Construction of modular tensor categories from Hopf algebras

Outline.

Thm. Ribbon factorisable Hopf algebras produce modular tensor categories (MTC)
Cor. Semisimple ——————————————— fusion ——— (MFC)

Where do we find such Hopf algebras?

Drinfeld double construction. If H is a f.d. Hopf algebra, we will produce a factorisable quasi-triangular Hopf algebra $D(H)$ – which in turn will produce a factorisable braided finite tensor category.

If H semisimple $\Rightarrow D(H)$ is a ss factorisable ribbon Hopf algebra – which in turn will produce a MFC.

Throughout these notes, I will ellaborate on two running examples:

(1) Semisimple example. If G is a finite gp, a ss ribbon fact Hopf algebra $D(G)$, called the double of G.

(2) Non-semisimple example. If $r \geq 3$ odd integer and $q = e^{\frac{2\pi i}{r}}$ a primitive root of unity, a non-semisimple ribbon factorisable Hopf algebra $u_q sl_2$, called the small sl_2-quantum group.

I : Hopf Algebras & Semisimplicity

Def. An *algebra* over a field $\mathbb{K}$ is a v.s. A w/ $\mu: A\otimes A \to A$ and unit $\eta: \mathbb{K} \to A$ which is associative & unital.

$$\mu\circ(\mu\otimes \mathrm{Id}) = \mu(\mathrm{Id}\otimes\mu), \qquad \mu(\mathrm{Id}\otimes\eta) = \mathrm{Id} = \mu(\eta\otimes \mathrm{Id})$$

Ex A. If G is a finite gp, $\mathbb{K}[G] = \bigoplus_{g\in G} \mathbb{K}\cdot g = \{\sum_{g\in G} \lambda_g \cdot g\}$ its group algebra

is a $\mathbb{K}$-algebra w/

$$G\times G \xrightarrow[\text{of } G]{\text{multip.}} G \quad \rightsquigarrow \quad \begin{array}{ccc} \mathbb{K}[G\times G] & \xrightarrow{\mathbb{K}[\ ]} & \mathbb{K}[G] \\ \wr\wr & & \\ \mathbb{K}[G]\otimes\mathbb{K}[G] & & \end{array}$$

and unit $\ast \xrightarrow{e} G \quad \rightsquigarrow \quad \eta: \mathbb{K} \longrightarrow \mathbb{K}[G]$, $\ 1 \longmapsto e$

Ex B. If $\mathfrak{g}$ a Lie algebra, its *universal envelopping algebra* is

$$U(\mathfrak{g}) = \frac{\bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}}{[x,y] = x\otimes y - y\otimes x}$$

w/ multiplication induced by $\mathfrak{g}^{\otimes n} \times \mathfrak{g}^{\otimes m} \xrightarrow{\otimes} \mathfrak{g}^{\otimes n+m}$.

$$\begin{cases} \text{Ex A} \quad \text{commutative} \iff G \text{ abelian} \\ \text{Ex B} \quad \text{————} \iff \mathfrak{g} \text{ ————}. \end{cases}$$

Def. A coalgebra (C, Δ, ε) is a vs C w/ $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \mathbb{K}$ which is coassociative & counital:

$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta \quad , \quad (\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id} = (\varepsilon \otimes \mathrm{Id})\Delta .$$

Ex A. Consider the diagonal map $\begin{array}{ccc} G & \longrightarrow & G \times G \\ g & \longmapsto & (g, g) \end{array} \quad \overset{\mathbb{K}[-]}{\rightsquigarrow}$

$$\begin{array}{ccccc} \mathbb{K}[G] & \longrightarrow & \mathbb{K}[G \times G] & \cong & \mathbb{K}[G] \otimes \mathbb{K}[G] \\ g & \longmapsto & & & g \otimes g \end{array}$$

and $G \to *$ the unique map $\rightsquigarrow \begin{array}{ccc} \mathbb{K}[G] & \longrightarrow & \mathbb{K} \\ g & \longmapsto & 1 \end{array}$

Ex B For $x \in \mathfrak{g}$, let $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $\varepsilon(x) = 0$.

For $x_1 \otimes \cdots \otimes x_r \in \mathfrak{g}^{\otimes n}$,

$$\Delta(x_1 \otimes \cdots \otimes x_r) = \Delta(x_1 \cdot \ldots \cdot x_r) := \Delta(x_1) \cdot \ldots \cdot \Delta(x_r).$$

$$\left\{ \begin{array}{l} \text{Ex A always cocommutative} \\ \text{Ex B} \ \underline{\qquad\qquad\qquad\qquad} \end{array} \right.$$

Def. Let B be a vs equipped simultaneously w/ an alg structure (B, μ, η) and a coalgebra str (B, Δ, ε). We say that B is a bialgebra if any of the following equivalent conditions holds:

1) Δ, ε are algebra maps
2) μ, η are coalgebra maps.

Ex A and Ex B are bialgebras

Def. Let B be a bialgebra. On $\mathrm{End}_{\mathbb{K}}(B) = \mathrm{Hom}_{\mathbb{K}}(B, B)$ we can consider the following product: given $f, g : B \to B$, its convolution $f * g$ is ~~defined~~ defined as the composite

$$\begin{tikzcd} B \arrow[r, "\Delta"] \arrow[rrr, bend right, "=: f*g"'] & B \otimes B \arrow[r, "f \otimes g"] & B \otimes B \arrow[r, "\mu"] & B \end{tikzcd}$$

It defines an algebra structure on $\mathrm{End}(B)$ w/ unit $\eta \circ \varepsilon : B \to B$.

