := \sum_{j+k=s} |f_{jk}|$.

Lemma: Let $\Delta_\rho = \{(x,y) \in \mathbb{R}^{2n} : x_l^2 + y_l^2 \leq \rho^2,\ l = 1, \dots, n\}$. Then in Δ_ρ we have $|f(x,y)| \leq \rho^s \|f\|$.

Lemma: The transformation from variables q, p to z, w changes the norm of a hom. pol of degree s at most $\sqrt{2}^s$.

Lemma: $\|\{f, g\}\| \leq sr\, \|f\|\, \|g\|$.

Lemma: The unique solution ϕ_s belonging to R_s of the equation $\mathcal{L}\phi_s = \psi_s$ satisfies $\|\phi_s\| \leq \frac{1}{\alpha_s}\|\psi_s\|$

Lemma: $\|\phi_s\| \leq a b^{s-1} \dfrac{(s+1)!}{\prod_{l=1}^{s} \alpha_l}$, $s \geq 1$.

Proposition: For any integer $r \geq 1$, there exist constants d_r, C_r: for ρ suf. small and for any orbit with $|I_l(0)| < \frac{2\rho^2}{9}$ $\forall l$ (i.e., the initial condition in $\Delta_{2\rho/3}$) we have

$$|I_l(t) - I_l(0)| < 3 d_r \rho^3 \qquad , \text{ for } |t| < T_r = \frac{d_r}{C_r \rho^r}.$$

Theorem: Consider the system $H = H_0 + H_1$, with $H_0 = \langle \omega, I\rangle$, and assume that ω satisfies the non-resonant condition $|\langle k, \omega\rangle| > \gamma |k|^{-\tau}$, for $k \in \mathbb{Z}^n$, $\gamma > 0$, $\tau \geq 0$. Then there exist $\beta_1, \beta_2 > 0$ and ρ^* such that, if $\rho < \frac{1}{3^{\tau+1}}\rho^*$, then for any orbit with initial point $(x(0), y(0)) \in \Delta_{2\rho/3}$ we have $|I_l(t) - I_l(0)| < \beta_1 \rho^2$ for $|t| < T^* = \beta \exp\left[\left(\frac{\rho^*}{\rho}\right)^{\frac{1}{\tau+1}}\right]$.

NEAR-IDENTITY CHANGE OF COORD.

Definition : A near-identity symplectic transformation is a transformation $\phi(\varepsilon, z) = z + O(\varepsilon)$ which is symplectic for each fixed ε.

Proposition : There is a 1 to 1 correspondence between near-id. sympl. transf. and time dependent Haml. vector fields (the correspondence is flow $\longrightarrow$ v.f.).

FORWARD ALGORITHM . LIE SERIES

Consider a near-id. sym. tf. $\phi \longleftrightarrow X_W$ for some function $W = W(\varepsilon, z)$. Let $H = H(\varepsilon, z)$ be a Hamiltonian and $G(\varepsilon, z) := H(\varepsilon, \phi(\varepsilon, z))$ the Hamiltonian in the new coordinates (G is called the Lie transform of H generated by W). Suppose that H, G, W has series expansion in ε :

$$H(\varepsilon, z) = H_0 + \varepsilon H_1 + \frac{\varepsilon^2}{2} H_2 + \dots$$

$$G(\varepsilon, z) = G_0 + \varepsilon G_1 + \frac{\varepsilon^2}{2} G_2 + \dots$$

$$W(\varepsilon, z) = W_1 + \varepsilon W_2 + \frac{\varepsilon^2}{2} W_3 + \dots$$

We want to obtain the new Hamilt. G. We use Lie series, a recursive set of formulas which relate these terms. This method introduces a set of functions $\{H_k^i\}_{i,k}$ so that $H_k = H_k^0$ and $G_i = H_0^i$:

