Notes on Representation Theory*

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Contents

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These notes on the representation theory of algebras, finite groups and semisimple Lie algebras were written during the summer of 2024. Sections 1–5 are taken from

- Etingof et al's lecture notes

The rest of the notes about semisimple Lie algebras, their classification, and their representation theory is taken from several sources:

- Hazewinkel-Gubareni-Kirichenko's book,
- these lecture notes by Kiyosi Igusa,
- Samelson's book,
- Humphrey's book,
- Fulton-Harris's book,
- Knapp's book.

1 Basics of Representation Theory

Let A be an algebra over some field $\Bbbk$ (of characteristic zero unless otherwise stated). A *representation* of A is a left A-module. A morphism or intertwiner between representations is an A-module homomorphism.

Example 1.1. The algebra A endowed with the left multiplication A-module structure is called the *(left) regular representation.*

Example 1.2. Given two A-modules V_1, V_2, the direct sum $V_1 \oplus V_2$ has a canonical A-module structure.

A representation $V \neq 0$ is called *irreducible* or *simple* if its only subrepresentations (aka A-submodules) are 0 and V. It is *indecomposable* if it is not isomorphic to the direct sum of two nonzero representations. Obviously, irreducible implies indecomposable, but in general, the converse is not true. A representation is called *semisimple* or *completely reducible* if it is a direct sum of simple ones.

Remark 1.3. Note that every finite dimensional representation V of an algebra contains an irreducible subrepresentation. Indeed, if V is simple, done. If not, it contains a proper subrepresentation $0 \neq W \subsetneq V$, of dimension strictly lower (but non-zero). If this is simple, done; else, and the process finishes by finite dimensionality.

Lemma 1.4 (Schur). *Let $\varphi : V \to W$ be a non-zero A-module homomorphism between two A-modules (over any field).*

1. *If V is simple, then φ is injective.*

2. *If W is simple, then φ is surjective.*

3. *If V and W are simple, then φ is an isomorphism.*

Proof. The proof is straightforward:

1. Since V is simple, its only submodules are $\{0\}$ and V itself. If φ were not injective, then its kernel would be a non-zero submodule of V, contradicting the simplicity of V. Therefore, φ must be injective.

2. Similarly, if W is simple and φ were not surjective, then the image of φ would be a proper submodule of W, contradicting the simplicity of W. Hence, φ is surjective.

3. If both V and W are simple, then φ is both injective and surjective, hence an isomorphism.

$\square$

Corollary 1.5 (Schur for algebraically closed fields)**.** *If $\Bbbk$ is algebraically closed, any A-module endomorphism $\varphi : V \to V$ of a simple A-module V must be $\varphi = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \Bbbk$.*

Proof. Suppose λ is an eigenvalue of φ. Then $\varphi - \lambda \cdot \mathrm{Id}$ is an A-module map. Since $\Bbbk$ is algebraically closed, λ must be a root of the characteristic polynomial of φ. Consequently, $\varphi - \lambda \cdot \mathrm{Id}$ is not an isomorphism because its determinant is zero. By Schur's lemma, $\varphi - \lambda \cdot \mathrm{Id}$ must be zero, so $\varphi = \lambda \cdot \mathrm{Id}$. $\square$

Corollary 1.6. *If A is commutative over an algebraically closed field, any simple A-module is 1-dimensional.*

Proof. Let V be a simple A-module. Since A is commutative, multiplication by any $a \in A$ is an intertwiner. By the previous corollary, multiplication by a must be scalar multiplication. Hence, every linear subspace of V is a submodule. Since V is simple, it must be 1-dimensional. $\square$

Example 1.7. For $A = \Bbbk[x]$, simple $\Bbbk[x]$-modules are 1-dimensional (as it is commutative). Indecomposable $\Bbbk[x]$-modules are determined by the Jordan normal forms. This example also shows that indecomposable does not imply irreducible even over algebraically closed fields.

Lemma 1.8. *Any finite dimensional A-module has a simple submodule (this does not hold for infinite dimensional modules).*

Given algebras A, B, an (A, B)-bimodule is a vector space V which is a left A-module and a right B-module, and both modules are compatible in the sense that $(av)b = a(vb)$. If V is an (A, B)-bimodule and W is a (B, C)-bimodule, then $V \otimes_B W$ is a (A, C)-bimodule in a natural way.

2 Representation Theory of $\mathfrak{sl}_2$

If $\mathfrak{g}$ is a Lie algebra, by a representation of $\mathfrak{g}$ we mean a representation of $U(\mathfrak{g})$, its universal enveloping algebra.

We will focus now on the complex Lie algebra

$$\mathfrak{sl}_2 := \{M \in \mathcal{M}_2(\mathbb{C}) : \mathrm{tr}(M) = 0\}.$$

Clearly this Lie algebra is generated by the matrices

$$X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the reader can easily check that they satisfy the relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H, \tag{1}$$

where $[u, v] := uv - vu$. Therefore, its universal enveloping algebra $U(\mathfrak{sl}_2)$ can be described as the quotient of the free $\mathbb{C}$-algebra generated by X, Y, and H subject to the relations above.

Given a finite-dimensional $U(\mathfrak{sl}_2)$-module V, a *weight vector* of *weight* $\lambda \in \mathbb{C}$ is a nonzero eigenvector of H with eigenvalue λ. The eigenspace $V[\lambda] := \ker(H - \lambda \mathrm{Id})$ is called a *weight space*. A weight λ is called a *highest weight* if

$$\mathrm{Re}\ \lambda \geq \mathrm{Re}\ \lambda'$$

for any other weight λ'. Equivalently, λ is a highest weight vector if $Xw = 0$. Note that every finite-dimensional representation W of $U(\mathfrak{sl}_2)$ has a highest weight vector. Indeed, since W is finite-dimensional and we are working over $\mathbb{C}$, the operator H has some eigenvector w with $Hw = \lambda w$. If $w \in \ker X$, we are done; else consider the sequence of vectors $(X^n w)_n$. It is easy to see from the relations defining $U(\mathfrak{sl}_2)$ that

$$H(X^n w) = X^n (H + 2n) w = (\lambda + 2n)(X^n w),$$

so we get a sequence of eigenvectors of H with different eigenvalues. This implies that there must be some n such that $X^n w \neq 0$ and $X^{n+1} w = 0$, as W is finite-dimensional. In this case $X^n w$ is the desired element.

It is easy to see from the defining relations of $U(\mathfrak{sl}_2)$ that for every $U(\mathfrak{sl}_2)$-module V, we have

$$X \cdot V[\lambda] \subset V[\lambda + 2], \quad Y \cdot V[\lambda] \subset V[\lambda - 2].$$

Lemma 2.1. *If V is a $U(\mathfrak{sl}_2)$-module, let λ be a highest weight and $v_0 \in V[\lambda]$ a highest weight vector. Set*

$$v_p := \frac{1}{p!} Y^p v_0, \quad k \geq p.$$

Then

$$H v_p = (\lambda - 2p) v_p, \quad X v_p = (\lambda - p + 1) v_{p-1}, \quad Y v_p = (p + 1) v_{p+1}.$$

Proof. To prove this, note that

$$Hv_p = H\left(\frac{1}{p!}Y^p v_0\right) = \frac{1}{p!}H(Y^p v_0).$$

Since H commutes with Y, we have

$$H(Y^p v_0) = (HY^p)v_0 = (Y^p(H + 2p))v_0 = (\lambda + 2p)(Y^p v_0),$$

so

$$Hv_p = (\lambda - 2p)v_p.$$

For the second claim, consider

$$Xv_p = X\left(\frac{1}{p!}Y^p v_0\right) = \frac{1}{p!}X(Y^p v_0).$$

Using the commutation relation $[X, Y] = H$, we get

$$X(Y^p v_0) = (Y^p X + [X, Y^p])v_0 = Y^p X v_0 + \text{terms involving } H.$$

Since $Xv_0 = 0$ (as v_0 is a highest weight vector),

$$X(Y^p v_0) = Y^p X v_0 = (p \cdot Y^{p-1}v_0),$$

so

$$Xv_p = (\lambda - p + 1)v_{p-1}.$$

Finally,

$$Yv_p = Y\left(\frac{1}{p!}Y^p v_0\right) = \frac{1}{p!}Y^{p+1}v_0,$$

so

$$Yv_p = (p + 1)v_{p+1}.$$

$\square$

Theorem 2.2. *For $n \geq 0$, let V_n denote a $(n+1)$-dimensional vector space with basis $(v_0, \ldots, v_n)$. Define a $U(\mathfrak{sl}_2)$-module structure on V_n as follows:*

$$\rho_n(X) = \begin{pmatrix} 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\rho_n(Y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n & 0 \end{pmatrix},$$

$$\rho_n(H) = \begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & n-2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}.$$

Then

1. *These matrices indeed define a $U(\mathfrak{sl}_2)$-module structure on V_n.*
2. *Each V_n is simple.*
3. *The modules V_n are pairwise non-isomorphic.*
4. *Every finite-dimensional simple $U(\mathfrak{sl}_2)$-module is isomorphic to one of the V_n.*

Each V_n is called the simple $U(\mathfrak{sl}_2)$-module with highest weight n.

Proof. (1) Computation: Verify the defining relations of $U(\mathfrak{sl}_2)$ hold for these matrices.

(2) First a claim: Any highest weight vector of V_n is proportional to v_0. Indeed, such a w must be a scalar multiple of some v_i as it is an eigenvector of H, and in particular of v_0 as none of the others belong to $\ker X$. Now to see that V_n is simple, let $V' \subset V_n$ be a subrepresentation. Let $v' \in V'$ be a highest weight vector for V'. Then this element viewed in V is also a highest weight vector. Since $v_0 \in V_n$ satisfies these properties, we must have that v' and v_0 are proportional, thus $v_0 \in V'$. But this means that $v_1 = Yv_0$, $v_2 = \frac{1}{2}Y^2 v_0$, etc. all belong to V' as well, so $V' = V$.

(3) By dimensions: The dimension of V_n is $n+1$. Since the modules V_n are pairwise non-isomorphic, this implies that V_n are pairwise non-isomorphic.

(4) If V is a simple $U(\mathfrak{sl}_2)$-module, let λ be a highest weight and $v_0 \in V[\lambda]$ a highest weight vector. Set $v_p := \frac{1}{p!}Y^p v_0$, $k \geq p$, so that the module structure is as in the Lemma above. If the v_p are non-zero, they must be linearly independent as they have different weights (eigenvalues). Since V

is finite-dimensional, there must be some n such that $v_n \neq 0$ and $Yv_n = 0$, so that $Yv_k = 0$ whenever $k > n$. Note that

$$0 = Xv_{n+1} = (\lambda - n)v_n,$$

hence $\lambda = n$, i.e., V has highest weight n. Therefore, $\mathrm{span}(v_0, \dots, v_n)$ defines a subrepresentation of V isomorphic to V_n. But since V is simple, they must be isomorphic. $\square$

Corollary 2.3. *The irreducible representation V_n with highest weight n is a direct sum of weight spaces,*

$$V_n = V[-n] \oplus V[-n+2] \oplus \cdots \oplus V[n-2] \oplus V[n],$$

each of them is one-dimensional.

Remark 2.4. V_1 is isomorphic to the standard 2-dimensional representation of $\mathfrak{sl}_2$, namely $\mathbb{C}^2$. The isomorphism $V_1 \cong \mathbb{C}^2$ is given by $v_0 \mapsto (1,0) =: z_1$ and $v_1 \mapsto (0,1) =: z_2$. Let $S^n\mathbb{C}^2$ denote the n-th symmetric power of $\mathbb{C}^2$, i.e., homogeneous polynomials in the unknowns z_1, z_2 of total degree n. Then it is not hard to see that

$$\begin{aligned}
(H \cdot P)(z_1, z_2) &= z_1 \frac{\partial P}{\partial z_1} - z_2 \frac{\partial P}{\partial z_2}, \\
(X \cdot P)(z_1, z_2) &= z_1 \frac{\partial P}{\partial z_2}, \\
(Y \cdot P)(z_1, z_2) &= z_2 \frac{\partial P}{\partial z_1}
\end{aligned}$$

defines an isomorphism of $U(\mathfrak{sl}_2)$-modules $V_n \cong S^n\mathbb{C}^2$.

Theorem 2.5 (e.g., Kassel V.4.6)**.** *Every finite-dimensional $U(\mathfrak{sl}_2)$-module is semisimple.*

Corollary 2.6. *Every finite-dimensional $U(\mathfrak{sl}_2)$-module is a direct sum of its weight spaces,*

$$V = \bigoplus_{n \in \mathbb{Z}} V[n],$$

where only finitely many $V[n]$ are non-zero.