Def. A bialgebra B is called a Hopf algebra if $\mathrm{Id} : B \to B$ is invertible in $\mathrm{End}(B)$. The inverse of Id (nec. unique if it exists) is called the antipode of B. and denoted by $S : B \to B$. By def

$$\mu(S \otimes \mathrm{Id})\Delta = \eta\,\varepsilon = \mu(\mathrm{Id} \otimes S)\Delta.$$

Ex A. $\begin{array}{ccc} G & \to & G \\ g & \mapsto & g^{-1} \end{array} \rightsquigarrow \begin{array}{ccc} \mathbb{K}[G] & \xrightarrow{S} & \mathbb{K}[G] \\ g & \longmapsto & g^{-1} \end{array}$ defines ~~the~~

antipode of $\mathbb{K}[G]$, making it into a Hopf algebra. Indeed

$$\begin{array}{ccccccc} \mathbb{K}G & \xrightarrow{\Delta} & \mathbb{K}G \otimes \mathbb{K}G & \xrightarrow{\mathrm{Id}\otimes S} & \mathbb{K}G \otimes \mathbb{K}G & \xrightarrow{\mu} & \mathbb{K}G \\ g & \longmapsto & g \otimes g & \longmapsto & g \otimes g^{-1} & \longmapsto & g \cdot g^{-1} = 1 \end{array}$$

precisely $(\eta\,\epsilon)(g) = \eta(1) = 1$.

Ex B. $S(x) := -x$ and in general

$$S(x_1 \otimes \cdots \otimes x_r) := (-1)^r\, x_r \otimes \cdots \otimes x_1$$

defines an antipode – also immediate to check.

The two running examples

Ex 1. Let G be a finite gp. Let $\mathbb{K}(G) := \mathrm{Hom}_{Set}(G, \mathbb{K})$,

this is a $\mathbb{K}$-v.s. w/ basis $(\varphi_g)_{g\in G}$, $\varphi_g(h) = \begin{cases} 1, & g = h \\ 0, & g \neq h \end{cases}$.

$\mathbb{K}(G)$ is an algebra w/ multiplication pointwise,

$$(f_1 \cdot f_2)g := f_1(g) \cdot f_2(g) \in \mathbb{K}.$$

and unit the constant map $\mathrm{const}_1 : \begin{array}{ccc} G & \to & \mathbb{K} \\ g & \mapsto & 1 \end{array}$. Note that

in terms of basis, $\mathrm{const}_1 = \sum_g \varphi_g$.

Let us define $D(G)$. As a vector space,

$$D(G) := \mathbb{K}[G] \otimes \mathbb{K}(G).$$

$D(G)$ is a Hopf algebra w/ the following structure maps:

$$(g \otimes \varphi_h) \cdot (\bar{g} \otimes \varphi_{\bar{h}}) := g\bar{g} \otimes \varphi_{\bar{g}^{-1} h g} \cdot \varphi_{\bar{h}}$$ multiplication

unit

$$1_{D(G)} = 1 \otimes \mathrm{const}_1$$

$$\Delta(g \otimes \varphi_h) = \sum_{x \in G} (g \otimes \varphi_x) \otimes (g \otimes \varphi_{hx^{-1}})$$ comultiplication

$$\varepsilon(g \otimes \varphi_h) = \begin{cases} 1, & h = e \\ 0, & h \neq e \end{cases}$$ counit

antipode

$$S(g \otimes \varphi_h) = g^{-1} \cdot \varphi_{gh^{-1}g^{-1}}$$

Exercise. Check that these maps define a Hopf alg str on $D(G)$.

Ex 2. Let $r \geq 3$ an integer and let $q = e^{\frac{2\pi i}{r}}$ a primitive rt of 1.

Put

$$r' = \frac{r}{\gcd(2,r)} = \begin{cases} r, & r \text{ odd} \\ \frac{r}{2}, & r \text{ even} \end{cases}$$

The <u>small quantum group</u> $u_q sl_2$ is defined as the $\mathbb{C}$-algebra generated by E, F, K w/ relations

$$KE = q^2 EK \quad , \quad KF = q^{-2} FK \quad , \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}} \quad ,$$

$$E^{r'} = 0 = F^{r'} \quad , \quad K^{r'} = 1 .$$

(in particular $K^{r'}$ is invertible w/ inverse $K^{r'-1}$, so K^{-1} exists; also $r \geq 2$ means that $q \neq q^{-1}$).

Note this is a "quatisation" / "deformation" of $U(sl_2)$.

The comultp., counit & antipode are determined by

$\Delta(K) = K \otimes K$	$\varepsilon(K) = 1$	$S(K) = K^{-1}$
$\Delta(E) = 1 \otimes E + E \otimes K$	$\varepsilon(E) = 0$	$S(E) = -EK^{-1}$
$\Delta(F) = F \otimes 1 + K^{-1} \otimes F$	$\varepsilon(F) = 0$	$S(F) = -KF$.