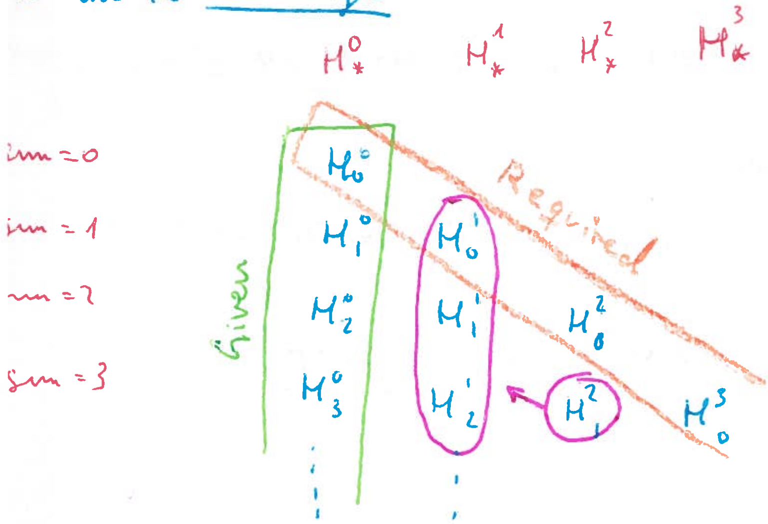
Theorem : The coefficients $\{H_k^i\}$ satisfy the recursive identities

$$\boxed{H_k^i = H_{k+1}^{i-1} + \sum_{s=0}^{k} \binom{k}{s} \{H_{k-s}^{i-1}, W_{s+1}\}}$$

and $G_i = H_0^i$; ie, G is $\boxed{G(\varepsilon, z) = \sum_{n=0}^{\infty} \left(\frac{\varepsilon^n}{n!}\right) H_0^n(z)}$

- This equalities give an algorithm to compute $G(\varepsilon, z)$. How to implement it?

We use the Lie triangle.



The latter formulae say that, to compute an element, we only need the entries of the left column and up, ie, for H_k^i we need (see the formula) $H_{k+1}^{i-1}, H_k^{i-1}, \dots, H_0^{i-1}$.

NORMALIZATION ALGORITHM

In many cases, the Hamiltonian is given, and we want to seek a change of variables to simplify it. When the Hamiltonian is in a sufficient simple form, it is said to be in "normal form".

Suppose that we follow the Lie transform algorithm using given generators $W_1, \dots, W_{n-1}$. We want to find W_n so that the new Hamilt. is in the simplest form. Let L_j^i the elements of the Lie triangle computed with $W_n = 0$, so

$$
\begin{array}{lll}
L_0^0 & & \\
L_1^0 & L_0^1 & \ddots \\
L_2^0 & L_1^1 & \\
\vdots & \vdots & \\
L_{i+j=n-1} & L^1_{i+j=n-2} & \\
L^0_{i+j=n} & \vdots &
\end{array}
\implies
$$

for a general W_n it must hold

$$
\begin{cases}
H_j^i = L_j^i , & i+j < n \quad (\text{rows sum}=0,\dots,n-1) \\
H_j^i = L_j^i + \{H_0^0, W_n\}, & i+j = n \quad (\text{row } n)
\end{cases}
$$

ie, W_n does not affect the terms in the first $n-1$ rows.

To obtain W_n so that a (unknown) H_0^n is in normal form, we need to solve

$$\boxed{\mathcal{L} W_n + H_0^n = L_0^n} \qquad \mathcal{L} = \{-, H_0^0\}$$

for W_n and H_0^n. It is called the homological equation.

NORMAL FORM AT AN EQUILIBRIUM

• **Setup**: Consider the case of an equilibrium of X_H, so that we can write $H = H_0 + \varepsilon H_1 + \frac{\varepsilon^2}{2} H_2 + \dots$, with $H_0 = \frac{1}{2} z^t S z$. Suppose $A = JS$ is diagonalizable

• Problem with the last lines: how the fuck to solve $\mathcal{L} W_n + H_0^n = L_0^n$ for both W_n and H_0^n ?

• Consider the case where $H_0 = \langle \omega, I \rangle = i \sum \omega_k z_k w_k = \frac{1}{2} \sum \omega_k (q_k^2 + p_k^2)$.

where z_k, w_k are complex coord. with

$$\{ z_k, w_\ell \} = \delta_{k\ell} \quad , \qquad \{ z_k, z_\ell \} = \{ w_k, w_\ell \} = 0 .$$

and ω non-resonant. Let $P = \mathbb{R}[z_1, \dots z_n, w_1, \dots w_n] = \mathbb{R}[z, w]$, and $\mathcal{L} = \{ \cdot, H_0^n \} : P \to P$.