Recall that $U(\mathfrak{sl}_2)$ admits a bialgebra structure which is determined by the condition that elements of $\mathfrak{sl}_2$ are primitive. As a bialgebra, the tensor product of representations is again a representation,

$$x \cdot (v \otimes w) := \Delta(x)(v \otimes w).$$

To determine how the tensor product splits as a sum of simple modules, let us define an important tool. Given a finite-dimensional $U(\mathfrak{sl}_2)$-module, its *formal character* is the Laurent polynomial

$$\mathrm{ch}(V) := \sum_{n\in\mathbb{Z}} \dim(V[n])t^n \in \mathbb{Z}[t,t^{-1}].$$

Lemma 2.7. *The formal character of $U(\mathfrak{sl}_2)$ is additive and multiplicative,*

$$\mathrm{ch}(V\oplus W) = \mathrm{ch}(V)+\mathrm{ch}(W), \qquad \mathrm{ch}(V\otimes W) = \mathrm{ch}(V)\mathrm{ch}(W).$$

Proof. The additive formula is immediate. For the multiplicative one, it suffices to check that the weight decomposition of the tensor product is given by

$$(V\otimes W)[k] = \bigoplus_{n+m=k} V[n]\otimes W[m].$$

That this is the case follows because if $v\in V[n]$ and $w\in W[m]$, then

$$H\cdot(v\otimes w) = H\cdot v\otimes w + v\otimes H\cdot w = nv\otimes w + v\otimes mw = (n+m)v\otimes w.$$

This concludes the proof. $\square$

Theorem 2.8. *Two $U(\mathfrak{sl}_2)$-modules V and W are isomorphic if and only if they have the same formal character,* $\mathrm{ch}(V)=\mathrm{ch}(W)$*.*

Proof. The "only if" direction is clear from the definition, so we only need to prove the "if" direction. We proceed by induction on the dimension of V and W. Notice that the condition $\mathrm{ch}(V)=\mathrm{ch}(W)$ implies that V and W have the same weight space decomposition and thus have the same dimension. Let n be the dimension of V and W. The case $n=0$ is vacuously true, so consider $n>0$. Let λ be a highest weight of both V and W. Then by the same argument as in the theorem stating that the V_n's are the only simple modules, we have that V contains a subrepresentation V' isomorphic to V_λ and W contains a subrepresentation W' isomorphic to V_λ. Consider an invariant inner product in V and W (it exists essentially because any representation of $\mathfrak{sl}_2$ comes from $SU(2)$, which is 1-connected and compact, so any representation admits a unitary inner product that descends to $\mathfrak{sl}_2$). Then decompose

$$V = V'\oplus(V')^\perp, \quad W = W'\oplus(W')^\perp.$$

Because the inner product is invariant, $\langle x\cdot v_1, v_2\rangle = \langle v_1, x\cdot v_2\rangle$, the orthogonals are also submodules. Additivity of the formal character implies that $\mathrm{ch}((V')^\perp)=\mathrm{ch}((W')^\perp)$, and since these subspaces have dimension less than n, we conclude by the induction hypothesis. $\square$

Corollary 2.9 (Clebsch-Gordan formula)**.** *The tensor product of two simple $U(\mathfrak{sl}_2)$-modules decomposes as*

$$V_n \otimes V_m \cong \bigoplus_{i=0}^{\min(n,m)} V_{n+m-2i} \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|}.$$

Proof. Just note that

$$\mathrm{ch}(V_n) = t^{-n} + t^{-n+2} + \cdots + t^{n-2} + t^n = \frac{t^{n+2} - t^{-n}}{t^2 - 1}.$$

The rest is a computation. □

Remark 2.10. We will see later on that the formal character actually defines a ring isomorphism

$$\mathrm{ch} : K(\mathsf{Mod}_{U(\mathfrak{sl}_2)}) \xrightarrow{\cong} \mathbb{Z}[t + t^{-1}],$$

where $\mathbb{Z}[t+t^{-1}] \subset \mathbb{Z}[t, t^{-1}]$ is the subring of palindromic Laurent polynomials.

3 General Results of Representation Theory

Unless otherwise stated, A is an algebra over an algebraically closed field.

Theorem 3.1. *Let (V_i) be a collection of non-isomorphic finite-dimensional simple A-modules. If W is a submodule of*

$$V := \bigoplus_i n_i V_i,$$

then W is isomorphic to $\oplus_i r_i V_i$, with $r_i \leq n_i$. The inclusion $W \hookrightarrow V$ is given by a direct sum of inclusions $r_i V_i \hookrightarrow n_i V_i$ which are determined by a $n_i \times r_i$ matrix with linearly independent columns.

Theorem 3.2. *(Density) Let V be a finite-dimensional simple A-module. Then the structure map*

$$\rho : A \to \mathrm{End}(V)$$

is surjective. That is, any endomorphism of V is "multiplication by" an element of A.

An A-module V is of *finite length* if there is a finite sequence of submodules

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that the quotients V_i/V_{i-1} are simple A-modules. Such a sequence is called a *Jordan-Hölder series* of V, and the quotients V_i/V_{i-1} are called *subquotients*.

Lemma 3.3. *Every finite-dimensional A-module V is of finite length.*

Proof. By induction on $\dim(V)$. The base case is clear. Now pick $V_1 \subseteq V$ a simple submodule, and set $U := V/V_1$, which is strictly of lower dimension. By the induction hypothesis, U has a Jordan-Hölder series of finite length $0 = U_0 \subset \cdots \subset U_n$. If $\pi : V \to V/V_1$ denotes the projection to the quotient, for $i \geq 2$ set $V_i := \pi^{-1}(U_i)$. Then

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

is a Jordan-Hölder series for V. □

We will later see that all Jordan-Hölder series have the same length and isomorphic subquotients. This is the content of the so-called Jordan-Hölder theorem.

The *Jacobson radical* $Rad(A)$ of a finite-dimensional algebra A is the set of all elements of A which act by 0 in all simple A-modules (and it is a two-sided ideal). It is not hard to see that $Rad(A)$ is the largest nilpotent two-sided ideal of A (recall that an ideal I is nilpotent if $I^n = 0$ for some n). Note that left ideals of A are exactly the subrepresentations of the (left) regular representation, and therefore $A/Rad(A)$ is another A-module. On the other hand, note that if V is a finite-dimensional representation of A, then $\mathrm{End}(V)$ is another representation, $(a \cdot f)(v) := a \cdot f(v)$, and if V is simple, then $\mathrm{End}(V)$ is *semisimple*, $\mathrm{End}(V) \cong (\dim(V))V$.

Theorem 3.4. *A finite-dimensional algebra A has finitely many simple A-modules (V_i, ρ_i) (up to isomorphism), these are finite-dimensional and there is an isomorphism of A-modules*

$$\oplus_i \rho_i : A/Rad(A) \overset{\cong}{\to} \bigoplus_i \mathrm{End}(V_i).$$

Corollary 3.5. *If (V_i) are the simple A-modules (up to isomorphism) of A, then*

$$\sum_i (\dim V_i)^2 \leq \dim A.$$

Of course, the equality holds when $Rad(A) = 0$. In this case, we say that A is *semisimple*. The following result justifies the name:

Theorem 3.6. *Let A be a finite-dimensional algebra. The following are equivalent:*

1. *A is a semisimple algebra.*
2. *Every finite-dimensional A-module is semisimple.*
3. *The regular representation A is semisimple.*

Corollary 3.7 (Converse of Schur's lemma)**.** *Let A be a semisimple algebra. If V is an A-module such that $\mathrm{End}_A(V) = \Bbbk$, then V is simple.*

Proof. Let $U \subset V$ be an A-submodule. Because A is semisimple, V admits a splitting $V = U \oplus W$ as A-modules. Consider the projection inclusion map $V \to U \to V$. If this map is 0, then $U = 0$; else, it must be a multiple of the identity by hypothesis and then $U = V$. □

We now introduce characters. If A is an algebra and V a finite-dimensional A-module, the *character* of V is the linear map

$$\chi_V : A \to \Bbbk \qquad , \qquad \chi_V(a) := \mathrm{tr}_V(\rho(a)).$$

Because the trace of a composition of endomorphisms is independent of the order, this means that the trace descends to a linear map

$$\chi_V : A/[A, A] \to \Bbbk.$$

Proposition 3.8. *Characters of non-isomorphic finite-dimensional simple A-modules are linearly independent (in A^*). Moreover, if A is finite-dimensional with simple finite-dimensional A-modules (V_i), then the collection (χ_{V_i}) forms a basis for $(A/[A, A])^*$.*

Theorem 3.9 (Jordan-Hölder)**.** *All Jordan-Hölder series of an A-module of finite length (e.g., finite-dimensional) have the same length and isomorphic subquotients (up to reordering).*

That common length is called the *length* of V.

Theorem 3.10 (Krull-Schmidt)**.** *Any A-module of finite length (e.g., finite-dimensional) of an algebra A can be uniquely decomposed into a direct sum of indecomposable A-modules, up to isomorphism and ordering.*

Theorem 3.11. *Let A, B be algebras. If V, W are irreducible representations of A, B respectively, then $V \otimes W$ is an irreducible representation of $A \otimes B$. Even more, all irreducible representations of the tensor product are of this form.*

4 Representations of Finite Groups

Let G be a finite group. A representation $G \to \mathrm{End}(V)$ is the same thing as a $\Bbbk[G]$-module V.

Theorem 4.1 (Maschke)**.** *The algebra $\Bbbk[G]$ is semisimple whenever char$(\Bbbk)$ does not divide $|G|$.*

Proof. It suffices to check that if V is a $\Bbbk[G]$-module and $W \subset V$ is a submodule, then there exists another submodule $W' \subset V$ such that $V = W \oplus W'$. Choose Z as a linear complement of W in V, so $V = W \oplus Z$ as vector spaces, and let $\pi : V \to W$ be the projection onto W, i.e., $\pi(w, z) = w$. Define

$$\bar{\pi} : V \to V, \quad \bar{\pi}(v) := \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1}v),$$

and set $W' := \ker \bar{\pi}$. It is easy to see that $\bar{\pi}_{|W} = \mathrm{Id}$ and $\bar{\pi}(V) \subset W$, so $\bar{\pi}^2 = \bar{\pi}$ and $\bar{\pi}$ is a projection along W', hence $V = W \oplus W'$, a priori only as vector spaces, but also as $\Bbbk[G]$-modules because if $h \in G$ and $y \in W'$,

$$\bar{\pi}\rho(h)y = \frac{1}{|G|} \sum_{g \in G} \rho(g)\pi\rho\left(g^{-1}h\right) y = \frac{1}{|G|} \sum_{\ell \in G} \rho(h\ell)\pi\rho\left(\ell^{-1}\right) y = \rho(h)\bar{\pi}y = 0.$$

This concludes. $\square$

Corollary 4.2. *The regular representation of $\Bbbk[G]$ decomposes as*

$$\Bbbk[G] \cong \bigoplus_i \mathrm{End}(V_i)$$

where (V_i) are all the simple $\Bbbk[G]$-modules, whenever char$(\Bbbk)$ does not divide $|G|$. Alternatively,

$$\Bbbk[G] \cong \bigoplus_i (\dim(V_i))V_i,$$

and therefore

$$|G| = \sum_i \dim(V_i)^2.$$

Remark 4.3. Actually, the converse of Maschke's theorem also holds, that is, $\Bbbk[G]$ is semisimple if and only if char$(\Bbbk)$ does not divide $|G|$.

Counterxample 4.4. If $G = \mathbb{Z}/p$ and char$(\Bbbk) = p$, then $\Bbbk[G]$ is not semisimple, because every simple $\Bbbk[G]$-module (which must be 1-dimensional as $\Bbbk[G]$ is commutative) has the trivial G-action. Indeed, the generator of G must act as multiplication by a p-th root of unity in $V \cong \Bbbk$, but in $\Bbbk$ any root of unity is the unity because $x^p - 1 = (x-1)^p$ in $\Bbbk$.

If V is a representation of $\Bbbk[G]$ and let χ_V be its character. Let us denote also by χ_V the restriction to G, i.e., $\chi_V : G \to V$. Because of the properties of the trace, the character χ_V is a class function. A *class function* of G is a set-theoretical map $\varphi : G \to \Bbbk$ which is invariant under conjugation, $\varphi(hgh^{-1}) = \varphi(g)$. We put

$$F_c(G, \Bbbk) := \mathrm{Hom}_{\mathsf{Set}}(G/\mathrm{conj}, \Bbbk)$$

for the set of class functions of G. Note that it is a vector space of dimension $|G/\mathrm{conj}|$.

Proposition 4.5. *Characters of simple $\Bbbk[G]$-modules form a basis for $F_c(G, \Bbbk)$, whenever char$(\Bbbk)$ does not divide $|G|$.*

Proof. By the proposition before the Jordan-Hölder theorem, it suffices to check that $(A/[A, A])^* \cong F_c(G, \Bbbk)$, which is readily verified. □

Corollary 4.6. *Characters determine representations (if char$(\Bbbk) = 0$):*

$$V \cong W \iff \chi_V = \chi_W.$$

Proof. It follows by expanding the characters in terms of irreducible representations using the additivity of the characters, the fact that $\Bbbk[G]$ is semisimple (i.e., all representations are sums of irreducible ones), and comparing coefficients of these expansions (so expansion in terms of basis elements) in the vector space $F_c(G, \Bbbk)$. □

Let us write $G^\vee$ for the set of irreducible representations of G.

Corollary 4.7.

$$|G^\vee| = |G/conj|,$$

that is, there are as many irreducible representations of G as conjugacy classes in G.

Examples 4.8. In what follows $\Bbbk = \mathbb{C}$.

1. **Finite Abelian Groups:** $G = \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$.

   Since G is abelian, $|G^\vee| = |G|$. Because every element of G is invertible, every element of G acts by automorphisms in any G-module. So in a simple G-module, which must be one-dimensional, we can view it as

   $$G \to \mathrm{Aut}(\mathbb{C}) \cong \mathbb{C}^\times.$$

In particular, $G^\vee$ has the structure of an abelian group: if $\rho, \rho' \in G^\vee$ are irreducible representations, so are $(\rho\rho')(g) := \rho(g)\rho'(g)$ and $\rho^{-1}(g) := \rho(g^{-1})$ (because all of them are one-dimensional). The group $G^\vee$ is called the *dual* or *character group* of G.

Let us start by $G = \mathbb{Z}/n$. An irreducible representation in this case is the same as a group homomorphism $\mathbb{Z}/n \to \mathbb{C}^\times$. Such a map must send 1 to an n-th root of unity. So if ρ denotes the representation that sends 1 to $e^{2\pi i/n}$, i.e.,

$$\rho(m) := e^{2\pi i m/n},$$

then $(\mathbb{Z}/n)^\vee = \{\rho^k\} \cong \mathbb{Z}/n$.

Now we have that for two abelian groups G_1 and G_2,

$$(G_1 \oplus G_2)^\vee \cong G_1^\vee \oplus G_2^\vee,$$

(basically because direct sum is the coproduct in abelian groups) so we conclude that for any finite abelian group we have

$$G^\vee \cong G.$$

2. **The Symmetric Group $\mathfrak{S}_3$:**

   In $\mathfrak{S}_n$, conjugacy classes are determined by cycle decomposition: two permutations are conjugate if and only if they have the same number of cycles of each length. The conjugacy classes are

$$[(1)] = \{1\}, \quad [(12)] = \{(12), (13), (23)\}, \quad [(123)] = \{(123), (132)\}.$$

   So there are three different irreducible representations. If d_i is the dimension of each of them, by the dimension formula we must have

$$d_1^2 + d_2^2 + d_3^2 = 6 = |\mathfrak{S}_3|,$$

   so two of them must be $d_i = 1$ and the third one $= 2$.

   The one-dimensional representations are the trivial representation $\mathbb{C}_+$ (i.e., $\rho(\sigma) = \mathrm{Id}$) and the sign representation $\rho(\sigma) = \mathrm{sign}(\sigma)$. The 2-dimensional representation can be visualized as representing the symmetries of the equilateral triangle with vertices $(1, 2, 3)$ at the points $(\cos 120°, \sin 120°)$, $(\cos 240°, \sin 240°)$, $(1, 0)$. For instance,

$$\rho((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho((123)) = \begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix}.$$

To check that this representation is irreducible, one argues as follows: consider any subrepresentation V. It must be the span of a subset of the eigenvectors of $\rho((12))$, which are the nonzero multiples of $(1,0)$ and $(0,1)$. But V must also be the span of a subset of the eigenvectors of $\rho((123))$, which are different vectors. Thus, V must be either 0 or the original 2-dimensional representation.