For ~~other~~ "words" in K, E, F, these def's are extended using the fact that Δ, ε are algebra hom & S is an algebra anti-homomorphism.

The category A-mod

Let A be a f.d. algebra over an alg closed field $\mathbb{K}$. The category A-mod of f.d. A-modules and A-module homomorphisms is a $\mathbb{K}$-linear category, which is by definition what we call a finite (abelian) category

The main goal of this talk series is to study how properties or structure on A translate to property / structure on A-mod.

I want to start by something that has nothing to do so far w/ Hopf structure.

Recall that a finite category $\mathcal{C}$ is semisimple if all obj are completely reducible, ie a direct sum of simple ones – S is simple if the only subobjects are 0 and S.

For what A do we have that A-mod is semisimple?

Recall. A two-sided ideal $I \subset A$ is nilpotent if $I^n = 0$ for some $n > 0$.

Definition. The Jacobson radical of A, $\mathrm{Jac}(A)$, is the largest nilpotent two-sided ideal of A,

$$\mathrm{Jac}(A) = \sum_{\substack{I \text{ two-sided} \\ \text{nilp. ideal of } A}} I$$

Def. Given an A-module (V,ρ), its <u>annihilator</u> is

$$\mathrm{Ann}(V) := \{ a \in A : \rho(a) \equiv \text{zero map} \}$$
$$= \{ a \in A : a \cdot V = 0 \}$$

<u>Proposition</u>. We have

$$\mathrm{Jac}(A) = \bigcap_{V \text{ simple}} \mathrm{Ann}(V)$$
$$= \{ a \in A : a \cdot V = 0 \ \forall\, V \text{ simple } A\text{-mod} \}.$$

<u>Pf</u>. $\subseteq$) Take V simple & $v \in V$ arbitrary. Then $\mathrm{Jac}(A) \cdot v \subset V$ is a submodule, and since V simple, $\mathrm{Jac}(A) \cdot v = 0$ or V. If $\mathrm{Jac}(A) \cdot v = V$ then $\exists\, a \in \mathrm{Jac}(A)$ such that $a \cdot v = v$. But then this means that $a^n \neq 0$ for any n!! Cannot be as $\mathrm{Jac}(A)$ is nilpotent. So $\mathrm{Jac}(A) v = 0$ and since v is arbitrary and so is V we conclude.

$\supseteq$). $\mathrm{Jac}(A)$ is the largest two-sided nilp. ideal of A, so it suffices to see that $\mathfrak{a} := \bigcap_{V \text{ simple}} \mathrm{Ann}(V)$ is a (two-sided) nilp. ideal. View A as an A-module (the <u>regular representation</u>), and consider a filtration of it,

$$0 = A_0 \subset A_1 \subset \dots \subset A_m = A$$

w/ A_i / A_{i-1} simple.

Now if $x \in \mathfrak{a}$, it acts by 0 on A_i/A_{i-1}, ie, $\rho(x) : A \to A$ maps A_i to A_{i-1}. Ie, $\mathfrak{a}^n = 0$. $\square$

<u>Definition</u>. An algebra A is called <u>semisimple</u> if $\mathrm{Jac}(A) = 0$.

<u>Theorem</u>. Let A be a fd algebra. Then the following are equivalent.

1) A is a semisimple algebra
2) A, as an A-module, is completely reducible
3) The category A-mod is semisimple.

Pf. <u>2) $\Rightarrow$ 3)</u> | . If V is a fd A-mod, let $e_1 \dots e_m$ be a basis and let

$$A^{\oplus m} \longrightarrow V \qquad (a_1, \dots a_m) \longmapsto \sum a_i e_i$$

obviously surjective; and immediate to see that it is an intertwiner. Ie V is a quotient A-module of $A^{\oplus m}$. By hyp, A is compl. red, ~~and~~ then so is $A^{\oplus m}$. Since quotients of comp. red mod are so as well, we conclude.

<u>3) $\Rightarrow$ 1)</u> | If any fd A-mod is comp. red, in particular A, ie $A = A_1 \oplus \cdots \oplus A_r$. If $a \in \mathrm{Jac}(A)$, then $a \cdot A_i = 0 \;\forall i$, so $a \cdot A = 0$, in particular $a = a \cdot 1 = 0$.

1) $\Rightarrow$ 2) Key fact. A fd algebra has finitely many simple A-modules (in fact at most $\dim A$) $V_1, \dots, V_r$, and there is an isomorphism

$$A/\mathrm{Jac}(A) \xrightarrow{\cong} \bigoplus_i V_i^{\oplus n_i}, \qquad n_i = \dim V_i.$$

So if $\mathrm{Jac}(A) = 0$ we are done. $\square$

Ex. A

Theorem (Maschke, 1899). Let G be a finite group. Then

$\mathbb{K}[G]$ semisimple $\iff$ char $\mathbb{K}$ does not divide $|G|$.

In particular if char $\mathbb{K} = 0$, $\mathbb{K}[G]$ always ss.

Ex 1. We will see later that $D(G)$ is a ss algebra from a more general result

Ex 2. $u_q \mathfrak{sl}_2$ is not semisimple.