Lemma: $\operatorname{Ker} \mathcal{L} = \operatorname{span} \{ z^\ell w^\ell \}$, $\operatorname{Im} \mathcal{L} = \operatorname{span} \{ z^\ell w^m : \ell \neq m \}$ $\Rightarrow$ $P = \operatorname{Ker} \mathcal{L} \oplus \operatorname{Im} \mathcal{L}$

Corollary (Very useful): To solve the homological equation, we can take H_0^n to be the terms of L_0^n in $\operatorname{Ker} \mathcal{L}$, i.e., terms in the with $z^\ell w^\ell$. (So in this way we can solve $\mathcal{L} W_n = L_0^n - H_0^n \in \operatorname{Im} \mathcal{L}$.) Moreover, $z^\ell w^\ell \sim I^\ell = I_1^{\ell_1} \dots I_n^{\ell_n}$, so the final series G will be a function of the actions only

$$G = \sum \omega_k I_k + H_2(I) + H_4(I) + \dots ,$$

called the Birkhoff normal form. Moreover, the system will be integrable with integrals $I_1, \dots I_n$.

XI

PERSISTENCE OF INVARIANT TORI

- Consider an integrable Hamiltonian system $h = h(I)$ in action-angle coordin. What happens if we perturbate the Hamiltonian to $H(I,\varphi) = h(I) + \varepsilon f(I,\varphi,\varepsilon)$? The KAM (Kolmogorov-Arnold-Moser) theorem predicts that a large measure of these tori persists for small $\varepsilon > 0$.

<u>Definition</u>: We say that a Hamiltonian is in the <u>Kolmogorov normal form</u> if it takes the form

$$H(I,\varphi) = \langle \omega, I \rangle + R(I,\varphi),$$

where R is at least quadratic in the actions.

- We'll show that a Hamiltonian $H = h(I) + \varepsilon f(I,\varphi,\varepsilon)$ can be brought into the Kolmogorov n.f.

("<u>Step 1</u>"): Under the condition $\det \frac{\partial^2 h}{\partial I^2} \neq 0$, choose I^* : $\omega = \frac{\partial I}{\partial I}(I^*)$ satisfies the diophantine equation $|\langle k, \omega \rangle| \geq \gamma |k|^{-\tau}$ $\forall k \in \mathbb{Z}^n - 0$, for some const. $\gamma > 0$, $\tau > n-1$. Translate I^* to 0 and Taylor:

$$\boxed{H(I,\varphi) = \langle \omega, I \rangle + A(\varphi) + \langle B(\varphi), I \rangle + \frac{1}{2} \langle C(\varphi) I, I \rangle + g(I,\varphi)}$$

with g cubic in the actions. Kolmog. n.f. is $A(\varphi) = B(\varphi) = 0$.

A canonical transformation will push the size of $A(\varphi), B(\varphi)$ from $O(\varepsilon)$ to $O(\varepsilon^2)$. We'll repeat the transf. and prove that the sequence of transformations converges.

<u>Theorem</u>: Consider the Hamiltonian $H(I,\varphi) = \frac{1}{2} \langle S I, I \rangle + \varepsilon f(I,\varphi,\varepsilon)$ on $\mathbb{R}^n \times \mathbb{T}^n$, where S is a real symmetric matrix and $f(I,\varphi,\varepsilon)$ a pol. of degree at most 2 in I, and analytic for φ and ε small. Assume S bounded and non-singular. For $\varepsilon = 0$, let I^* be an unperturbated torus with frequency $\omega = S I^*$ satisfying $|\langle k, \omega \rangle| \geq \gamma |k|^{-\tau}$, $\gamma > 0$, $\tau > n-1$ $\forall k \in \mathbb{Z}^n - 0$.

Then, there exists $\varepsilon_* > 0$: $\forall$ $|\varepsilon| < \varepsilon_*$ the Hamiltonian $H(I,\varphi)$ possesses an invariant torus ε-close to I^*, $0 < a < 1$, and the flow on the torus is quasi-periodic with frequency vector ω.