3. **The Quaternion Group** $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

   TODO.

If V, W are representations of G, then so are V^* and $V \otimes W$:

$$\rho_{V^*}(g) := (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*,$$

and

$$\rho_{V \otimes W}(g) := \rho_V(g) \otimes \rho_W(g),$$

which follows from the usual bialgebra structure on $\Bbbk[G]$. It is clear that

$$\chi_{V^*}(g) = \chi_V(g^{-1}),$$

and

$$\chi_{V \otimes W} = \chi_V \chi_W.$$

Lemma 4.9. *Let V be a finite-dimensional complex representation of G. Then $V \cong V^*$ (as G-modules) if and only if $\chi_V(G) \subset \mathbb{R}$.*

Proof. We have $\chi_V(g) = \sum_i \lambda_i$, the sum of the eigenvalues of $\rho(g)$ (e.g., from the Jordan normal form). These eigenvalues must be roots of unity because $\rho(g)^{|G|} = \rho(g^{|G|}) = \rho(1) = \mathrm{Id}$. So for complex representations,

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi_V(g)},$$

and the claim follows as characters determine representations. □

Recall that a *Hermitian inner product* in a complex vector space is a complex-valued bilinear form $\langle -, - \rangle$ which is antilinear in the second slot, conjugated symmetric, and positive definite.

We now define a Hermitian inner product on the space $F_c(G, \mathbb{C})$ of class functions. Set

$$(f_1, f_2) := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Recall that characters of irreducible representations of G form a basis for $F_c(G, \mathbb{C})$. The following theorem says in particular that this basis is orthonormal with respect to the above-defined inner product.

Theorem 4.10 (Schur Orthogonality Relations)**.** *For any complex G-modules V, W, we have*

$$(\chi_V, \chi_W) = \dim \operatorname{Hom}_G(V, W).$$

In particular, if V, W are irreducible,

$$(\chi_V, \chi_W) = \begin{cases} 1, & \text{if } V \cong W, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 4.11. *A finite-dimensional complex G-module V is simple if and only if $(\chi_V, \chi_V) = 1$.*

Proof. The "only if" part follows from the previous theorem. For the "if" part, use the previous theorem and the converse of Schur's lemma. □

Corollary 4.12. *Let (V_i) be the set of finite-dimensional simple G-modules and let $W = \bigoplus_i n_i V_i$. Then*

$$n_i = (\chi_W, \chi_{V_i}).$$

Proof.

$$(\chi_W, \chi_{V_i}) = \sum_j n_j (\chi_{V_j}, \chi_{V_i}) = n_i.$$

□

Recall that the *centralizer* $C(g)$ of an element $g \in G$ is the set of elements of the group that commute with g.

Here is another character formula:

Theorem 4.13. *Let $g, h \in G$. Then*

$$\sum_{V \in G^\vee} \chi_V(g)\overline{\chi_V(h)} = \begin{cases} |C(g)|, & \text{if } g \text{ is conjugated to } h, \\ 0, & \text{otherwise.} \end{cases}$$

A complex finite-dimensional representation V of G is *unitary* if it is endowed with a G-invariant Hermitian form, i.e., a Hermitian form $(-,-)$ for which G acts by unitary operators (isometries), i.e., $(gv, gw) = (v, w)$. Every complex finite-dimensional representation V of a finite group admits a unitary structure: if B is a positive definite form in V, then

$$(v, w) := \sum_{g \in G} B(gv, gw)$$

is the desired structure. This gives an alternative proof of Maschke's theorem, as the unitary structure allows us to take the orthogonal complement of any subspace, and therefore every finite-dimensional G-module is completely reducible (i.e., semisimple).

If the finite-dimensional G-module V is simple, then a unitary structure on V is essentially unique, namely any other differs by scaling by a positive real number. Indeed, if B, B' are two Hermitian forms in V, both non-degenerate, consider the composition of antilinear isomorphisms

$$V \overset{\cong}{\to} V^* \overset{\cong}{\leftarrow} V,$$

where each of the maps is the polarity, i.e., $v \mapsto B(v, -)$. The composition is then a G-module map $T : V \to V$, which satisfies $B(v, w) = B'(Tv, w)$. By Schur's lemma, $T = \lambda \cdot \mathrm{Id}$ and since both forms are positive definite, the only option is that $\lambda > 0$.

Let V be a finite-dimensional simple G-module and let $(v_1, \ldots, v_n)$ be an orthonormal basis with respect to the (essentially unique) G-invariant Hermitian product. The *matrix elements* of V are the functions

$$t_{ij}^V : G \to \mathbb{C}, \qquad t_{ij}^V(g) := (gv_i, v_j).$$

If $\mathrm{Fun}(G, \mathbb{C}) = \mathrm{Hom}_{\mathsf{Set}}(G, \mathbb{C})$, then one can show that the matrix elements form an orthonormal basis with respect to a Hermitian product on $\mathrm{Fun}(G, \mathbb{C})$ defined in the same way as for the space of class functions.

Remark 4.14. Let us explain how the representation theory of a finite group can be summarized in the so-called *character table*. The characters of all the irreducible representations of a finite group can be arranged into a character table, with conjugacy classes of elements as the columns, and characters as the rows. More specifically, the first row in a character table lists representatives of conjugacy classes, the second one the numbers of elements in the conjugacy classes, and the other rows list the values of the characters on the conjugacy classes. Note that the rows and columns of a character table are orthonormal with respect to the appropriate inner products. Also, note that in any character table, the row corresponding to the trivial representation consists of ones, and the column corresponding to the neutral element consists of the dimensions of the representations.

Example 4.15. This is the character table of $\mathfrak{S}_3$, which we discussed above:

$\mathfrak{S}_3$	Id	(12)	(123)
#	1	3	2
$\mathbb{C}_+$	1	1	1
$\mathbb{C}_-$	1	-1	1
$\mathbb{C}^2$	2	0	-1

This is obtained by explicitly computing traces in the irreducible representations.

Character tables also allow us to easily describe the tensor product of representations in terms of irreducible ones: if

$$V_i \otimes V_j = \bigoplus_k N_{ij}^k V_k,$$

then by a previous corollary, we know that

$$N_{ij}^k = (\chi_{V_i \otimes V_j}, \chi_{V_k}) = (\chi_{V_i}\chi_{V_j}, \chi_{V_k}).$$

Example 4.16. Following up the previous example, we have

$\mathfrak{S}_3$	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$
$\mathbb{C}_+$	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$
$\mathbb{C}_-$		$\mathbb{C}_+$	$\mathbb{C}^2$
$\mathbb{C}^2$			$\mathbb{C}_+ \oplus \mathbb{C}_- \oplus \mathbb{C}^2$

5 More on Representations of Finite Groups

Recall that a complex number $z \in \mathbb{C}$ is an *algebraic number* (resp. *algebraic integer*) if it is a root of a monic polynomial with rational (resp. integer) coefficients. Alternatively, $z \in \mathbb{C}$ is an algebraic number (resp. algebraic integer) if it is an eigenvalue of a matrix with rational (resp. integer) coefficients. The equivalence between the two approaches is of course the characteristic polynomial.

The set of algebraic numbers is denoted by $\overline{\mathbb{Q}}$ whereas the set of algebraic integers is denoted by $\mathbb{A}$.

Lemma 5.1. *The set $\mathbb{A}$ is a ring, and $\overline{\mathbb{Q}}$ is a field (concretely, it is the algebraic closure of $\mathbb{Q}$).*

Proof. The fact that both are rings follows because if $A \in \mathcal{M}_n(\mathbb{C})$ and $B \in \mathcal{M}_n(\mathbb{C})$ are square matrices, v is an eigenvector of A with eigenvalue α and w is an eigenvector of B with eigenvalue β, then $\alpha \pm \beta$ is an eigenvalue of $A \otimes I_m + I_n \otimes B$ (with eigenvector $v \otimes w$) and $\alpha\beta$ is an eigenvalue of $A \otimes B$ (with eigenvector $v \otimes w$). Now $\overline{\mathbb{Q}}$ is a field because if $\alpha \neq 0$ is a root of a polynomial $p(x)$ of degree d, then α^{-1} is a root of $x^d p(x^{-1})$. □ □

Lemma 5.2 (Integral Root Theorem)**.** *The set* $\mathbb{A} \cap \overline{\mathbb{Q}} = \mathbb{Z}$*, that is, any rational root of a monic integral polynomial is an integer.*

Proof. If z is a root of an integral monic polynomial $p(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n$ of degree n and $z = p/q$ with p, q coprime integers, then expanding $p(z) = 0$ and clearing denominators we have $p^n = -q(a_1 p^{n-1} + \cdots + a_n q^{n-1})$, so q divides p^n. But p and q are coprime, hence $q = \pm 1$. □ □

Every algebraic number α has a *minimal polynomial* , which is the monic polynomial with rational coefficients of the smallest degree such that $p(\alpha) = 0$. Any other polynomial $q(x)$ with rational coefficients such that $q(\alpha) = 0$ is divisible by $p(x)$. Roots of $p(x)$ are called the *algebraic conjugates* of α; they are roots of any polynomial q with rational coefficients such that $q(\alpha) = 0$. Note that any algebraic conjugate of an algebraic integer is obviously also an algebraic integer. Therefore, by the Vieta theorem (aka Cardano formulas), the minimal polynomial $p(x)$ of an algebraic integer has integer coefficients.

Theorem 5.3 (Frobenius Divisibility)**.** *Let G be a finite group and let V be a finite-dimensional simple G-module. Then* $\dim V$ *divides $|G|$, the order of G.*

Proof. Let us denote the conjugacy classes of G by $C_1, \ldots, C_n$ and their representatives by $g_1, \ldots, g_n$. Define

$$\lambda_i := \frac{\chi_V(g_i) \cdot |C_i|}{\dim V}.$$

First, we claim that these are algebraic integers. Indeed, for C a conjugacy class of G, let $P := \sum_{h \in C} h$. Then P is a central element of $\mathbb{Z}[G]$, so it acts on V by some scalar λ, which is an algebraic integer (since $\mathbb{Z}[G]$ is a finitely generated $\mathbb{Z}$-module, any element of $\mathbb{Z}[G]$ is integral over $\mathbb{Z}$, i.e., satisfies a monic polynomial equation with integer coefficients). On the other hand, taking the trace of P in V, we get $|C|\chi_V(g) = \lambda \dim V$.

Now consider the element

$$Q := \sum_i \lambda_i \overline{\chi_V(g_i)}.$$

The values $\chi_V(g_i)$ are sums of roots of unity because so are the eigenvalues of $\rho(g)$ (because the order of g is finite), so each $\chi_V(g_i)$ is an algebraic integer; hence Q is also an algebraic integer since $\mathbb{A}$ is a ring. Therefore, we have

$$Q = \sum_i \lambda_i \overline{\chi_V(g_{C_i})} = \sum_i \frac{|C_i|\chi_V(g_{C_i})\overline{\chi_V(g_{C_i})}}{\dim V} = \sum_{g \in G} \frac{\chi_V(g)\overline{\chi_V(g)}}{\dim V} = \frac{|G|(\chi_V, \chi_V)}{\dim V} = \frac{|G|}{\dim V}.$$

So $Q \in \mathbb{A}$, and $\frac{|G|}{\dim V} \in \mathbb{Q}$, so we conclude that Q is an integer by the previous lemma. $\square$ $\square$

Recall that a group is *solvable* if there is a sequence of nested normal subgroups $\{e\} = G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = G$ such that the subquotients are abelian. The following theorem can be proven using representation theory.

Theorem 5.4 (Burnside)**.** *Any group G of order $p^n q^m$, where p and q are primes, is solvable.*

The following theorem is a consequence of the theorem above about the irreducible representations of the tensor product of algebras.

Theorem 5.5. *Let G, H be groups and let (V_i), (W_i) be the collection of their irreducible representations, respectively. Then $(V_i \otimes W_j)$ is the collection of irreducible representations of $G \times H$.*

A *virtual representation* is an element of $\mathbb{Z}[G^\vee]$, that is, a linear combination of simple G-modules, $V = \sum_i n_i V_i$. The character of V is $\chi_V := \sum_i n_i \chi_{V_i}$.

Lemma 5.6. *Let V be a virtual representation. If $(\chi_V, \chi_V) = 1$ and $\chi_V(e) > 0$, then χ_V is the character of a simple G-module.*

Proof. Let $V_1, V_2, \ldots, V_m$ be the irreducible representations of G, and $V = \sum n_i V_i$. Then by the orthonormality of characters, $(\chi_V, \chi_V) = \sum_i n_i^2$. So $\sum_i n_i^2 = 1$, meaning that $n_i = \pm 1$ for exactly one i, and $n_j = 0$ for $j \neq i$. But $\chi_V(e) > 0$, so $n_i = +1$ and we are done. $\square$ $\square$

We now study the so-called restricted and induced representations. Let G be a group and let $H \subset G$ be a subgroup. If V is a G-module (over a ground field $\Bbbk$), then V is trivially an H-module by restricting to elements of H, and it is denoted by $\mathrm{Res}_H^G(V)$ and will be called the *restricted representation* . In fact, this defines a functor

$$\mathrm{Res}_H^G : \mathsf{Mod}_G \to \mathsf{Mod}_H.$$

Now if V is an H-module, aka an $\Bbbk[H]$-module, then we can define a G-module by extension of scalars,

$$\mathrm{Ind}_H^G(V) := V \otimes_{\Bbbk[H]} \Bbbk[G],$$

and it will be called the *induced representation* . Note that there is a linear isomorphism

$$\Bbbk[G] \overset{\cong}{\to} \Bbbk[G]^* \qquad , \qquad g \mapsto \delta_g$$

which is in fact of G-modules: $h\delta_g(x) = \delta_g(h^{-1}x) = \delta_{hg}(x)$. This implies the following isomorphism of G-modules:

$$V \otimes_{\Bbbk[H]} \Bbbk[G] \overset{\cong}{\to} V \otimes_{\Bbbk[H]} \Bbbk[G]^* \overset{\cong}{\to} \mathrm{Hom}_H(\Bbbk[G], V),$$

where the G-action on $\mathrm{Hom}_H(\Bbbk[G], V)$ is given by $(g \cdot \varphi)(x) := \varphi(xg)$. Also note that

$$\dim(\mathrm{Ind}_H^G(V)) = \dim(V)\frac{|G|}{|H|}$$

(this is an integer by Lagrange's theorem), roughly because if g, g' belong to the same H-coset, $g' = hg$; then $g \otimes v = g \otimes hh^{-1}v = g' \otimes h^{-1}v$. Similarly, this construction defines a functor

$$\mathrm{Ind}_H^G : \mathsf{Mod}_H \to \mathsf{Mod}_G.$$

Since this is an instance of restriction-(co)extension of scalars in modules, by the general adjunctions there we get

Theorem 5.7 (Frobenius Reciprocity)**.** *Let $H \subset G$. There are adjunctions*

$$\mathrm{Ind}_H^G : \mathsf{Mod}_H \leftrightarrows \mathsf{Mod}_G : \mathrm{Res}_H^G$$

and

$$\mathrm{Res}_H^G : \mathsf{Mod}_G \leftrightarrows \mathsf{Mod}_H : \mathrm{Ind}_H^G\,.$$

That is, if V is a G-module and W is an H-module, then there are bijections

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(W), V) \overset{\cong}{\to} \mathrm{Hom}_H(W, \mathrm{Res}_H^G(V))$$

and

$$\mathrm{Hom}_H(\mathrm{Res}_H^G(V), W) \overset{\cong}{\to} \mathrm{Hom}_G(V, \mathrm{Ind}_H^G(W)).$$

Taking dimensions and using the Schur orthogonality relations we get

Corollary 5.8 (Frobenius Reciprocity for Characters)**.** *In the notation of the theorem above, we have*

$$\langle \chi_{\mathrm{Res}_H^G(V)}, \chi_W \rangle_G = \langle \chi_V, \chi_{\mathrm{Ind}_H^G(W)} \rangle_H.$$

In particular, if V, W are irreducible, then the multiplicity of V in $\mathrm{Ind}_H^G(W)$ equals the multiplicity of W in $\mathrm{Res}_H^G(V)$.