<u>About the proof</u>. If H is a f.d Hopf algebra, let

$$\mathcal{I}_\ell(H) := \{ x \in H : hx = \varepsilon(h)\cdot x \ \ \forall h \in H \}$$

be the space of <u>left integrals</u>, which happens to be a 1-dim lin subspace. One can prove the following generalisation of Maschke's thm for arbitr. f.d Hopf algebras:

$$H \text{ semisimple} \iff \varepsilon(\mathcal{I}_\ell(H)) \neq 0.$$

When $H = u_q\mathfrak{sl}_2$, then $\mathcal{I}_\ell(u_q\mathfrak{sl}_2)$ can be seen to be spanned by

$$\Lambda := E^{r-1}F^{r-1}\cdot \sum_{i=0}^{r-1} K^i$$

Because $\varepsilon(E) = \varepsilon(F) = 0$, $\varepsilon(\mathcal{I}_\ell(u_q\mathfrak{sl}_2)) = 0$, so $u_q\mathfrak{sl}_2$ is non-SS.

Next I want to explain how str / prop on a ~~Hopf~~ algebra A translates into str / prop of A-mod.

<u>Def</u>. A monoidal structure on A-mod is said to be <u>$\mathbb{K}$-extended</u> if $\otimes = \otimes_{\mathbb{K}}$, $\mathbb{1} = \mathbb{K}$ and the associator and left/right unitors as in Vect.

<u>Proposition</u>. Let A be a ~~..~~ f.d algebra. There is a bijection

$$\left\{ \begin{matrix} \text{bialgebra} \\ \text{str on } A \end{matrix} \right\} = \left\{ \begin{matrix} \mathbb{K}\text{-extended} \\ \text{monoidal str} \\ \text{on } A\text{-mod} \end{matrix} \right\}$$

<u>About the pf</u>. If $(A, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra, then view $V \otimes W = V \otimes_{\mathbb{K}} W$ as an A-module w/

$$a \cdot (v \otimes w) := \Delta(a) \cdot (v \otimes w) = \sum_{(a)} a_{(1)} v \otimes a_{(2)} w .$$

(here I used Sweedler's notation $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$). Then ~~twisted~~ $\mathbb{K}$ get an A-module structure with $a \cdot \lambda := \varepsilon(a) \cdot \lambda$.

Conversely, given a $\mathbb{K}$-extended monoidal str on A-mod, take A viewed as A-module, and define $\Delta : A \to A \otimes A$ as

$$\Delta(a) := a \cdot (1_A \otimes 1_A) \in A \otimes A$$

($A \otimes A$ is an A-module). Similarly since $\mathbb{K}$ is an A-module define $\varepsilon : A \to \mathbb{K}$ as

$$\varepsilon(a) := a \cdot 1_{\mathbb{K}} \in \mathbb{K} .$$

$\square$

<u>Remark</u>. If H is a f d Hopf algebra, then it is automatic that S is invertible. In fact if $\operatorname{char} \mathbb{K} \neq 0$, then S has finite order by a classical theorem of Radford. In fact, if $S \neq \mathrm{Id}$, then the order is even.

The antipode plays a major role in the duality of A-mod.

<u>Remark</u>. Recall that being left/right rigid is a property, not structure.

<u>Proposition</u>. Let B be a f.d bialgebra. Then

$$\begin{array}{c} B \text{ is a Hopf algebra} \\ \text{(ie, the antipode exists)} \end{array} \iff B\text{-mod is rigid.}$$

<u>About the proof</u>. If B is a Hopf algebra, then set for $V \in B$-mod

$$V^\vee := V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$$

$$^\vee V := V^*$$

and the A-module structure on $V^\vee$ and $^\vee V$ as

$$(a.f)(v) := f(S(a).v) \quad , \quad (a.f)(v) = f(S^{-1}(a).v)$$

resp. Conversely, suppose that B-mod is rigid, ie every object V has left and right duals $V^\vee$ and $^\vee V$. Ie we have B-linear maps

$$ev_V : V^\vee \otimes_{\mathbb{K}} V \longrightarrow \mathbb{K} \quad , \quad coev_V : \mathbb{K} \longrightarrow V \otimes_{\mathbb{K}} V^\vee$$

satisfying the zigzag eq's, similarly for the right duality. Since these maps are in particular $\mathbb{K}$-linear, this means that as $\mathbb{K}$-vector spaces $V^\vee \cong V^* \cong {}^\vee V$. In particular the $^\vee(V^\vee) \cong V$ becomes

$$\begin{aligned} V &\longrightarrow V^{**} \\ v &\longmapsto (\omega \mapsto \omega(v)) \end{aligned}$$

View A as A-mod. For $a \in A$ define $S(a)$ as the elmt in A corresponding to $\omega \mapsto (a.\omega)(1_A)$ in A^{**}. Ie $S(a)$ is the unique elmt of A st

$$(a.\omega)(1) = \omega(S(a)) \qquad \forall \omega \in A^* .$$

□

The upshot is that we have built a rough dictionary

A	A-mod
semisimple	semisimple
bialgebra	monoidal ($\mathbb{k}$-extended)
antipode	rigid

So if we start w/ a Hopf algebra H, then H-mod is a finite monoidal rigid linear category, $\otimes$ is bilinear w/ unit $\mathbb{k}$ simple. This is precisely what we call a <u>finite tensor category</u>.

If H is semisimple, then H-mod is additionally semisimple, and it is exactly what by definition is a <u>fusion category</u>.

