Relations between quantum integers

There are three popular definitions of quantum integers in the literature:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + q^{n-5} + \cdots + q^{-n+3} + q^{-n+1}$$

$$\langle n \rangle_q := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = q^{\frac{n-1}{2}} + q^{\frac{n-3}{2}} + q^{\frac{n-5}{2}} + \cdots + q^{\frac{-n+3}{2}} + q^{\frac{-n+1}{2}}$$

$$\{n\}_q := \frac{1 - q^n}{1 - q} = q^{n-1} + q^{n-2} + \cdots + q + 1 .$$

Lemma. We have

1) $[n]_q = \langle n \rangle_{q^2}$

2) $[n]_q = q^{-n+1} \{n\}_{q^2}$

3) $\langle n \rangle_q = q^{\frac{-n+1}{2}} \cdot \{n\}_q$

Pf. 1) By def

2) $$q^{-n+1}\{n\}_{q^2} = q^{-n+1}\left(q^{2n-2} + q^{2n-4} + \cdots + q^2 + 1\right) = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$$

$$= [n]_q$$

3) Direct from 1) & 2). $\square$

- Now for $\| = [\,,\ \langle$ or $\{$ let $\|n\|_q! := \|n\|_q \cdot \|n-1\|_q \cdots \|2\|_q \cdot \|1\|_q$.

<u>Lemma</u>. We have

1) $[n]_q! = \langle n \rangle_{q^2}!$

2) $[n]_q! = q^{\frac{-n(n-1)}{2}} \{n\}_{q^2}!$

3) $\langle n \rangle_q! = q^{\frac{-n(n-1)}{4}} \{n\}_q!$

<u>Pf</u>. 1) By def

2) $$[n]_q! = q^{-n+1} \cdot q^{-(n-1)+1} \cdots q^{-2+1} q^{-1+1} \{n\}_{q^2}!$$
$$= q^{n} q^{-n} q^{-(n-1)} \cdots q^{-2} \cdot q^{-1} \{n\}_{q^2}!$$
$$= q^{-\sum_{i=1}^{n-1} i} \{n\}_{q^2}!$$
$$= q^{-\frac{(n-1)n}{2}} \{n\}_{q^2}!$$

3) Follows from the previous lemma. $\square$