Example 5.9. Put $G = \mathfrak{S}_3$ and $H = \mathbb{Z}/2 = \langle (12) \rangle$. Applying the previous corollary, we compute

$$\mathrm{Ind}_H^G(\mathbb{C}_+) = \mathbb{C}_+ \oplus \mathbb{C}^2 \qquad , \qquad \mathrm{Ind}_H^G(\mathbb{C}_-) = \mathbb{C}_- \oplus \mathbb{C}^2,$$

because $\mathrm{Res}_H^G(\mathbb{C}^2) = \mathbb{C}_+ \oplus \mathbb{C}_-$.

We can also compute the character of the induced representation:

Theorem 5.10 (Mackey Formula)**.** *Let $H \subset G$ and let V be an H-module. The character of the induced representation $\mathrm{Ind}_H^G(V)$ is given by*

$$\chi_{\mathrm{Ind}_H^G(V)}(g) = \sum \chi_V(xgx^{-1}),$$

where the sum runs over the right cosets $[x] \in H \backslash G$ such that $xgx^{-1} \in H$.

6 Representation theory of $\mathfrak{S}_n$

Since we know that there are as many irreducible representations of a finite group as conjugacy classes in the group, let us start by recalling the conjugacy classes of $\mathfrak{S}_n$. Given $\sigma \in \mathfrak{S}_n$, write it as a product of disjoint cycles of lengths $k_1 \geq k_2 \geq \cdots \geq k_p$, where we include the 1's in the list for fixed points. The *length type* of σ is the tuple $(k_1, k_2, \ldots, k_p)$. For example, if $\sigma \in \mathfrak{S}_{10}$, $\sigma = (2578)(1369)$, then its length type is $(4, 4, 1, 1)$.

Lemma 6.1. *Two permutations $\sigma, \sigma' \in \mathfrak{S}_n$ are conjugated if and only if they have the same length type.*

If n is a positive integer, a *partition* of n is a tuple $\lambda = (k_1, k_2, \ldots, k_p)$ as above, with $k_i \geq k_{i+1}$ and $n = \sum_i k_i$. Therefore, there are as many irreducible representations of $\mathfrak{S}_n$ as partitions λ of n.

To every partition λ, we will attach a *Young diagram* Y_λ, which is a picture in the plane consisting of k_1 boxes in a row, below which are k_2

boxes, and so on. A *Young tableau* T_λ corresponding to Y_λ is the result of filling the numbers $1, \dots, n$ into the squares of Y_λ in some way (without repetitions).

This is an example of a Young tableau for a permutation of $\mathfrak{S}_{10}$ of cycle type $(5, 4, 1)$:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 4 & 7 & 8 \\ \hline 3 & 5 & 6 & 9 & \multicolumn{1}{c}{} \\ \cline{1-4} 10 & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array}$$

For every Young tableau T_λ, we will define two subgroups P_λ, Q_λ in $\mathfrak{S}_n$: the *row subgroup* P_λ is the subgroup of permutations that permute elements in the same row; and the *column subgroup* Q_λ is the subgroup of permutations that permute elements in the same column. Note that $P_\lambda \cap Q_\lambda = \{\mathrm{Id}\}$. Now define the *Young projectors*

$$a_\lambda := \frac{1}{|P_\lambda|} \sum_{\sigma \in P_\lambda} \sigma, \qquad b_\lambda := \frac{1}{|Q_\lambda|} \sum_{\sigma \in Q_\lambda} \mathrm{sign}(\sigma)\sigma.$$

Also set $c_\lambda = a_\lambda b_\lambda$, which is called the *Young symmetriser*, and note that this element is nonzero as $P_\lambda \cap Q_\lambda = \{\mathrm{Id}\}$.

Now consider the left regular representation $\mathbb{C}[\mathfrak{S}_n]$. The *Specht module* associated to T_λ is the subrepresentation

$$V_\lambda := \mathbb{C}[\mathfrak{S}_n] c_\lambda \subset \mathbb{C}[\mathfrak{S}_n].$$

Theorem 6.2. *The Specht modules V_λ exhaust all irreducible representations of $\mathfrak{S}_n$,*

$$\mathfrak{S}_n^\vee = \{V_\lambda\}_\lambda.$$

Examples 6.3.

1. If $\lambda = (n)$, then $P_\lambda = \mathfrak{S}_n$ and $Q_\lambda = \mathrm{Id}$, so c_λ is the symmetriser, and V_λ is the one-dimensional trivial representation.

2. If $\lambda = (1, \dots, 1)$, then $Q_\lambda = \mathfrak{S}_n$ and $P_\lambda = \mathrm{Id}$, so c_λ is the antisymmetriser, and V_λ is the one-dimensional sign representation.

3. If $n = 3$ and $\lambda = (2, 1)$, then $V_\lambda = \mathbb{C}^2$, the representation that permutes the vertices of the equilateral triangle.

Let us now introduce a lexicographic order in the set of partitions of a given n: we put $\lambda > \mu$ if the first non-vanishing $\lambda_i - \mu_i$ is positive. This allows us to describe the induced representation $\mathrm{Ind}_{P_\lambda}^{\mathfrak{S}_n}(\mathbb{C}) \cong \mathbb{C}[\mathfrak{S}_n] a_\lambda$ in terms of irreducible ones:

Proposition 6.4. *We have*

$$\mathrm{Ind}_{P_\lambda}^{\mathfrak{S}_n}(\mathbb{C}) = \bigoplus_{\mu \geq \lambda} K_{\mu\lambda} V_\mu,$$

for some non-negative integers $K_{\mu\lambda}$ which are called the Kostka numbers.

We can also compute the character of $\mathrm{Ind}_{P_\lambda}^{\mathfrak{S}_n}(\mathbb{C})$. For $m > 0$ and $x = (x_1, \ldots, x_N)$, let

$$H_m(x) := \sum_i x_i^m.$$

Now, let C be the conjugacy class of $\mathfrak{S}_n$ having i_ℓ cycles of length ℓ for all $\ell > 0$.

Theorem 6.5. *Let N be the number of parts of λ (i.e., $\lambda = (\lambda_1, \ldots, \lambda_N)$). If $\sigma \in \mathfrak{S}_n$ has i_ℓ cycles of length ℓ for $\ell > 0$, then $\chi_{\mathrm{Ind}_{P_\lambda}^{\mathfrak{S}_n}(\mathbb{C})}(\sigma)$ is the coefficient of $x^\lambda := \prod_j x_j^{\lambda_j}$ in the polynomial*

$$\prod_{m \geq 1} H_m(x)^{i_m}.$$

Let us write $\Delta(x) := \prod_{1 \leq i < j \leq N}(x_i - x_j)$ for the *Vandermonde polynomial*. The following theorem gives a formula for the character of the irreducible representations of $\mathfrak{S}_n$:

Theorem 6.6 (Frobenius character formula)**.** *Let N be the number of parts of λ. If $\sigma \in \mathfrak{S}_n$ has i_ℓ cycles of length ℓ for $\ell > 0$, then $\chi_{V_\lambda}(\sigma)$ is the coefficient of $\prod_j x_j^{\lambda_j + N - j}$ in the polynomial*

$$\Delta(x) \prod_{m \geq 1} H_m(x)^{i_m}.$$

Because $\dim(V_\lambda) = \chi_{V_\lambda}(\mathrm{id})$, we can use the character formula to compute the dimensions of the Specht modules. We need a definition before: for a square (i, j) in a Young diagram λ, where $i, j \geq 1, i \leq \lambda_j$, define the *hook length* of (i, j) to be the number $h(i, j)$ of squares (i', j') in λ with $i' \geq i, j' = j$ or $i' = i, j' \geq j$.

Theorem 6.7 (Hook length formula)**.** *The dimension of the Specht modules is given by*

$$\dim(V_\lambda) = \frac{n!}{\prod_{i \leq \lambda_j} h(i, j)}.$$

We now move on to study how the representation theory of $\mathfrak{S}_n$ interacts with that of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ (and also with that of $GL(V)$), which is known as *Schur-Weyl duality*. First, we need a result of independent interest. Let R be a ring and $S \subset R$ a subring. The *centraliser* of S is

$$C_R(S) := \{r \in R : rs = sr \ \forall s \in S\}.$$

It is always the case that $S \subset C_R(C_R(S))$, but whether equality holds depends on the specific situation. Now our main case of interest is when $R = \mathrm{End}(V)$, the ring of endomorphisms of a finite-dimensional complex vector space V. If $A \subset \mathrm{End}(V)$ is a subring, then V inherits a canonical structure of A-module: $f \cdot v := f(v)$. It is readily verified that $C_{\mathrm{End}(V)}(A) = \mathrm{End}_A(V)$.

Theorem 6.8 (Double centraliser theorem)**.** *Let E be a finite-dimensional complex vector space, let $A \subset \mathrm{End}(E)$ be a semisimple subalgebra, and let $B := C_{\mathrm{End}(E)}(A) = \mathrm{End}_A(E)$. Then:*

1. *$C_{\mathrm{End}(E)}(C_{\mathrm{End}(E)}(A)) = A$, that is, $\mathrm{End}_B(E) = A$.*

2. *B is semisimple.*

3. *If (U_i) is the set of simple A-modules, then setting $W_i := \mathrm{Hom}(U_i, E)$, we have that (W_i) is the set of simple B-modules.*

4. *View E as an $A \otimes B$-module, $(f \otimes g) \cdot v := f(g(v))$. Then E decomposes as*
$$E \cong \bigoplus_i U_i \otimes W_i.$$

For a proof, see here.

We want to apply the previous theorem to the situation where $E = V^{\otimes n}$, and A is the image of $\mathbb{C}[\mathfrak{S}_n]$ in $\mathrm{End}(V^{\otimes n})$ (here we view $\mathbb{C}[\mathfrak{S}_n]$ acting on $V^{\otimes n}$ by permuting the factors). The key observation is that A is semisimple: for note that A is the quotient of $\mathbb{C}[\mathfrak{S}_n]$ modulo the kernel of $\rho : \mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}(V^{\otimes n})$ (which is an ideal). But in general if $I \subset D$ is a two-sided ideal (of an arbitrary finite-dimensional algebra D), we have that $Rad(D/I) = (Rad(D) + I)/I$. In particular, if D is semisimple, so is D/I. Hence $A = \mathbb{C}[\mathfrak{S}_n]/\ker(\rho)$ is semisimple.

On the other hand, there is a natural action $\rho : \mathfrak{gl}(V) \to \mathrm{End}(V^{\otimes n})$ of $\mathfrak{gl}(V)$ on $V^{\otimes n}$, namely

$$\rho(f) = \sum_i \mathrm{Id} \otimes \cdots \otimes f \otimes \cdots \otimes \mathrm{Id}.$$

This action extends to $U(\mathfrak{gl}(V))$ via the universal enveloping algebra adjunction

$$\mathrm{Hom}_{\mathrm{Alg}}(U(\mathfrak{g}), A) \cong \mathrm{Hom}_{\mathrm{LieAlg}}(\mathfrak{g}, \mathcal{L}A).$$

Lemma 6.9. *The image of $U(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$ equals $B = \mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}(V^{\otimes n})$.*

Recall that A is the image of $\mathbb{C}[\mathfrak{S}_n]$ in $\mathrm{End}(V^{\otimes n})$ and B is the image of $U(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes n})$. By the double centraliser theorem, these are centralisers of each other. We also have that B is semisimple.

Theorem 6.10 (Schur-Weyl duality for $\mathfrak{gl}(V)$)**.** *The algebras A and B are centralisers of each other, B is semisimple, and*

$$V^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes M_\lambda,$$

where the sum is taken over partitions of n, the V_λ are Specht modules for $\mathfrak{S}_n$, and M_λ are irreducible representations of $\mathfrak{gl}(V)$, or zero.

There is an action $\rho : GL(V) \to \mathrm{End}(V^{\otimes n})$ given by

$$\rho(f) = f \otimes \cdots \otimes f \otimes \cdots \otimes f,$$

which commutes with the $\mathfrak{S}_n$-action. It is not hard to see that the image of $GL(V)$ in $\mathrm{End}(V^{\otimes n})$ spans B, and again by the double centraliser theorem we get:

Theorem 6.11 (Schur-Weyl duality for $GL(V)$)**.** *We have a decomposition*

$$V^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes L_\lambda$$

as a representation of $\mathfrak{S}_n \times GL(V)$, where the sum is taken over partitions of n, the V_λ are Specht modules for $\mathfrak{S}_n$, and $L_\lambda = \mathrm{Hom}_{\mathfrak{S}_n}(V_\lambda, V^{\otimes n})$ are distinct irreducible representations of $GL(V)$, or zero.

7 Intermezzo: Semisimple Lie Algebras

Before studying the representation theory of semisimple Lie algebras, we will review their classification.