II. QUASI-TRIANGULAR & FACTORISABLE HOPF ALGEBRAS

Today we will introduce an important structure on Hopf algebras.

Def. Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. A quasi-triangular structure on H is a choice of an invertible elmt $R \in H \otimes H$, called the universal R-matrix, satisfying

$$(\Delta \otimes \mathrm{Id})\, R = R_{13} R_{23}$$

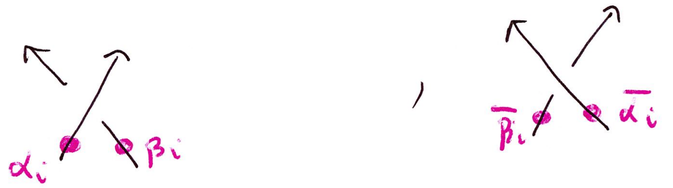
$$(\mathrm{Id} \otimes \Delta)\, R = R_{13} R_{12}$$

$$\Delta^{op} = R \cdot \Delta(-) \cdot R^{-1}$$

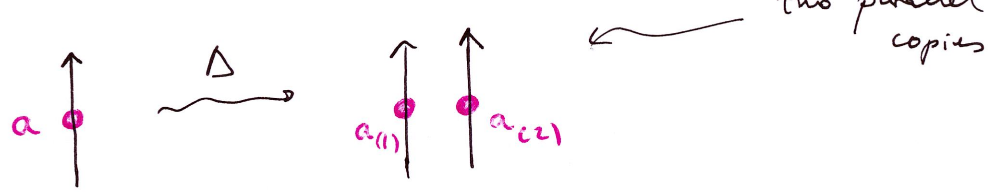
where $R_{12} = R \otimes 1 \in H^{\otimes 3}$, $R_{23} = 1 \otimes R \in H^{\otimes 3}$, and $R_{13} = (\mathrm{flip} \otimes \mathrm{Id})\, R_{23}$.

The pair (H, R) is a qt Hopf alg

Knot-theoretical interpretation. R represents the positive crossing, R^{-1} the neg one: if $R = \sum_i \alpha_i \otimes \beta_i$ and $R^{-1} = \sum_i \bar{\alpha}_i \otimes \bar{\beta}_i$, then

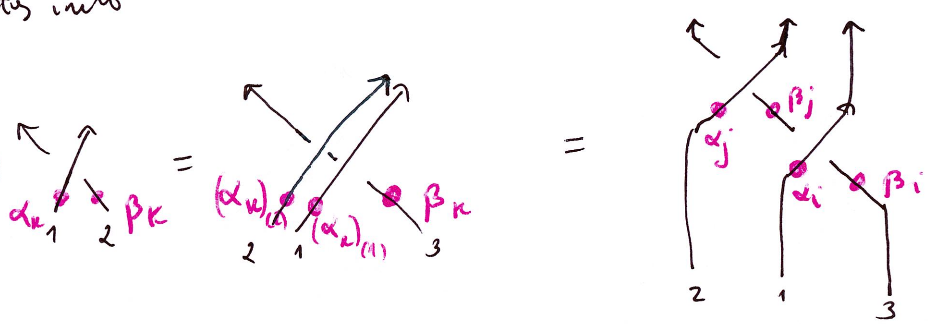


In this picture, Δ corresponds w/ "strand doubling"



Eg the first axiom translates into

$(\Delta \otimes \mathrm{Id})\, R = (\mathrm{double} \otimes \mathrm{Id})$



<u>Main properties</u>. 1) R satisfies the quantum Yang-Baxter equation :

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}$$

(in the knot-theoretical picture, this amounts to the braid relation)

2) $$(S \otimes \mathrm{Id})R = R^{-1} = (\mathrm{Id} \otimes S^{-1})R$$

3) $$(S \otimes S)R = R$$

4) $$(\varepsilon \otimes \mathrm{Id})R = 1 = (\mathrm{Id} \otimes \varepsilon)R$$

Recall that if H is a fd Hopf algebra, H-mod was a finite tensor category: a monoidal linear rigid ~~tensor~~ finite category w/ $\otimes$ bilin and simple unit.

We want to see what the structure of quasi-triang translates into A-mod.

<u>Proposition</u>. Let H be a fd Hopf alg. There is a bijection

$$\left\{ \begin{array}{c} \text{quasi-triangular} \\ \text{structures on } H \end{array} \right\} = \left\{ \text{braidings on } H\text{-mod} \right\}$$

<u>About the proof</u>. Given (H,R) qt, then a braiding is induced as

$$c_{V,W} : V \otimes W \longrightarrow W \otimes V$$
$$v \otimes w \longmapsto c_{V,W}(v \otimes w) := R_{21}(w \otimes v)$$

Conversely, if H-mod is endowed w/ a braiding $c_{V,W}$, view H as H-module and define

$$R := \mathrm{flip}\left(c_{H,H}(1_H \otimes 1_H) \right)$$

$\square$

<u>Remark</u>. A qt Hopf alg (H,R) is <u>triangular</u> if $R^{-1} = R_{21}$. In this case, H-mod is a symmetric monoidal category.

Triangular Hopf algebras are "degenerate" in the sense that there is no difference between the positive and negative crossing in the knot-theoretical picture.