Proposition: There* are canonical transformations bringing $H(I,\varphi) = \langle \omega, I\rangle + A(\varphi) + \langle B(\varphi), I\rangle + \frac{1}{2}\langle C(\varphi) I, I\rangle$ to the form $H'(I,\varphi) = \langle \omega, I\rangle + A'(\varphi) + \langle B'(\varphi), I\rangle + \frac{1}{2}\langle C'(\varphi) I, I\rangle$, where

$$A' = \exp\left(L_{\langle Y,\varphi\rangle} \hat{A}\right), \qquad \hat{A} = \frac{1}{2}\left\langle C\left(\frac{\partial X}{\partial \varphi} + \xi\right), \left(\frac{\partial X}{\partial \varphi} + \xi\right)\right\rangle + \left\langle B, \left(\frac{\partial X}{\partial \varphi} + \xi\right)\right\rangle,$$

$$\langle B'(\varphi), I\rangle = \sum_{j\geq 1} \frac{j}{(j+1)!} L^{j}_{\langle Y,\varphi\rangle} \langle \hat{B}, I\rangle, \qquad \hat{B} = B + C\left(\frac{\partial X}{\partial \varphi} + \xi\right)$$

$$\langle C'(\varphi) I, I\rangle = \langle C(\varphi) I, I\rangle + \sum_{j\geq 1} \frac{1}{j!} L^{j}_{\langle Y,\varphi\rangle} \langle C I, I\rangle,$$

where X, Y, ξ are solution to

$$\partial_\omega X + A = 0, \qquad \partial_\omega Y + \hat{B} = 0, \qquad \overline{C}\xi + \overline{B + C\frac{\partial X}{\partial \varphi}} = 0$$

provided that $\overline{A} = 0$, $\overline{\hat{B}} = 0$

Lemma: let $\partial_\omega := \{-, \langle \omega, I\rangle\} = \langle \omega, \partial_\varphi(-)\rangle$. The equation $\partial_\omega W = g$ has solution only if $\overline{g} = 0$. Moreover, it is unique through the choice $\overline{W} = 0$.

: Here we need to use the multivariable Fourier series;

$$g(I,\varphi) = \sum_{k\in\mathbb{Z}^n} g_k(I)\, e^{i\langle k,\varphi\rangle}, \qquad \text{where } g_k(I) = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} g(I,\varphi)\, e^{-i\langle k,\varphi\rangle}\, d\varphi.$$

are the Fourier coefficients

Lemma (Iterative): Let $H(I,\varphi)$ be of the form

$$H = \langle \omega, I\rangle + A + \langle B, I\rangle + \frac{1}{2}\langle CI, I\rangle \qquad , \qquad A,B,C = A(\varphi), B(\varphi), C(\varphi)$$

and assume:

1) $\exists\, \sigma, \varepsilon > 0 \;:\; \max(\|A\|_\sigma, \|B\|_\sigma) \le \varepsilon$

2) $\exists\, 0 < m \le 1 \;:\; m|x| \le |\bar{C}x| \quad \forall x \in \mathbb{R}^n$.

3) $\forall\; w = w(\varphi)$ vector valued with $\|w\|_\sigma < \infty \Rightarrow \|Cw\|_\sigma \le \frac{1}{m}\|w\|_\sigma$.

4) $|\langle k, \omega\rangle| \geq \gamma |k|^{\tau} \quad \forall k \in \mathbb{Z}^n - 0$, for some $\gamma > 0$, $\tau > n-1$.

Let $0 < d \le \frac{1}{6}$ and $\sigma_* > 0 : (1-3d)\sigma \geq \sigma_*$. Then there exists $\Lambda > 0$: if $m > 0$ is such that $\frac{\Lambda \varepsilon}{m^5 d^{3\tau+4}} \le 1$, then there exists a canonical transformation $(I,\varphi) = \Psi(\varphi', I')$ with

$$|I - I'| \le \frac{\Lambda\varepsilon}{m^2 d^{2\tau+2}} \frac{\rho}{\sigma} < d\cdot\rho \quad , \quad |\varphi - \varphi'| \le \frac{\Lambda\varepsilon}{m^2 d^{2\tau+2}} < d\cdot\sigma$$

$(I', \varphi') \in \mathbb{T}^n_{(1-2d)\sigma} \times \Delta_{(1-2d)\rho}$ which brings the Hamiltonian to the form

$$H = \langle \omega, I\rangle + A' + \langle B', I\rangle + \frac{1}{2}\langle C' I, I\rangle$$

satisfying 1), 2), 3) with other $\varepsilon', \sigma', m'$.