A Lie algebra (over a field $\Bbbk$ of characteristic zero unless otherwise stated) is *abelian* if it has a trivial bracket, $[\mathfrak{g}, \mathfrak{g}] = 0$. If V is a vector space, a Lie subalgebra of $\mathfrak{gl}(V) = \mathrm{End}(V)$ is called a *linear Lie algebra*. By a theorem of Ado, every finite-dimensional Lie algebra is isomorphic to a linear Lie algebra, i.e., every finite-dimensional Lie algebra has a faithful finite-dimensional representation.

If A is an algebra, a *derivation* is a linear map $\delta : A \to A$ satisfying $\delta(xy) = \delta(x)y + x\delta(y)$. The set of derivations $Der(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$. Now if $A = \mathfrak{g}$ is a Lie algebra and $z \in \mathfrak{g}$,

$$\mathrm{ad}_z := [z, -] : \mathfrak{g} \to \mathfrak{g}$$

is a derivation by the Jacobi identity and therefore we have the *adjoint representation*

$$\mathrm{ad} : \mathfrak{g} \to Der(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g}).$$

An *ideal* in a Lie algebra $\mathfrak{g}$ is a vector subspace I such that $[I, \mathfrak{g}] \subset I$. For instance, the *derived algebra* $[\mathfrak{g}, \mathfrak{g}]$ is always an ideal of $\mathfrak{g}$. Also, the *centre* $Z(\mathfrak{g}) := \ker(\mathrm{ad})$ is also an ideal.

A Lie algebra is *simple* if it is nonabelian and has no ideals except for 0 and $\mathfrak{g}$. Hence if $\mathfrak{g}$ is simple, $Z(\mathfrak{g}) = 0$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

The *derived series* of $\mathfrak{g}$ is

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots \supset \mathfrak{g}^{(n)} \supset \cdots$$

with $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$. The Lie algebra is called *solvable* if $\mathfrak{g}^{(n)} = 0$ for some n.

Similarly, the *lower central series* of $\mathfrak{g}$ is

$$\mathfrak{g} \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots \supset \mathfrak{g}^n \supset \cdots$$

with $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{(n-1)}]$. The Lie algebra is called *nilpotent* if $\mathfrak{g}^n = 0$ for some n. Note that $\mathfrak{g}^{(n)} \subset \mathfrak{g}^n$ (this can be shown inductively), hence nilpotent implies solvable.

An element $x \in \mathfrak{g}$ is *ad-nilpotent* if ad_x is a nilpotent endomorphism, $\mathrm{ad}_x^n = 0$ for some n. Note that if $\mathfrak{g}$ is nilpotent, then every element is ad-nilpotent. The converse also holds:

Theorem 7.1 (Engel)**.** *A finite-dimensional Lie algebra is nilpotent if and only if all elements are ad-nilpotent.*

The *solvable radical* $Rad(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the largest solvable ideal of $\mathfrak{g}$. A Lie algebra is *semisimple* if its solvable radical is zero, i.e., if it has no nonzero solvable ideal. Also recall that the *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{k}, \quad \kappa(x, y) := \mathrm{tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)).$$

The Killing form is associative in the sense that $\kappa([x, y], z) = \kappa(x, [y, z])$.

Theorem 7.2. *Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. Then the following are equivalent:*

1. *$\mathfrak{g}$ is a semisimple Lie algebra.*

2. *$\mathfrak{g}$ has no nonzero abelian ideals.*

3. *The Killing form of $\mathfrak{g}$ is nondegenerate.*

4. *$\mathfrak{g}$ is a direct sum of simple Lie algebras.*

Examples 7.3. The following are simple complex Lie algebras (and will turn out to be the only ones):

1. $\mathfrak{a}_n = \mathfrak{sl}(n+1)$, $n \geq 1$.
   The subalgebra of $\mathfrak{gl}(n+1)$ of traceless matrices, $tr(X) = 0$. We have $\dim(\mathfrak{a}_n) = (n+1)^2 - 1$. Note that we have removed $\mathfrak{a}_0 = \mathfrak{sl}_1$ since it is trivial, hence not simple.

2. $\mathfrak{b}_n = \mathfrak{so}(2n+1)$, $n \geq 2$.
   The subalgebra of $\mathfrak{gl}(2n+1)$ of skew-symmetric matrices, $X^T = -X$. We have $\dim(\mathfrak{b}_n) = n(2n+1)$. Note that we have removed $\mathfrak{b}_i = \mathfrak{a}_i$ for $i = 0, 1$.

3. $\mathfrak{c}_n = \mathfrak{sp}(2n)$, $n \geq 3$.
   The subalgebra of $\mathfrak{gl}(2n)$ of symplectic matrices,
   $$X^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} X.$$
   We have $\dim(\mathfrak{c}_n) = n(2n+1)$. Note that we have removed $\mathfrak{c}_i = \mathfrak{b}_i$ for $i = 1, 2$.

4. $\mathfrak{d}_n = \mathfrak{so}(2n)$, $n \geq 4$.
   The subalgebra of $\mathfrak{gl}(2n)$ of skew-symmetric matrices, $X^T = -X$. We have $\dim(\mathfrak{d}_n) = n(2n-1)$. Note that we have removed $\mathfrak{d}_i = \mathfrak{a}_i$ for $i = 1, 3$ and $\mathfrak{d}_2 \cong \mathfrak{a}_2 \oplus \mathfrak{a}_2$, which is semisimple but not simple.

5. The exceptional Lie algebras $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4$ and $\mathfrak{g}_2$ of dimensions 78, 133, 248, 52 and 14, respectively.

7.1 Abstract Root Systems, Cartan Matrices, and Dynkin Diagrams

Roughly speaking, the classification theorem that we are chasing after will state that there are one-to-one correspondences between four different pieces of data:

1. Semisimple Lie algebras (up to isomorphism).
2. (Abstract) Root systems (up to isomorphism).
3. (Abstract) Fundamental systems (up to isomorphism).
4. (Abstract) Cartan matrices (up to isomorphism).
5. (Abstract) Dynkin diagrams (up to isomorphism).

With the equivalence of these five pieces of data at hand, we will classify abstract Cartan matrices, which will lead to a classification of semisimple Lie algebras. To begin with, we will forget about Lie algebras for a moment and focus on defining (2)-(5) and studying their equivalences. Later on, we will associate to each semisimple Lie algebra one object of (2)-(5), but a priori (2)-(5) have nothing to do with Lie algebras.

Let V be a Euclidean vector space, and let $(-,-)$ be its inner product. For $\alpha, \beta \in V$, the mapping

$$s_\alpha : V \to V, \quad s_\alpha(\beta) := \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

is called the *reflection* in α, since it is the reflection with respect to the hyperplane $P_\alpha = \{\beta \in V : (\alpha, \beta) = 0\}$.

An *(abstract) root system* is a pair (V, Φ) where V is a finite-dimensional Euclidean vector space, and Φ is a finite set of nonzero vectors, called *roots*, such that:

1. Φ spans V.
2. If $\alpha \in V$, then $-\alpha \in V$ and no other real multiple of α is in Φ.
3. For any two roots $\alpha, \beta \in \Phi$, the reflection in α of β is another root,

$$s_\alpha(\beta) = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi.$$

4. (Integrality condition) For any two roots $\alpha, \beta \in \Phi$,

$$\langle \alpha, \beta \rangle := 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z},$$

   that is, β and its reflection $s_\alpha(\beta)$ differ by an integer multiple of α. These integers $\langle \alpha, \beta \rangle$ are called the *Cartan integers*.

The *rank* of (V, Φ) is the dimension of V. An abstract root system is *irreducible* or *indecomposable* if it cannot be decomposed into a union $\Phi_1 \amalg \Phi_2$ of two disjoint orthogonal nonempty subsets of Φ. Any abstract root system decomposes as the union of irreducible root systems, and the vector space decomposes as an orthogonal sum of the irreducible ones.

An *abstract root system isomorphism* $\varphi : (V, \Phi) \to (V', \Phi')$ is a similarity $\varphi : V \to V'$ (isometry up to a constant factor, i.e., $(\varphi(\alpha), \varphi(\beta)) = \lambda \cdot (\alpha, \beta)$ for some $\lambda > 0$ for all $\alpha, \beta \in \Phi$) such that $\varphi(\Phi) = \Phi'$. That is, φ is a linear isomorphism that takes one set of roots onto the other and preserves the Cartan integers, $\langle \varphi(\alpha), \varphi(\beta) \rangle = \langle \alpha, \beta \rangle$ for all $\alpha, \beta \in \Phi$.

Lemma 7.4 (Finiteness)**.** *For any roots $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$, we have that*

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}.$$

Note that

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\frac{(\alpha, \beta)^2}{|\alpha|^2 |\beta|^2} = 4\cos^2\theta$$

where θ is the angle between the two roots. This yields the following table of possibilities, assuming $|\beta| \geq |\alpha|$ and $\beta \neq \pm\alpha$:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\lvert\beta\rvert / \lvert\alpha\rvert$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	$\sqrt{2}$
-1	-2	$3\pi/4$	$\sqrt{2}$
1	3	$\pi/6$	$\sqrt{3}$
-1	-3	$5\pi/6$	$\sqrt{3}$

If $\beta \neq \pm\alpha$ are roots, the α-string of roots containing β is the set $(\beta + \mathbb{Z}\alpha) \cap \Phi$.

Lemma 7.5 (String property)**.** *The α-string of roots containing β is of the form $\{\beta + i\alpha : -p \leq i \leq q\}$ where $p, q \geq 0$ and $p - q = \langle \beta, \alpha \rangle \leq 4$.*

The table and this property allow us to determine the root systems (at least in low dimensions).

A subset Δ of Φ is called a *fundamental system* or *base* for the root system (V, Φ) if it is a basis for V and every root β can be written as an integral linear combination of elements of Δ with all the integer coefficients having simultaneously the same sign. Given a choice of base, roots in Δ are called *simple*. Roots that are positive (resp. negative) linear combinations of simple roots are called *positive* (resp. *negative*), and denoted Φ^+ (resp. Φ^-), so that $\Phi = \Phi^+ \amalg \Phi^-$. It can be shown that every abstract root system has a base.

Given an abstract root system (V, Φ), the subgroup of $GL(V)$ generated by all reflections s_α for all $\alpha \in \Phi$ is called the *Weyl group* of Φ and denoted $W(\Phi)$.

Proposition 7.6 (Properties of the Weyl group)**.** *We have:*

1. *The Weyl group $W(\Phi)$ is a finite group.*
2. *$W(\Phi)$ permutes the roots.*
3. *Given two sets of positive roots, there exists an element of the Weyl group that takes one to the other.*
4. *Given two sets of simple roots, there exists a unique element of the Weyl group that takes one to the other.*
5. *$W(\Phi)$ is generated by reflections of simple roots.*
6. *Given a root α, there exists a simple root α_i and $s \in W(\Phi)$ such that $\alpha = s(\alpha_i)$.*

Note that (4) in the above proposition says that there is a bijection between sets of simple roots and elements of the Weyl group. The Weyl group is an example of a *Coxeter group*.

An important property of irreducible root systems is the following:

Proposition 7.7. *If Φ is an irreducible root system, then at most two different lengths occur in Φ, and furthermore all roots of a given length are conjugate under the action of the Weyl group.*

When two different lengths occur, we talk of *short* and *long* roots, so every root is therefore either short or long.

It turns out that a fundamental system for an (abstract) root system determines it. For this, we want to take one level of abstraction up and isolate the notion of a fundamental system. Here is the definition: an *abstract fundamental system* is a pair (V, Δ) where V is a finite-dimensional Euclidean space and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a choice of basis with the property that

$$\langle \alpha_i, \alpha_j \rangle := 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}_{\leq 0}$$

(in fact, as above, the only values this can take are $0, -1, -2, -3$). As for root systems, two abstract fundamental systems are *isomorphic* if there is a similarity between the Euclidean spaces taking one basis onto the other or, equivalently, a linear isomorphism taking one basis onto the other and preserving the Cartan integers. The notion of irreducibility goes as for root systems.

The properties above imply the following:

Theorem 7.8. *The passage from an abstract root system to a fundamental system explained above induces a bijection*

$$\frac{\begin{pmatrix}\text{abstract root}\\ \text{systems}\end{pmatrix}}{\text{isomorphism}} \xrightarrow{\cong} \frac{\begin{pmatrix}\text{abstract fundamental}\\ \text{systems}\end{pmatrix}}{\text{isomorphism}}$$

which makes corresponding irreducible abstract root systems with irreducible abstract fundamental systems.

Proof. The well-definedness follows from the notions of isomorphism, the fact that every root system has a base, that any two fundamental systems are related by an element of the Weyl group (hence an isometry), and the property that distinct simple roots α, β satisfy that $(\alpha, \beta) \leq 0$ because $\alpha - \beta$ can be seen not to be a root, and one can show that if $(\alpha, \beta) > 0$ then either $\alpha - \beta$ is a root or $\alpha = \beta$. The fundamental system of a root system uniquely determines it: indeed, by the proposition above about the properties of the Weyl group, (5) says that this is generated by the fundamental system, and by (6) the rest of the roots can be recovered by applying the Weyl group to the fundamental system. This proves both injectivity and surjectivity. $\square$

To each abstract fundamental system, we will associate a so-called Cartan matrix. Here is the relevant definition: an *abstract Cartan matrix* is an integral square matrix $C = (c_{ij}) \in \mathcal{M}_n(\mathbb{Z})$ satisfying:

1. $c_{ii} = 2$,

2. $c_{ij} \leq 0$ for all $i \neq j$,

3. $c_{ij} = 0$ if and only if $c_{ji} = 0$,

4. C is symmetrisable, i.e., there exists a diagonal matrix D with positive entries such that DCD^{-1} is a symmetric, positive definite matrix.

Two abstract Cartan matrices are isomorphic if one is conjugated to the other by a permutation matrix (the base change matrix between two bases that differ by some permutation of indices), that is, they differ by changing the enumeration of the indices of one of the matrices. An abstract Cartan matrix is *reducible* if, for some enumeration of indices, the matrix is block diagonal (with more than one block), otherwise it is *irreducible*. Obviously, several abstract Cartan matrices can be arranged as the blocks of a block-diagonal matrix yielding a new abstract Cartan matrix. The converse also holds: any abstract Cartan matrix, after a suitable enumeration of indices (i.e., is isomorphic to), can be written in block-diagonal form with each block an irreducible abstract Cartan matrix.