Or in terms of the braiding, $c_{W,V} c_{V,W} = \mathrm{Id}_{V\otimes W}$ for any V, W.

I am interested in the extreme opposite, ie where R is "maximally non-deg".

<u>Def</u>. Let (H,R) be a fd qt Hopf algebra, and let $Q := R_{21} R \in H \otimes H$, the so-called monodromy element. The <u>Drinfeld map</u> is the $\mathbb{K}$-linear map

$$\begin{aligned} \Phi : H^* &\longrightarrow H \\ \omega &\longmapsto (\mathrm{Id} \otimes \omega) Q \end{aligned}$$

This map is never the zero map: for the counit ε we have

$$\Phi(\varepsilon) = (\mathrm{Id} \otimes \varepsilon)(R_{21} \cdot R) = \underbrace{(\mathrm{Id} \otimes \varepsilon) R_{21}}_{1} \cdot \underbrace{(\mathrm{Id} \otimes \varepsilon) R}_{1} = 1,$$

so $\mathrm{Im}\, \Phi$ is at least 1 dim.

<u>Def</u>. A fd qt Hopf alg is <u>factorisable</u> if the Drinfeld map is an isomorphism.

<u>Remark</u>. If (H,R) is triangular, then $Q = 1 \otimes 1$ and then for any $\omega \in H^*$,

$$\Phi(\omega) = (\mathrm{Id} \otimes \omega)(1 \otimes 1) = \underbrace{\omega(1)}_{\in \mathbb{K}} \cdot 1,$$

ie $\mathrm{Im}\, \Phi$ is 1-dim — the minimum possible. In this sense being factorisable is the opposite

I want to give an alternative characterisation of being factorisable.

Proposition. Let (H,R) be a f d qt Hopf alg. Then

$$H \text{ factorisable} \iff \exists\ (e_i), (v_i) \text{ bases for } H \text{ st}$$
$$Q = \sum_{i=1}^{n} v_i \otimes e_i .$$

Pf $\Leftarrow$) Let (e_i^*) dual basis of (e_i). Then the Drinfeld map

$$\Phi : H^* \longrightarrow H$$
$$\omega \longmapsto (\mathrm{Id} \otimes \omega)\, Q = (\mathrm{Id} \otimes \omega)\left(\sum v_i \otimes e_i\right)$$
$$e_k^* \longmapsto \sum v_i \otimes e_k^*(e_i) = v_k$$

ie Φ sends a basis to a basis, ie Φ is an iso.

$\Rightarrow$) Let (e_i) basis for H, (e_i^*) dual basis, and define

$v_k := \Phi(e_k^*)$. A basis for $H \otimes H$ is given by $(e_p \otimes e_q)$, so

$Q = \sum_{p,q} \lambda_{pq}\, e_p \otimes e_q$. Now observe that

$$v_k = \Phi(e_k^*) = (\mathrm{Id} \otimes e_k^*)\left(\sum_{pq} \lambda_{pq}\, e_p \otimes e_q\right) = \sum \lambda_{p,q}\, e_k^*(e_q)\, e_p = \sum_p \lambda_{pk}\, e_p$$

So $Q = \sum_{pq} \lambda_{pq}\, e_p \otimes e_q \underset{q=k}{=} \sum_{p,k} \lambda_{pk}\, e_p \otimes e_k = \sum_k v_k \otimes e_k$. $\square$

Now the question is: ~~[illegible]~~ what does factorisability translate into when taking H-mod?

Def. Let $\mathcal{C}$ be a braided finite tensor cat (FTC). An object $T \in \mathcal{C}$ is <u>transparent</u> if $c_{X,T}\, c_{T,X} = \mathrm{Id}_{T\otimes X}$ $\forall X \in \mathcal{C}$. The <u>Müger centre</u> $Z_2(\mathcal{C})$ of $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ consisting of transparent objects. We say that the Müger centre is <u>trivial</u> if every transparent object is isomorphic to a direct sum of finitely many copies of $1 \in \mathcal{C}$; in this case $Z_2(\mathcal{C}) \simeq \mathrm{vect}$.
$$1 \longleftrightarrow \Bbbk .$$

Def. Let $\mathcal{C}$ be a FTC. The <u>Drinfeld centre</u> of $\mathcal{C}$ is the following cat:

$$Z(\mathcal{C}): \begin{cases} \underline{\text{obj}}. \ \text{pairs } (X, \xi_{X,-}) \text{ where } X \in \mathcal{C} \text{ and } \xi_{X,-} \text{ is a nat transf} \\ \qquad \xi_{X,Y} : X \otimes Y \longrightarrow Y \otimes X \qquad \text{(half-braiding)} \\ \qquad \text{st } \xi_{X,1} = \mathrm{Id}_X \text{ and } \xi_{X,U\otimes V} = (\mathrm{Id}_U \otimes \xi_{X,V})(\xi_{X,U} \otimes \mathrm{Id}_V) \\ \underline{\text{arrows}}. \ \text{arrows in } \mathcal{C} \text{ compatible w/ half-braidings.} \end{cases}$$

It turns out that $Z(\mathcal{C})$ is always a braided FTC.