XII

• (Norms) : We will consider the following norms :

1) For $v \in \mathbb{R}^n$, $|v| := \sum_{j=1}^{n} |v_j|$ (the norm 1)

2) For $f : \mathbb{T}^n \to \mathbb{R}$ analytic, $\|f\|_\sigma := \sum_{k \in \mathbb{Z}^n} |f_k| e^{|k|\sigma}$, $\sigma > 0$ (it is a family of norms).

Here f_k are the coefficients of the Fourier series of f. It is called weighted Fourier norm

3) For $g : \mathbb{T}^n \to \mathbb{R}^n$ vector valued, $g = (g_1, \dots, g_n)$, $\|g\|_\sigma := \sum_{j=1}^{n} \|g_j\|_\sigma$.

• (More stuff) : Also consider:

4) $\Delta_\rho(0) := \{ I \in \mathbb{C}^n : |I_j| < \rho \ \forall j = 1..n \}$

5) $\mathbb{T}^n_\sigma = \{ \varphi \in \mathbb{C}^n : |\operatorname{Im} \varphi_j| < \sigma \ \forall j = 1..n \}$

6) For $f : \Delta_\rho(0) \to \mathbb{R}$, $|f|_\rho := \sup_{I \in \Delta_\rho(0)} |f(I)|$

7) For $g : \Delta_\rho(0) \times \mathbb{T}^n_\sigma \to \mathbb{R}$, $|g|_{\rho,\sigma} := \sup_{(I,\varphi) \in \Delta_\rho(0) \times \mathbb{T}^n_\sigma} |g(I,\varphi)|$

8) For $f : \Delta_\rho(0) \times \mathbb{T}^n_\sigma \to \mathbb{R}$, with $f_k = f_k(I)$ the Fourier coeff, set $\|f\|_{(\rho,\sigma)} := \sum_{k \in \mathbb{Z}^n} |f_k|_\rho e^{|k|\sigma}$

Lemma: Consider the domain $\Delta_\rho(0) \times \mathbb{T}^n_\sigma$. Let $w(\varphi), v(\varphi)$ analytic vector functions and $C(\varphi)$ a $n \times n$ matrix with entries $c_{ij}(\varphi)$ analytic. Then

1) $\| \langle w, I \rangle \|_{(\rho,\sigma)} \le \rho \|w\|_\sigma$

2) $\| \langle w, I \rangle \|_{(\rho,\sigma)} \le D \cdot \rho$ for some $D > 0$ $\Rightarrow$ $\|w\|_\sigma \le n \cdot D$

3) $\| \langle C(\varphi) I, I \rangle \|_{(\rho,\sigma)} \le D \cdot \rho^2 \Rightarrow \|c_{ij}\|_\sigma \le D$

4) $\| \langle w, v \rangle \|_\sigma \le \|w\|_\sigma \|v\|_\sigma$ (Cauchy - Schwarz)

5) $\|\bar{f}\|_\sigma \le \|f\|_\sigma$, and $\|f - \bar{f}\| \le \|f\|_\sigma$.

<u>Lemma</u>: Set $g(I,\varphi)$ with $\bar{g}(I)=0$, $\|g\|_{(1-\delta)(\rho,\sigma)}$ bounded for $0\leq\delta<1$, and ω diophantine. Set $f(I,\varphi)$, with $\bar{f}(I)=0$ which solves $\partial_\omega f=g$. Then for all $0<\alpha<1-\delta$,

1) $\|f\|_{(1-\delta-\alpha)(\rho,\sigma)} \leq \frac{1}{\gamma}\left(\frac{\tau}{e\alpha\sigma}\right)^{\tau} \|g\|_{(1-\delta)(\rho,\sigma)}$

2) $\left\|\frac{\partial f}{\partial \varphi}\right\|_{(1-\delta-\alpha)(\rho,\sigma)} \leq \frac{1}{\gamma}\left(\frac{\tau+1}{e\alpha\sigma}\right)^{\tau+1} \|g\|_{(1-\delta)(\rho,\sigma)}$

We construct an infinite sequence of canonical transformations $\Psi^{(k)}$, such that $(\varphi^{(k-1)},I^{(k-1)})=\Psi^{(k)}(\varphi^{(k)},I^{(k)})$ and with their respective ε_k, σ_k, m_k, α_k, ... It turns out that $\frac{\varepsilon_k}{\varepsilon_0}=\left(\frac{1}{k+1}\right)^{2(3\tau+4)}$, so $\varepsilon_k\to0$ when $k\to\infty$.