It is important to note that the diagonal matrix D from (4) in the above definition is essentially unique:

Lemma 7.9. *Let C be an abstract Cartan matrix in block-diagonal form with each of the blocks an irreducible abstract Cartan matrix. Then the associated matrix D is unique up to a multiplicative scalar on each block.*

Proposition 7.10. *If C is an abstract Cartan matrix and $i \neq j$, then*

1. $c_{ij}c_{ji} < 4$.

2. $c_{ij} \in \{0, -1, -2, -3\}$.

In particular, if $c_{ij} = -2$ or -3, then $c_{ji} = -1$.

We will now associate, to every abstract fundamental system (V, Δ), an abstract Cartan matrix $C_{(V,\Delta)}$. If $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is the set of simple roots, then define

$$C_{(V,\Delta)} := (\langle \alpha_i, \alpha_j \rangle).$$

Theorem 7.11. *We have:*

1. *$C_{(V,\Delta)}$ is indeed an abstract Cartan matrix.*

2. *Isomorphic abstract fundamental systems give isomorphic Cartan matrices.*

3. *An abstract fundamental system is determined, up to isomorphism, by its Cartan matrix.*

4. *Any abstract Cartan matrix arises as the Cartan matrix of an abstract fundamental system.*

5. *A fundamental system* (V, Δ) *is irreducible if and only if the Cartan matrix* $C_{(V,\Delta)}$ *is irreducible.*

The previous theorem says that there is a bijection:

$$\frac{\begin{pmatrix}\text{abstract fundamental}\\ \text{systems}\end{pmatrix}}{\text{isomorphism}} \cong \frac{\begin{pmatrix}\text{abstract Cartan}\\ \text{matrices}\end{pmatrix}}{\text{isomorphism}}.$$

Indeed, (1) – (3) says that the passage $(V, \Delta) \mapsto C_{(V,\Delta)}$ is well-defined, (4) says that the map is injective, and (5) that it is surjective. Furthermore, (6) says that this bijection makes correspond irreducible fundamental systems with irreducible Cartan matrices.

About the proof. (1) is standard, see e.g., Knapp Proposition 2.52. Note that one can take $D = \operatorname{diag}(|\alpha_1|, \ldots, |\alpha_l|)$ to check that the Cartan matrix is symmetrisable. (2) is obvious by definition. (3) First, we can recover the norms of the simple roots from the diagonal matrix D (up to a proportionality constant). With the entries of the Cartan matrix and these values, the inner product is fully determined (up to a constant). The root system is recovered from the simple roots because of (5) and (6) from the proposition about the properties of the Weyl group. (4) requires a case-by-case analysis of the Dynkin diagrams. (5) See Knapp Proposition 2.54. $\square$

The datum of an abstract Cartan matrix can be encoded in a planar graph called the Dynkin diagram. Here is the definition: if $C \in \mathcal{M}_n(\mathbb{Z})$ is an abstract Cartan matrix, the *abstract Dynkin diagram* associated to C is the planar graph $D(C)$ that has n (the size of C) vertices, and the vertices i and j are connected by $c_{ij}c_{ji} = 0, 1, 2, 3$ edges. If $c_{ij}c_{ji} = 2, 3$ and $|c_{ij}| < |c_{ji}|$, then the corresponding double/triple edge is decorated with an arrow from i to j. An isomorphism of Dynkin diagrams is a graph isomorphism that preserves the orientation of the additional arrows.

The Dynkin diagram fully determines the Cartan matrix: indeed, if there are no edges between vertices i and j, then $c_{ij}c_{ji} = 0$ and so one of them is 0, but by the defining property $c_{ij} = c_{ji} = 0$. Now if one edge connects i and j, $c_{ij}c_{ji} = 1$ and necessarily $c_{ij} = c_{ji} = -1$. If two or three edges connect i and j, $c_{ij}c_{ji} = 2, 3$, then necessarily one of them equals 1 and the other 2,3; and this is decided by the direction of the arrow: from i to j if $|c_{ij}| < |c_{ji}|$.

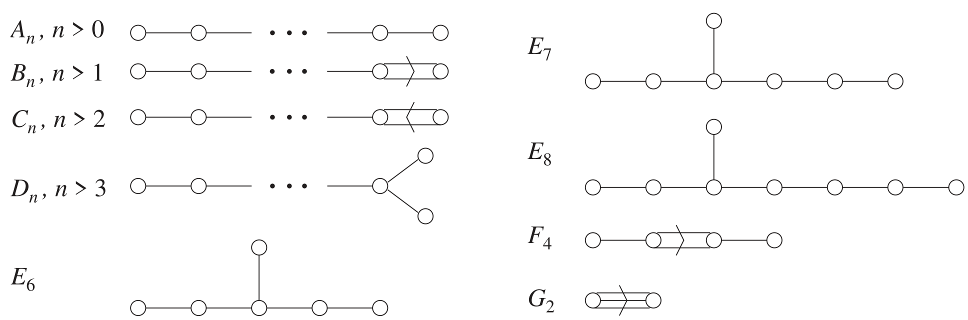
This establishes a bijection

$$\frac{\begin{pmatrix}\text{abstract Cartan}\\ \text{matrices}\end{pmatrix}}{\text{isomorphism}} \cong \frac{\begin{pmatrix}\text{abstract Dynkin}\\ \text{diagrams}\end{pmatrix}}{\text{isomorphism}}$$

(surjectivity comes from the definition). Obviously, irreducible Cartan matrices correspond to connected Dynkin diagrams.

The classification of root systems, fundamental systems, and Cartan matrices is then reduced to the classification of connected Dynkin diagrams. See Samelson's book for a readable proof.

Theorem 7.12 (Classification of Dynkin diagrams)**.** *The following list exhausts all possible connected Dynkin diagrams, and all diagrams are pairwise non-isomorphic:*



In the figure, n indicates the number of vertices. The diagrams A_n, B_n, C_n, D_n are called the classical diagrams, whereas E_6, E_7, E_8, F_4, G_2 are called the exceptional diagrams. For completeness, let us also write down the Cartan matrices corresponding to these Dynkin diagrams:

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \qquad B_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix},$$

$$C_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \quad D_n = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & & & \vdots \\ 0 & -1 & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & 2 & -1 & -1 \\ \vdots & & 0 & -1 & 2 & 0 \\ 0 & \dots & 0 & -1 & 0 & 2 \end{pmatrix},$$

$$E_6 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \qquad E_7 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

7.2 Classification of Complex Semisimple Lie Algebras

To close the circle, and finish what we came to do here, we would like to explain how to construct a root system out of a semisimple Lie algebra, and how the root system allows us to reconstruct the Lie algebra.

Let $\mathfrak{h}$ be an abelian complex finite-dimensional Lie algebra, and let (V, ρ) be a finite-dimensional $\mathfrak{h}$-module. Given $\alpha \in \mathfrak{h}^*$, set

$$V_\alpha := \{v \in V : \rho(h)v = \alpha(h)v \ \forall h \in \mathfrak{h}\} = \bigcap_{h \in \mathfrak{h}} \ker(\rho(h) - \alpha(h)\mathrm{Id}).$$

If $V_\alpha \neq 0$, the corresponding $\alpha \in \mathfrak{h}^*$ is called a *weight*, and V_α is called a *weight space*. Write $\Pi(\rho)$ for the set of weights.

Theorem 7.13 (Weight Space Decomposition)**.** *Let $\mathfrak{h}$ be an abelian complex finite-dimensional Lie algebra, and let (V, ρ) be a finite-dimensional $\mathfrak{h}$-module. Then $\Pi(\rho)$ is nonempty. Moreover, if each $\rho(h)$ is semisimple for all $h \in \mathfrak{h}$, the V decomposes as a direct sum of weight spaces,*

$$V = \bigoplus_{\alpha \in \Pi(\rho)} V_\alpha.$$

Proof. Let $h_1, \dots, h_n$ be a basis for $\mathfrak{h}$. $\rho(h_1)$, as a complex endomorphism, has some eigenvalue α_1, and let $E_1 := V_{\alpha_1}$ for the corresponding eigenspace. By the key observation, all $\rho(h) : V \to V$ preserve E_1, i.e., they restrict to maps $E_1 \to E_1$. Consider such a restriction for $\rho(h_2) : E_1 \to E_1$. The same story: it has an eigenvalue α_2 with corresponding eigenspace $E_2 \subset E_1$. The moral is that we get a sequence $0 \neq E_n \subset \cdots \subset E_1$ such that h_i acts as multiplication by α_i on E_i. Define $\alpha : \mathfrak{h} \to \mathbb{C}$ by $\alpha(h_i) = \alpha_i$. Then by construction $0 \neq E_n \subset V_\alpha$, i.e., $\alpha \in \Pi(\rho)$.

For the decomposition, one argues similarly: since $\rho(h_1)$ is diagonalizable, let $\mu_1, \dots, \mu_r$ be the different eigenvalues of $\rho(h_1)$, and let $V = \oplus_i V_{\mu_i}$ be the eigenspace decomposition. We can again restrict $\rho(h_2)$ to each of the pieces, $\rho(h_2) : V_{\mu_i} \to V_{\mu_i}$. This restriction is again diagonalizable, so each V_{μ_i} decomposes as $V_{\mu_i} = V_{\mu_i, \lambda_1^i} \oplus \cdots \oplus V_{\mu_i, \lambda_k^i}$ where the λ_j^i are the different eigenvalues. The story continues again. In total, we get a decomposition $V = V_1 \oplus \cdots \oplus V_N$ such that h_j acts by a scalar $\alpha_{ij} \in \mathbb{C}$ on V_i. Define $\alpha_i : \mathfrak{h} \to \mathbb{C}$ by $\alpha_i(h_j) := \alpha_{ij}$. Note that all the α_i are pairwise distinct, because the eigenvalues were chosen to be distinct for each h_j. It readily follows that $V_i = V_{\alpha_i}$, hence we are done. □ □

We will apply the previous result in the following context: let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra. A *Cartan subalgebra* is a maximal abelian sub-Lie algebra $\mathfrak{h}$ consisting only of ad-semisimple elements. In a semisimple Lie algebra, such a Cartan subalgebra always exists, and it is nontrivial. If $\mathfrak{g}$ is semisimple, then it is not solvable (i.e., $\mathfrak{g}$ has no solvable ideals; in particular, $\mathfrak{g}$ cannot be solvable). Not solvable implies not nilpotent as we saw before, and by Engel's theorem, there must be an element x which is not ad-nilpotent. The Jordan-Chevalley decomposition theorem says that any endomorphism decomposes as the sum of a semisimple (=diagonalizable) endomorphism and a nilpotent one. Hence for $\mathrm{ad}(x)$, the semisimple part cannot be zero. The subalgebra generated by all ad-semisimple elements, which is abelian, is the Cartan subalgebra.

We now fix for once and for all a choice of Cartan subalgebra $\mathfrak{h}$ and consider the restriction of the adjoint representation to $\mathfrak{h}$,

$$\mathrm{ad} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}).$$

A *nonzero weight* for the adjoint representation is called a *root*, and the set of roots is denoted by Φ. Each of the weight spaces is a *root space*, and by the weight space decomposition theorem, we get the so-called *root space decomposition*:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}.$$

Note that $\mathfrak{g}_0 = \ker(\mathrm{ad}_{|\mathfrak{h}}) =: C_{\mathfrak{g}}(\mathfrak{h})$, the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Theorem 7.14. *Any Cartan subalgebra is self-centralizing, i.e.,* $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Therefore, the root space decomposition can be rewritten as

$$\boxed{\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.}$$

Proposition 7.15. *(Properties) We have:*

1. *If* $\alpha, \beta \in \mathfrak{h}^*$*, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$*,*
2. *If* $\alpha, \beta \in \mathfrak{h}^*$ *and* $\beta \neq -\alpha$*, then the respective root spaces are orthogonal,* $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$*,*
3. *The restriction of* κ *to* $\mathfrak{h}$ *is also non-degenerate.*

Property 3. says that the Killing form gives rise to an isomorphism

$$\mathfrak{h} \xrightarrow{\cong} \mathfrak{h}^* \qquad , \qquad h \mapsto \kappa(h, -).$$

Given $\alpha \in \mathfrak{h}^*$, its image under the inverse of this isomorphism is denoted t_α. In other words, t_α is the unique element such that $\alpha = \kappa(t_\alpha, -)$. This allows us to define a bilinear form on $\mathfrak{h}^*$:

$$(\alpha, \beta) := \kappa(t_\alpha, t_\beta).$$

Of course, the theorem will be that these roots give rise to an abstract root system. Now we have to construct a (real) Euclidean vector space for these roots to live, so far we are working over $\mathbb{C}$. In order to understand why we can restrict to reals from complex numbers, we first need to understand how integers appear in quotients involving the defined inner product in $\mathfrak{h}^*$. The answer is: every complex semisimple Lie algebra $\mathfrak{g}$ has a copy of $\mathfrak{sl}(2,\mathbb{C})$ inside, so that $\mathfrak{g}$ can be seen as an $\mathfrak{sl}(2,\mathbb{C})$-module, and integers will emerge from the fact that the $\mathfrak{sl}(2,\mathbb{C})$ triple acts by integers on a simple $\mathfrak{sl}(2,\mathbb{C})$-module. The following is Proposition 1.15.8 in Hazewinkel-Gubareni-Kirichenko:

Proposition 7.16. *(Properties) Let $\alpha \in \Phi$. We have:*

1. *Φ spans $\mathfrak{h}^*$,*
2. *If $\alpha \in \Phi$, then $-\alpha \in \Phi$,*
3. *If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, then $[x,y] = \kappa(x,y)t_\alpha$,*
4. *$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_0 = \mathfrak{h}$ is a 1-dimensional linear subspace spanned by t_α, i.e., $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} \cdot t_\alpha$,*
5. *$\alpha(t_\alpha) = \kappa(t_\alpha, t_\alpha) \neq 0$,*
6. *For any $x \in \mathfrak{g}_\alpha$, there exists an element $y \in \mathfrak{g}_{-\alpha}$ such that the elements $x, y, h := [x,y]$ span a 3-dimensional simple subalgebra S_α isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ via*

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   We will write $x_\alpha, y_\alpha, h_\alpha$ to emphasize the dependency on the choice of root.

7. *$h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$ and $h_\alpha = h_{-\alpha}$.*
8. *$\dim \mathfrak{g}_\alpha = 1$. Hence $S_\alpha \cong \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathbb{C}h_\alpha$.*

Each of the h_α is called a *coroot*. The (additive) subgroup generated by the coroots is called the *coroot lattice*.