Def. Let $\mathcal{C}$ be a braided ftc, and let $\boxtimes$ be the Deligne-Kelly product of finite categories. Define $G: \mathcal{C} \boxtimes \mathcal{C} \longrightarrow Z(\mathcal{C})$ as the composite

$$\mathcal{C} \boxtimes \mathcal{C} \xrightarrow{T_+ \boxtimes T_-} Z(\mathcal{C}) \boxtimes Z(\mathcal{C}) \xrightarrow{\text{tensor prod}} Z(\mathcal{C})$$

w/ $T_+(X) = (X, c_{X,-})$ and $T_-(X) = (X, c^{-1}_{-,X})$.

<u>Theorem</u> (Shimizu, 2016). Let $\mathcal{C}$ be a braided FTC. Then

$$\mathcal{Z}_2(\mathcal{C}) \text{ is trivial} \iff G \text{ is an equivalence}.$$

<u>Def</u>. A braided FTC $\mathcal{C}$ is called <u>factorisable</u> if ~~they~~ of the equivalent conditions of the above thm holds.

Let's return to our main task :

<u>Theorem</u>. Let (H, R) be a ~~fd~~ qt Hopf algebra. Then

$$H \text{ factorisable} \iff H\text{-mod factorisable}.$$

<u>About the proof</u>. By a result of Schneider, H factorisable iff $D(H) \cong (H \otimes H)^{F_R}$, where $(H \otimes H)^{F_R}$ is a Hopf alg w/ the same underlying alg str of $H \otimes H$ but w/ Δ and S "twisted" by a given "2-cocycle" $F_R \in H^{\otimes 4}$.

But then

$$\begin{aligned} H\text{-mod} \boxtimes H\text{-mod} &\simeq (H \otimes H)\text{-mod} \\ &\simeq D(H)\text{-mod} \\ &\simeq \mathcal{Z}(H\text{-mod}) \end{aligned}$$

$\square$

Recall that a FTC is *pivotal* if equipped w/ a monoidal nat iso

$$\omega : \mathrm{Id} \overset{\cong}{\Longrightarrow} (-)^{\vee\vee}$$

What is the counterpart of this str in the Hopf alg world?

- Write $G(H) = \{x \in H : \Delta(x) = x \otimes x\}$ for the set of *grouplike* elements of a Hopf alg H. By counitality it follows that if $x \in G(H)$ then $\varepsilon(x) = 1$. Also if $x \in G(H)$, then x is invertible w/ $x^{-1} = S(x)$ (follows from antipode axiom).

Def. A *pivotal Hopf algebra* is a pair (H, k) where H is a Hopf alg and $k \in G(H)$ is a choice of a gplike elmt, called the *balancing element* or *pivot*, st it implements S^2 by conjugation:

$$S^2(x) = k \cdot x \cdot k^{-1} \qquad \forall x \in H.$$

The balancing elmt is ~~not~~ unique; it is determined upto multiplication by an elmt in $G(H) \cap Z(H)$.

Proposition. Let H be a Hopf alg. There is a bijection

$$\left\{ \begin{array}{c} \text{pivots} \\ \text{in } H \end{array} \right\} = \left\{ \begin{array}{c} \text{pivotal} \\ \text{structures} \\ \text{on } H\text{-mod} \end{array} \right\}$$

About the pf. Given (H, k), define $\omega_V : V \to V^{**}$ as

$$\omega_V(v) := (\phi \mapsto \phi(k \cdot v))$$

Conversely, given a pivotal str on H-mod, take H as a H-mod; then $\kappa \in H$ is defined as the image of 1_H under the composite

$$\begin{array}{ccccccc} H & \xrightarrow{\omega_H} & H^{\vee\vee} = H^{**} & \underset{\text{canonical }\Bbbk\text{-lin iso}}{\cong} & H \\ 1_H & \longmapsto & & & \kappa \end{array}$$

$\square$

So far we have the following picture: qt pivotal Hopf algebras produce braided pivotal FTC. For such categories we defined a twist θ_X as the composite

$$X \underset{\cong}{\xrightarrow{\omega_X}} X^{\vee\vee} \underset{\cong}{\xrightarrow{D^{-1}}} X$$

where D^{-1} denotes the inverse of the <u>Drinfeld isomorphism</u>

$$\underbrace{X \xrightarrow{\mathrm{coev}_{X^\vee} \otimes \mathrm{Id}} X \otimes X^{\vee\vee} \otimes X^\vee \xrightarrow{\mathrm{Id} \otimes c_{X^{\vee\vee}, X^\vee}} X \otimes X^\vee \otimes X^{\vee\vee} \xrightarrow{\widetilde{\mathrm{ev}}_X \otimes \mathrm{Id}} X^{\vee\vee}}_{D}$$

A braided pivotal FTC is a <u>finite ribbon category</u> if the twists are self-dual in the sense that $\theta_{X^\vee} = (\theta_X)^\vee$.

(Alternatively: for a braided pivotal ftc, define the left and right twists, and demand that they coincide. You get automatically $\theta_{X^\vee} = (\theta_X)^\vee$.)

Ie, for a braided piv FTC, being ribbon is a <u>property</u>. What is the analogous property for Hopf alg?