If $\rho_k=(1-3\alpha_k)\rho_{k-1}$, then the transformation $\Psi^{(k)}:\Delta_{\rho_k}(0)\times\mathbb{T}^n_{\sigma_k}\longrightarrow\Delta_{\rho_{k-1}}(0)\times\mathbb{T}^n_{\sigma_{k-1}}$ is analytic and so is any finite composite $\hat{\Psi}^{(k)}:=\Psi^{(1)}\circ\dots\circ\Psi^{(k)}:\Delta_{\rho_k}(0)\times\mathbb{T}^n_{\sigma_k}\longrightarrow\Delta_{\rho_0}(0)\times\mathbb{T}^n_{\sigma_0}$

The sequence $\hat{\Psi}^{(k)}$ converges to $\Psi^{(\infty)}:\Delta_{\rho_0/2}(0)\times\mathbb{T}^n_{\sigma_0/2}\to\Delta_{\rho_0}(0)\times\mathbb{T}^n_{\sigma_0}$ with absolute convergence $\Rightarrow$ uniform convergence in compacts $\Rightarrow$ analyticity of $\Psi^{(\infty)}$.

The sequence of Hamiltonians $H^{(k)}=H^{(0)}\circ\Psi^{(k)}$ converges to $H^{(\infty)}=H^{(0)}\circ\Psi^{(\infty)}$ and by construction it is in normal form.

XIII

NEKHOROSHEV THEORY

• Consider a Hamiltonian of the form $H(\varphi, I) = h(I) + \varepsilon V(\varphi)$, $(\varphi, I) \in \mathbb{T}^n \times \mathbb{R}^n$, where

$$h(I) = \frac{1}{2}\sum_{j=1}^{n} I_j^2 \qquad , \qquad V(\varphi) = \sum_{|k|\leq K} v_k \, e^{i\langle k,\varphi\rangle}$$

with $K \geq 2$. Assume that $\mathrm{span}_{\mathbb{Z}}\{k : |k| \leq K\}$ has dim n.

• Let $S_\varepsilon := \{(I,\varphi) \in \mathbb{R}^n \times \mathbb{T}^n : H(\varphi, I) = \varepsilon\}$. For $\varepsilon = 0$ it is a $(n-1)$-dim sphere in the action space. For small ε it is a topological sphere (smoothly $\simeq$ sphere). In particular, for $\varepsilon = \frac{1}{2}$, it will lie in a spherical shell around S_1.

<u>Theorem</u>: Consider the Hamiltonian $H(\varphi, I) = h(I) + \varepsilon V(\varphi)$ in a domain $D \times \mathbb{T}^n$, $D = \{I \in \mathbb{R}^n : \|I\|^2 \leq 1 + 2\varepsilon\}$ and assume that $V(\varphi)$ is bounded, $\sup_{\varphi \in \mathbb{T}^n} |V(\varphi)| \leq 1$. Then there exist positive constants μ_*, T_* depending on ε, K, n such that, if $8\lambda\mu_*\varepsilon < 1$, then for every orbit $(I(t), \varphi(t))$ with $I(0) \in D$

$$\| I(t) - I(0) \| \leq (\mu_* \varepsilon)^{1/4}$$

for all t : $|t| \leq \frac{T_*}{\varepsilon} \exp\left[\left(\frac{1}{\mu_*\varepsilon}\right)^{1/a}\right]$, $a = 2(n^2+n)$.

• To solve the homological equation $L_{H_0} W_s + Z_s = g_s$, we need $g_k = w_k \, i\langle k, I\rangle + z_k$.

- If for $k \in \mathbb{Z}^n$ $\exists\, I : \langle k, I\rangle = 0 \Rightarrow$ set $w_k = 0$, $z_k = g_k$.
- Otherwise set $z_k = 0$, $w_k = \frac{g_k}{i\langle k, I\rangle}$.