I want to explain using the above proposition how the Cartan integers are indeed integers. Let $\alpha, \beta \in \Phi$. The *α-root string through β* is defined as

$$\mathfrak{g}_{\beta\alpha} := \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}.$$

It follows from the property that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and the one stating that $[x, y] = \kappa(x, y)t_\alpha$ that $\mathfrak{g}_{\beta\alpha}$ is invariant under the actions of $x_\alpha, y_\alpha, h_\alpha$, and so $\mathfrak{g}_{\beta\alpha}$ is in fact an $S_\alpha \cong \mathfrak{sl}_2$-module. Now, the elements of $\mathfrak{g}_{\beta\alpha}$ satisfy $[h, x] = (\beta + n\alpha)(h)x$ for $h \in \mathfrak{h}$, in particular for h_α. In particular, the eigenvalues of $\mathrm{ad}(h_\alpha)$ in $\mathfrak{g}_{\beta\alpha}$ must be

$$(\beta + n\alpha)(h_\alpha) = \beta(h_\alpha) + 2n,$$

where we have used that $\alpha(h_\alpha) = 2\alpha(t_\alpha)/\kappa(t_\alpha, t_\alpha) = 2$. But, from the representation theory of $\mathfrak{sl}_2$, we know that h_α has integral eigenvalues. Hence $\beta(h_\alpha) \in \mathbb{Z}$. But we have

$$\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)},$$

hence $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

Let us now explain how to construct the real Euclidean vector space where our roots will live (now they live in $\mathfrak{h}^*$, which is a complex vector space!). The first observation is that Φ spans $\mathfrak{h}^*$: suppose it does not. Then there is some $0 \neq h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. But then for any $x \in \mathfrak{g}_\alpha$, $[h, x] = \alpha(h)x = 0$, and for any $h' \in \mathfrak{h}$ we also have $[h, h'] = 0$ since $\mathfrak{h}$ is abelian. So h commutes with all generators of $\mathfrak{g}$, hence it lives in the centre $Z(\mathfrak{g})$. But $Z(\mathfrak{g}) = 0$ because $\mathfrak{g}$ is semisimple. Therefore, choose a basis $\alpha_1, \ldots, \alpha_n$ of $\mathfrak{h}^*$ consisting of roots. Now I claim that any other root $\beta \in \Phi$ is a *rational* linear combination of the α_i's. For let $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$, with $c_i \in \mathbb{C}$. For each j, $(\beta, \alpha_j) = \sum_{i=1}^n c_i(\alpha_i, \alpha_j)$, i.e.,

$$\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i=1}^n \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} c_i.$$

Consider the right-hand side of the previous equation as a system of n linear equations with unknowns c_i with integer coefficients, since these are Cartan numbers. Because the α_i's form a basis for $\mathfrak{h}^*$ and $(-,-)$ is nondegenerate, the matrix $((\alpha_i, \alpha_j))$ is nonsingular, hence neither is the matrix $((\alpha_i, \alpha_j)/(\alpha_j, \alpha_j))$. Hence the system has a unique solution over $\mathbb{Q}$.

This implies that $\Phi \subset \mathrm{span}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n) \subset \mathfrak{h}^*$. Let $E_{\mathbb{Q}} := \mathrm{span}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_n)$, a rational vector space. Now I claim that $(-,-)$ is positive definite on this rational vector space. This readily follows from the following: if $\lambda \in \mathfrak{h}^*$,

$$(\lambda, \lambda) = \kappa(t_\lambda, t_\lambda) = \mathrm{Tr}(\mathrm{ad}\, t_\lambda \, \mathrm{ad}\, t_\lambda) = \sum_{\beta \in \Phi} \beta(t_\lambda)^2 = \sum_{\beta \in \Phi} (\beta, \lambda)^2,$$

where in the third equality we have used that $\operatorname{ad} t_\lambda$ acts on $\mathfrak{g}_\beta$ by multiplication by $\beta(\operatorname{ad} t_\lambda)$. So $(-,-)$ is positive semi-definite, and since it is non-singular, it has to be positive definite. The real Euclidean vector space we were looking for is then $E := E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.

Theorem 7.17. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be a choice of Cartan subalgebra. Then the construction above of (E, Φ) is an abstract root system. Moreover, the Lie algebra is simple if and only if the root system is irreducible.*

In fact, one can also show that the construction of the root system is independent of the choice of Cartan subalgebra (this is a very deep result).

Theorem 7.18. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{h}_1, \mathfrak{h}_2$ be two Cartan subalgebras. Then there exists an automorphism $\varphi : \mathfrak{g} \to \mathfrak{g}$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$.*

The dimension of any Cartan subalgebra is called the *rank* of the Lie algebra.

The two previous theorems combined establish a well-defined map

$$\frac{\begin{pmatrix}\text{complex semisimple}\\ \text{Lie algebras}\end{pmatrix}}{\text{isomorphism}} \to \frac{\begin{pmatrix}\text{abstract root}\\ \text{systems}\end{pmatrix}}{\text{isomorphism}}$$

The last thing we want to do is to prove that this map is a bijection. Let us start proving that it is surjective.

Let $\mathfrak{g}$ be a (complex finite-dimensional) semisimple Lie algebra, and consider a fundamental system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. For every i, consider generators $x_i := x_{\alpha_i}$, $y_i := y_{\alpha_i}$, and $h_i := h_{\alpha_i}$ of S_{α_i}.

Proposition 7.19. *The elements x_i, y_i, h_i for $i = 1, \ldots, n$ span $\mathfrak{g}$, and they satisfy the following relations:*

1. $[h_i, h_j] = 0$,
2. $[x_i, y_i] = h_i$,
3. $[x_i, y_j] = 0$ *if* $i \neq j$,
4. $[h_i, x_j] = c_{ji} x_j$,
5. $[h_i, y_j] = -c_{ji} y_j$,
6. $(\operatorname{ad} x_i)^{1-c_{ji}}(x_j) = 0$ *if* $i \neq j$,

7. $(\operatorname{ad} y_i)^{1-c_{ji}}(y_j) = 0$ *if* $i \neq j$,

where $C = (c_{ij})$ *is the Cartan matrix associated to* Δ.

The generators in the preceding proposition are called the *standard generators* or *Chevalley generators*. The relations (1)-(7) are called the *Serre relations*. These form in fact a complete set of relations for any semisimple Lie algebra:

Theorem 7.20 (Serre)**.** *Any complex, rank* n *semisimple Lie algebra* $\mathfrak{g}$ *is isomorphic to the free Lie algebra with generators* x_i, y_i, h_i *for* $i = 1, \ldots, n$ *subject to the Serre relations.*

Serre's theorem gives injectivity of the above passage. The following theorem establishes its surjectivity.

Theorem 7.21 (Existence Theorem)**.** *Let* C *be an abstract Cartan matrix of size* n*, and let* $\mathfrak{g}$ *be the free Lie algebra with generators* x_i, y_i, h_i *for* $i = 1, \ldots, n$ *subject to the Serre relations. Then* $\mathfrak{g}$ *is a finite-dimensional semisimple Lie algebra with Cartan matrix* C.

Note that Serre's theorem allows lifting isomorphisms of root systems to Lie algebra isomorphisms. Finally, we arrive at the complete classification of finite-dimensional, complex semisimple Lie algebras.

Theorem 7.22. *The above constructions give bijections*

$$\frac{\begin{pmatrix}\text{complex semisimple}\\ \text{Lie algebras}\end{pmatrix}}{\text{isomorphism}} \xrightarrow{\cong} \frac{\begin{pmatrix}\text{abstract root}\\ \text{systems}\end{pmatrix}}{\text{isomorphism}} \xrightarrow{\cong} \frac{\begin{pmatrix}\text{abstract fundamental}\\ \text{systems}\end{pmatrix}}{\text{isomorphism}}$$

$$\xrightarrow{\cong} \frac{\begin{pmatrix}\text{abstract Cartan}\\ \text{matrices}\end{pmatrix}}{\text{isomorphism}} \xrightarrow{\cong} \frac{\begin{pmatrix}\text{abstract Dynkin}\\ \text{diagrams}\end{pmatrix}}{\text{isomorphism}}$$

Under these bijections, simple Lie algebras correspond to irreducible abstract root systems, irreducible abstract fundamental systems, irreducible abstract Cartan matrices, and connected Dynkin diagrams.

Theorem 7.23 (Classification of Simple Lie Algebras)**.** *The simple Lie algebras* $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n, \mathfrak{d}_n, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4$*, and* $\mathfrak{g}_2$ *have associated Dynkin diagrams* $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$*, and* G_2*, respectively. Therefore, these are the only simple complex Lie algebras, up to isomorphism.*

Any finite-dimensional complex semisimple Lie algebra is, therefore, isomorphic to a finite direct sum of copies of these simple Lie algebras. This completes our classification.

8 Representation Theory of Complex Semisimple Lie Algebras

Let us put into practice what we learned about semisimple Lie algebras to completely classify its representation theory. The first step is to realize that it suffices to focus on simple $\mathfrak{g}$-modules:

Theorem 8.1 (Weyl)**.** *Any finite-dimensional representation of a finite-dimensional complex semisimple Lie algebra is semisimple (i.e., completely reducible).*

The main theorem will say that simple $\mathfrak{g}$-modules are in one-to-one correspondence with the so-called dominant weights. For the rest of this section, let us fix a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, root system Φ, fundamental system Δ, and Weyl group W.

Recall that given a representation V of $\mathfrak{g}$, a *weight* is an element $\lambda \in \mathfrak{h}^*$ such that $V_\lambda = \{v \in V \mid hv = \lambda(h)v \ \forall h \in \mathfrak{h}\} \neq 0$, and V_λ is called a *weight space*. We denote $\Pi = \Pi(V)$ for the set of weights of V. It is immediate that for a root $\alpha \in \Phi$, we have $\mathfrak{g}_\alpha(V_\lambda) \subseteq V_{\alpha+\lambda}$: indeed if $x_\alpha \in \mathfrak{g}_\alpha$ and $v \in V_\lambda$, we have

$$h \cdot x_\alpha \cdot v = [h, x_\alpha] \cdot v + x_\alpha \cdot h \cdot v = \alpha(h)x_\alpha \cdot v + \lambda(h)x_\alpha \cdot v = (\alpha + \lambda)(h)x_\alpha \cdot v.$$

A *highest weight* for V is a weight $\lambda \in \Pi$ (i.e., $V_\lambda \neq 0$) such that $\lambda + \alpha \notin \Phi$ for all $\alpha \in \Phi_+$ (i.e., $V_{\lambda+\alpha} = 0$ for all positive roots). Clearly, any finite-dimensional representation has a highest weight. A *maximal vector* (of weight λ) is an element $0 \neq v^+ \in V_\lambda$ which is killed by all $\mathfrak{g}_\alpha$ for any $\alpha \in \Phi_+$, i.e., $x_\alpha \cdot v^+ = 0$. Clearly, any nonzero vector of highest weight is a maximal vector, but the converse is not true. A *standard cyclic module* (of highest weight λ) is a representation V which is generated by a maximal vector of weight λ, i.e., $V = U(\mathfrak{g}) \cdot v^+$. The name is, of course, correct: λ is indeed a highest weight for V. Indeed, if

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Phi_-} \mathfrak{g}_\alpha,$$

the root space decomposition is rewritten as

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$$

(where $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$ is the so-called *Borel subalgebra* of $\mathfrak{g}$). By the PBW theorem, we then have

$$V = U(\mathfrak{g})v^+ = U(\mathfrak{n}^-)U(\mathfrak{h})U(\mathfrak{n})v^+ = U(\mathfrak{n}^-)v^+,$$

because the $U(\mathfrak{n})$ part gives multiples of v^+ and the $U(\mathfrak{h})$ part kills v^+, so all other weights of V must be lower than λ. In particular, this also implies that $\dim V_\lambda = 1$, and that in the weight space decomposition $V = \oplus V_\beta$ of V, the rest of weights are smaller, i.e., they are all of the form $\lambda - \sum_i k_i\alpha_i$ with $k_i \geq 0$.

Proposition 8.2 (Properties)**.** *Let V be a standard cyclic module of highest weight λ. Then:*

1. *V is indecomposable,*
2. *Any quotient of V is again a standard cyclic module of highest weight λ,*
3. *V has a unique maximal (proper) submodule, and a corresponding unique irreducible quotient,*
4. *If V is irreducible, then v^+ is the unique maximal vector in V (up to nonzero multiples), i.e., the maximal weight λ of V is unique.*

Now the obvious question is: are there standard cyclic modules out there? The answer is affirmative, and there is a unique one for each possible weight $\lambda \in \mathfrak{h}^*$:

Theorem 8.3 (Existence and Uniqueness of Irreducible Standard Cyclic Modules)**.** *Let $\lambda \in \mathfrak{h}^*$. Then there exists, up to isomorphism, a unique irreducible standard cyclic module $V(\lambda)$ of highest weight λ (which might be infinite-dimensional).*

I want to explain how to construct such a module $V(\lambda)$. The key observation is that such a $V(\lambda) = U(\mathfrak{g})v^+$, viewed as a $\mathfrak{b}$-module, contains a 1-dimensional submodule spanned by v^+. Then: consider $D_\lambda := \mathbb{C}v^+$ viewed as a $\mathfrak{b}$-module, with $h \cdot v^+ := \lambda(h)v^+$ for $h \in \mathfrak{h}$ and $x \cdot v^+ := 0$ for $x \in \mathfrak{n}$. Now from a $\mathfrak{b}$-module we can pass to a $\mathfrak{g}$-module by the induced representation (extension of scalars):

$$Z(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} D_\lambda.$$

This module is called the *Verma module* of weight λ. It is not hard to see that $Z(\lambda)$ is a standard cyclic module of weight λ. Now, if $Y(\lambda)$ is the maximal proper submodule of $Z(\lambda)$, then $V(\lambda) := Z(\lambda)/Y(\lambda)$ is irreducible and a standard cyclic module of highest weight λ by the above proposition.