<u>Definition</u>: The <u>normal form Hamiltonian</u> is

$$Z^{(r)}(I,\varphi) = h(I) + Z_1(I,\varphi) + \cdots + Z_r(I,\varphi) + R^{(r+1)}(I,\varphi)$$

where $Z_s(I,\varphi)$ contains only harmonics for which $\langle k, I\rangle \neq 0$ $\forall I$ in the domain.

denote F_K the class of functions which don't contain Fourier harmonics with $|k| > K$.

Lemma: $f \in F_K$, $g \in F_{K'}$ $\Rightarrow$ $f+g \in F_{\max(K,K')}$, $fg \in F_{K+K'}$, $\{f,g\} \in F_{K+K'}$.

Definition: a) A resonance module is a subgroup $M \subset \mathbb{Z}^n$: $\mathrm{span}_{\mathbb{R}} M \cap \mathbb{Z}^n = M$

b) A function $Z(I,\varphi)$ is in normal form w.r.t. M if its Fourier expansion only has harmonics with $k \in M$

$$Z(I,\varphi) = \sum_{k\in M} Z_k(I)\, e^{i\langle k,\varphi\rangle}.$$

c) A set $V \subset \mathbb{R}^n$ is a non-resonance domain of type (M, α, δ, N) if $|\langle k, \underbrace{\omega(I)}_{I \text{ in our problem}}\rangle| > \alpha$ for all $I \in V_\delta$, $k \in \mathbb{Z}^n - M$, $|k| \le N$

Proposition: Let $V \subset \mathbb{R}^n$ be a non-resonant domain of type (M, α, δ, rK), where r is the order of the normalization and K the one appearing in the summation. Then there are constants A, D depending on V s.t. if $\mu = \frac{Ar\varepsilon}{\alpha\delta} \le \frac{1}{2}$, then $H = h(I) + \varepsilon V(\varphi)$ can be transformed to the normal form, with $Z_s(I,\varphi)$ in normal form w.r.t. M and $Z_s \in F_{sK}$. Moreover, for any $\lambda \in \mathbb{R}^n$: $|\lambda| = 1$, and $\lambda \perp M$ (i.e. $\langle \lambda, k\rangle = 0$ $\forall k \in M$) $\exists$ a first integral of $H^{(r)} - R^{(r)} = h + Z_1 + \dots + Z_r$ given by $\phi_0 = \langle \lambda, I\rangle$.

Definition: Let $I \in V$ and let $\Pi_M(I) := \{ I' \in V : I - I' \in \mathrm{span}_{\mathbb{R}} M \}$. Then for initial condition $I(0)$

$$\Pi_M(I(0)) = I(0) + \mathrm{span}_{\mathbb{R}} M$$

is called the plane of fast drift

Definitions: Let $N \in \mathbb{N}$, and let $s := \dim_{\mathbb{Z}} M$

a) Let $M \subset \mathbb{Z}^n$ be a resonance module and let $M_N := \{ k \in M : |k| \le N \}$. We say that M is an N-module if M_N contains $s = \dim_{\mathbb{Z}} M$ independent vectors.

b) Let M be a N-module of $\dim M = s$. We associate numbers called resonance parameters

$$0 < \beta_0 < \beta_1 < \dots < \beta_n \quad , \quad 0 < \delta_1 < \dots < \delta_n$$

c) $\Sigma_M := \{ I \in \mathbb{R}^n : \langle k, I \rangle = 0 \ \forall k \in M \}$ is the resonant manifold

d) $Z_M := \{ I \in \mathbb{R}^n : |\langle k, I \rangle| < \beta_s$ for s independent $k \in M \}$ is a resonant zone of multiplicity s

e) $Z_s^* := \bigcup_{M : \dim M = s} Z_M$ are the resonant regions of multiplicity s $\quad (1 \leq s \leq n)$

f) $B_M := Z_M - Z^*_{s+1}$ are the resonant blocks.

g) Given $I \in B_M$, $\Pi_M(I) := I + \mathrm{span}_{\mathbb{R}} M$ is the resonant plane

h) $\Pi_{M,\delta_s}(I) := \{ I' \in \mathbb{R}^n : \mathrm{dist}(I', \Pi_M(I)) < \delta_s \}$ are the extended resonant planes

i) The connected component of $\Pi_{M,\delta_s}(I) \cap Z_M$ containing I is the cylinder $C_{M,\delta_s}(I)$.

The base of the cylinder is $\partial Z_M \cap C_{M,\delta_s}(I)$.