Corollary 8.4. *Any finite-dimensional simple $\mathfrak{g}$-module is isomorphic to a standard cyclic module $V(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

Proof: Suppose V is a finite-dimensional simple $\mathfrak{g}$-module. Then V has at least a maximal vector v^+ of some weight λ. The submodule $U(\mathfrak{g})v^+$ it generates must be, by irreducibility, the whole thing, $V = U(\mathfrak{g})v^+$, i.e., V is an irreducible standard cyclic module. By the uniqueness, $V \cong V(\lambda)$. $\square$

The remaining question is: for what $\lambda \in \mathfrak{h}^*$ does the representation $V(\lambda)$ happen to be finite-dimensional? Let us write $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ for the fundamental system. For every i, consider generators $x_i := x_{\alpha_i}$, $y_i := y_{\alpha_i}$, and $h_i := h_{\alpha_i}$ of S_{α_i}. Recall that $h_1, \ldots, h_n$ form a basis for $\mathfrak{h}$. An *integral weight* is an element $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_i) \in \mathbb{Z}$ for all i (recall that $\lambda(h_i) = \langle \lambda, \alpha_i \rangle$). An integral weight is *dominant* if $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all i. The *fundamental dominant weights* are the elements of the dual basis of the h_i's, $\lambda_i(h_j) = \delta_{ij}$. We write Λ for the set (lattice) of integral weights, and Λ^+ for the set of dominant weights. It is clear that

$$\Lambda \cong \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_n \quad \text{and} \quad \Lambda^+ \cong \mathbb{N}\lambda_1 \oplus \cdots \oplus \mathbb{N}\lambda_n.$$

Also note that the Weyl group W acts on Λ, since it acts by isometries (or just look at the reflection formula). One more definition: the *root lattice* is the subgroup $\Lambda_r \subset \Lambda$ spanned by Φ.

Theorem 8.5. *Let $\lambda \in \mathfrak{h}^*$. Then the unique irreducible standard cyclic module $V(\lambda)$ of highest weight λ is finite dimensional if and only if λ is a dominant weight. In such a case, the Weyl group W acts on $\Pi(V(\lambda))$ permuting the weights, and the weight spaces of each orbit have the same dimension,* $\dim V_\mu = \dim V_{\sigma\mu}$ *for $\sigma \in W$.*

The upshot of the above results is:

Theorem 8.6. *There is a one-to-one correspondence*

$$\Lambda^+ \xrightarrow{\cong} \frac{\begin{pmatrix}\text{finite-dimensional} \\ \text{simple } \mathfrak{g}\text{-modules}\end{pmatrix}}{\text{isomorphism}}, \qquad \lambda \mapsto V(\lambda).$$

The set of finite-dimensional $\mathfrak{g}$-modules is, in fact, a semiring endowed with the direct sum and tensor product of $\mathfrak{g}$-modules. Given a set X, recall that the *free commutative monoid* generated by X is $\mathbb{N}[X]$, i.e., finite expressions $\sum_i n_i x_i$. Since Λ^+ is also a commutative monoid, let us denote the basic elements $1 \cdot \lambda \in \mathbb{N}[\Lambda^+]$ as $e(\lambda)$ (otherwise $\lambda + \mu$ could refer to $\lambda + \mu \in \Lambda^+$ or $1\lambda + 1\mu \in \mathbb{N}[\Lambda^+]$). Note that $\mathbb{N}[\Lambda^+] = \mathbb{N}[\{\text{irreducibles}\}]$.

Corollary 8.7. *There is a commutative monoid isomorphism*

$$\mathbb{N}[\Lambda^+] \xrightarrow{\cong} \frac{\begin{pmatrix}\text{finite-dimensional}\\ \mathfrak{g}\text{-modules}\end{pmatrix}}{\text{isomorphism}}, \qquad \sum_i n_i e(\mu_i) \mapsto \bigoplus_i n_i V(\mu_i)$$

Now, since Λ^+ is a commutative monoid, $\mathbb{N}[\Lambda^+]$ is, in fact, a commutative semiring (this is the analogous of the group ring for monoids/semirings), with multiplication dictated by $e(\lambda)e(\mu) := e(\lambda + \mu)$. Since Λ^+ even has a basis, then we have that

$$\mathbb{N}[\Lambda^+] \cong \mathbb{N}[\mathbb{N}\lambda_1 \oplus \cdots \oplus \mathbb{N}\lambda_n] \cong \mathbb{N}[u_1, \ldots, u_n],$$

the semiring of polynomials with non-negative integral coefficients. However, the map from the preceding corollary fails to be a semiring map, since it would imply that $V(\lambda) \otimes V(\mu) \cong V(\lambda + \mu)$, which is very false in general, cf. the Clebsch-Gordan formula for $\mathfrak{sl}_2$. We can, however, modify the previous monoid isomorphism and lift it to a semiring isomorphism. The observation is that the variable u_i corresponds to $e(\lambda_i)$ which in turn corresponds to $V(\lambda_i)$.

Theorem 8.8. *The unique semiring map*

$$\mathbb{N}[u_1, \ldots, u_n] \xrightarrow{\cong} \frac{\begin{pmatrix}\text{finite-dimensional}\\ \mathfrak{g}\text{-modules}\end{pmatrix}}{\text{isomorphism}}, \qquad u_i \mapsto V(\lambda_i)$$

is an isomorphism.

Proof. The key observation is that if v, v' are maximal weight vectors of $V(\lambda), V(\mu)$ of weight λ, μ, then $v \otimes v'$ is a maximal weight vector of $V(\lambda) \otimes V(\mu)$ of weight $\lambda + \mu$. Since $\dim V(\lambda)_\lambda = 1 = \dim V(\mu)_\mu$, this implies that $\dim(V(\lambda) \otimes V(\mu))_{\lambda+\mu} = 1$, so in the decomposition of $V(\lambda) \otimes V(\mu)$ in irreducibles, $V(\lambda + \mu)$ must appear exactly once, and all other representations occur with smaller weight.

To see surjectivity of the map, it suffices to consider irreducible representations by the additivity of the map. Inductively, given $V = V(\sum n_i \lambda_i)$, the image $V(\lambda_1)^{\otimes k_1} \otimes \cdots \otimes V(\lambda_n)^{\otimes k_n}$ of $u_1^{n_1} \cdots u_n^{k_n}$ is the sum of $V(\sum n_i \lambda_i)$ and other irreducible with smaller highest weight, which by induction are already in the image. Injectivity is similar. $\square$

8.1 The Representation Ring $\mathrm{Rep}(\mathfrak{g})$

The *representation ring* $\mathrm{Rep}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is given by the Grothendieck construction applied to the semiring of $\mathfrak{g}$-modules. That is, $\mathrm{Rep}(\mathfrak{g})$ is the quotient of the free abelian group generated by finite-dimensional $\mathfrak{g}$-modules modulo the subgroup spanned by the relation $[V \oplus W] = [V] + [W]$. Of course, this is the same thing as the Grothendieck ring of the category of finite-dimensional $\mathfrak{g}$-modules, $\mathrm{Rep}(\mathfrak{g}) = K(\mathsf{Mod}_{\mathfrak{g}})$. This becomes a commutative ring with the product $[V] \cdot [W] := [V \otimes W]$.

Applying the Grothendieck construction to the semiring isomorphism above we get an isomorphism

$$\mathbb{Z}[u_1, \dots, u_n] \xrightarrow{\cong} \mathrm{Rep}(\mathfrak{g}).$$

Next we would like to give another description of the representation ring, via the so-called *formal character*.

Let Λ be the lattice of integral weights, which is a free abelian group generated by the fundamental dominant weights λ_i's. Given a dominant weight λ, we have already mentioned that $V(\lambda)$ is finite-dimensional and all weights μ of $V(\lambda)$ are, in fact, integral weights, which are smaller. By Weyl's theorem, this implies that any finite-dimensional representation has only integral weights, i.e., $\Pi \subset \Lambda$. We consider the integral group ring $\mathbb{Z}[\Lambda]$, and as above, we put $e(\lambda)$ for their basis elements, with product $e(\lambda)e(\mu) := e(\lambda + \mu)$.

Given a finite-dimensional representation V of $\mathfrak{g}$, the *formal character* of V is

$$\mathrm{ch}(V) := \sum_{\mu \in \Pi} \dim(V_\mu) e(\mu) \in \mathbb{Z}[\Lambda].$$

The action of the Weyl group on Λ extends to the group ring in the obvious way, $\sigma e(\lambda) := e(\sigma\lambda)$.

Proposition 8.9. *The formal character satisfies the following properties:*

1. *It is additive,* $\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$*,*
2. *It is multiplicative,* $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V) \cdot \mathrm{ch}(W)$*,*
3. *It is invariant under the action of the Weyl group,* $\sigma(\mathrm{ch}(V)) = \mathrm{ch}(V)$*,*
4. *Any element of* $\mathbb{Z}[\Lambda]$ *which is fixed by the action of the Weyl group can be expressed, in a unique way, as an integral linear combination of* $\mathrm{ch}(V(\lambda))$ *for* $\lambda \in \Lambda^+$*.*

Items (1) and (2) imply that there is a well-defined ring homomorphism

$$\mathrm{ch} : \mathrm{Rep}(\mathfrak{g}) \to \mathbb{Z}[\Lambda],$$

item (3) says that the image of ch is contained in the subring $\mathbb{Z}[\Lambda]^W$ of W-invariants, so we get a ring homomorphism

$$\mathrm{ch} : \mathrm{Rep}(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^W,$$

and (4) that the previous map is a bijection, hence we obtain

Theorem 8.10. *The formal character induces a ring isomorphism*

$$\mathrm{ch} : \mathrm{Rep}(\mathfrak{g}) \xrightarrow{\cong} \mathbb{Z}[\Lambda]^W.$$

Furthermore, by the Weyl theorem, the semiring of isomorphism classes of finite-dimensional representations is cancellative, i.e., a $\mathfrak{g}$-module isomorphism $V \oplus Z \cong W \oplus Z$ implies $V \cong W$. Therefore, two representations V, W are isomorphic if and only if $[V] = [W]$ in $\mathrm{Rep}(\mathfrak{g})$, and by the above theorem we get

Theorem 8.11. *Two finite-dimensional $\mathfrak{g}$-modules V, W are isomorphic if and only if they have the same formal character,* $\mathrm{ch}(V) = \mathrm{ch}(W)$*.*

Example 8.12. For $\mathfrak{sl}_2$, the irreducible representations are $V(m)$, $m \geq 0$, with highest weight m, so $\mathbb{Z}[\Lambda] = \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$, $t = e(1)$. The generator of the Weyl group $W = \mathbb{Z}/2$ acts by $t \mapsto t^{-1}$, hence $\mathbb{Z}[\Lambda]^W$ consists of palindromic polynomials, i.e.,

$$\mathrm{Rep}(\mathfrak{sl}_2) \cong \mathbb{Z}[t + t^{-1}]$$

via the formal character.

Now, the weight space decomposition of $V(m)$ gives

$$\mathrm{ch}_{V(m)} = t^m + t^{m-2} + t^{m-4} + \cdots + t^{-m} = \frac{t^{m+1} - t^{-m-1}}{t - t^{-1}}.$$

This implies that

$$\begin{aligned}
\mathrm{ch}(V(n) \otimes V(m)) &= \mathrm{ch}(V(n)) \cdot \mathrm{ch}(V(m)) \\
&= \frac{(t^{n+1} - t^{-n-1})(t^{m+1} - t^{-m-1})}{(t - t^{-1})^2} \\
&= \frac{t^{m+n+2} - t^{m-n} + t^{-m-n-2} - t^{n-m}}{(t - t^{-1})^2} \\
&= \frac{t^{m+n+1} - t^{-m-n-1}}{t - t^{-1}} + \frac{t^{m+n-1} - t^{-m-n+1}}{t - t^{-1}} + \cdots + \frac{t^{m-n+1} - t^{-m+n-1}}{t - t^{-1}} \\
&= \mathrm{ch}(V(n+m) + V(n+m-2) + V(n+m-4) + \cdots + V(m-n)),
\end{aligned}$$

and we obtain the Clebsch-Gordan formula, which was given above, using the preceding theorem.

Example 8.13. For $\mathfrak{a}_n = \mathfrak{sl}_{n+1}$, we have fundamental weights $\lambda_1, \ldots, \lambda_n$ with corresponding irreducible standard cyclic modules $V = \mathbb{C}^{n+1}$ the standard representation, and $\Lambda^m V$ for $m \leq n$. So

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] = \mathbb{Z}[t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}] = \mathbb{Z}[t_1, \ldots, t_n, t_{n+1}]/(t_1 \cdots t_{n+1} - 1).$$

The Weyl group $W = \mathfrak{S}_{n+1}$ acts permuting the indices. Hence $\mathbb{Z}[\Lambda]^W$ consists of symmetric polynomials, i.e.,

$$\mathrm{Rep}(\mathfrak{sl}_{n+1}) \cong \mathbb{Z}[s_1, \ldots, s_n]$$

where s_i is the i-th elementary symmetric function on $t_1, \ldots, t_{n+1}$.

8.2 Weyl's Character Formula, Dimension Formula, and Kostant's Formula

I want now to give a concrete formula for the formal character of the irreducible representations. Recall that we are writing $\lambda_1, \ldots, \lambda_n$ for the fundamental dominant weights. The weight $\delta := \sum_i \lambda_i$ is called the *minimal strongly dominant weight*. It can easily be seen that $\delta = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha$. Now, given an integral weight $\mu \in \Lambda$, define

$$\omega(\mu) := \sum_{\sigma \in W} \mathrm{sgn}(\sigma) e(\sigma\mu) \in \mathbb{Z}[\Lambda],$$

where $\mathrm{sgn}(\sigma) := (-1)^p$ where p is the minimum number of reflections that gives σ, in other words $\mathrm{sgn}(\sigma) = \det(\sigma)$.

Theorem 8.14. *Weyl's character formula: For any dominant weight $\lambda \in \Lambda^+$, the formal character of $V(\lambda)$ is given by*

$$\mathrm{ch}(V(\lambda)) = \frac{\omega(\lambda + \delta)}{\omega(\delta)}.$$

Example 8.15. For $\mathfrak{sl}_2$, we have that $e(m) = t^m$ and hence $\omega(m) = t^m - t^{-m}$. On the other hand, $\delta = 1$, hence

$$\mathrm{ch}(V(m)) = \frac{\omega(m+1)}{\omega(1)} = \frac{t^{m+1} - t^{-m-1}}{t - t^{-1}},$$

as we saw before.

Theorem 8.16. *Dimension formula: For any dominant weight $\lambda \in \Lambda^+$, the dimension of $V(\lambda)$ is given by*

$$\dim(V(\lambda)) = \prod_{\alpha \in \Phi_+} \frac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle} = \prod_{\alpha \in \Phi_+} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)}.$$

Theorem 8.17. *Kostant's formula: For any dominant weight $\lambda \in \Lambda^+$ and any integral weight $\mu \in \Lambda$, the dimension of the weight space $V(\lambda)_\mu$ is given by*

$$\dim(V(\lambda)_\mu) = \sum_{\sigma \in W} \operatorname{sgn}(\sigma) p(\sigma(\lambda + \delta) - (\mu + \delta)),$$

where for an integral weight ν, we have written $p(\nu)$ for the number of ways that ν can be written as a sum of positive roots.

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