<u>Def</u>. Let H be a Hopf algebra. The element

$$u := \mu\,(S \otimes \mathrm{Id})\, R_{21} = \sum_i S(\beta_i)\, \alpha_i$$

is called the <u>Drinfeld element</u>. It is invertible w/ inverse

$$u^{-1} = \mu(\mathrm{Id} \otimes S^2) R_{21} = \sum \beta_i \cdot S^2(\alpha_i) .$$

<u>Proposition</u>. Let (H, R, κ) be a qt pivotal Hopf algebra, and consider the corresponding braided pivotal FTC H-mod. We have

$$\kappa^2 = u \cdot S(u^{-1}) \iff H\text{-mod is a finite ribbon category} .$$

<u>About the pf</u>. Let $v := \kappa^{-1} \cdot u$ (κ is invertible as it is gplike), in particular v is also invertible. The corresponding twist is defined as

$$\begin{aligned} \theta_W : W &\longrightarrow W \\ w &\longmapsto v^{-1} w \end{aligned} .$$

Conversely, if H-mod is ribbon, view H as H-mod and let

$$\begin{aligned} \theta_H : H &\longrightarrow H \\ 1_H &\mapsto \theta_H(1_H) =: v' \end{aligned} .$$

Then define $\kappa := v' \cdot u$.

$\square$

The element v in the proof is called the <u>ribbon element</u>. Implicitly in the proof I am using a classical theorem of Kauffman and Radford that establishes a bijection between

$$\left\{ \begin{array}{c} \text{pivots } \kappa \text{ in } H \\ \text{satisfying} \\ \kappa^2 = u \cdot S(u^{-1}) \end{array} \right\} = \underbrace{\left\{ \begin{array}{l} \text{central elmts } v \in H \text{ st} \\ \bullet\ v^2 = u S(u) \\ \bullet\ S(v) = v \\ \bullet\ \varepsilon(v) = 1 \\ \bullet\ \Delta(v) = (R_{21} R)^{-1} \cdot (v \otimes v) \end{array} \right\}}_{}$$

set of ribbon elmts

$$\kappa \longmapsto \kappa^{-1} \cdot u$$

<u>Def</u>. If $\kappa^2 = u \cdot S(u^{-1})$, then (H, R, κ) is called a <u>ribbon Hopf algebra</u> (or (H, R, v)).

Ex 1. $D(G)$ turns out to have a str of ribbon Hopf alg:

$$\boxed{R = \sum_{g\in G} g \otimes \varphi_g}$$, the universal R-matrix

It is easy to see that R satisfies the axioms, eg

$$(\Delta \otimes \mathrm{Id}) R = \sum_{g\in G} g \otimes g \otimes \varphi_g$$

$$R_{13} R_{23} = \left(\sum_g g \otimes 1 \otimes \varphi_g\right) \left(\sum_h 1 \otimes h \otimes \varphi_h\right)$$

$$= \sum_{g,h} g \otimes h \otimes \varphi_g \varphi_h$$

↳ only $\neq 0$ if $g = h$, which is φ_g

$$= \sum_g g \otimes g \otimes \varphi_g$$

In this case we have that $\underline{\kappa = 1}$, or equivalently

$$v = u = \sum S(\beta_i)\, \alpha_i = \sum_g S(\varphi_g)\cdot g = \sum_g \varphi_{g^{-1}} \cdot g$$

$$= \sum g\, \varphi_{g^{-1} g\, g^{-1}} = \underline{\underline{\sum_g g\, \varphi_{g^{-1}}}} \quad \left(= \underline{\underline{\sum_g g^{-1} \varphi_g}} \right)$$

We will later see that $D(G)$ is in fact factorisable and semisimple

Ex 2. $u_q sl_2$ also happens to have a ribbon structure: recall that

$r \geq 3$ integer and $r' = \begin{cases} r/2, & r \text{ even} \\ r, & r \text{ odd} \end{cases}$.

If $[k] = \dfrac{q^k - q^{-k}}{q - q^{-1}}$ and $[k]! = [k] \cdots [1]$, then

$$\boxed{R = \frac{1}{r'} \sum_{a,b,c=0}^{r'-1} \frac{(q-q^{-1})^a}{[a]!} q^{\frac{a(a-1)}{2} - 2bc} K^b E^a \otimes K^c F^a}$$

~~The formula for~~

On the other hand, $\boxed{\kappa = K}$. The formula for the ribbon elmt is more complicated: for r odd we have

$$v = \frac{i^{\frac{r-1}{2}}}{\sqrt{r}} \sum_{a,b=0}^{r} \frac{(q^{-1}-q)^a}{[a]!} q^{-\frac{(a+3)a}{2} + \frac{r+1}{2}(b-1)^2} F^a E^a K^{-a-b}$$

(Beliakova – De Renzi)

We already saw that it is not ss.

By a result of Lyubashenko, $u_q sl_2$ is factorisable iff r' odd.

(ie $r \not\equiv 0 \bmod 4$)

* Next time we will see that these choices of R come from a general contruction.

~~The RHS is the set of ribbon elements, and in this case (H, R, v) (or $((H,R,v)$) is called a ribbon Hopf algebra.~~

- Recall that a <u>modular tensor category</u> (MTC) is a factorisable ribbon finite tensor category. A semisimple modular tensor category is called a <u>modular fusion category</u>.

The upshot of the last results is that

<u>Theorem</u>. If H is a factorisable ribbon fd Hopf algebra, then H-mod is a MTC. If H is furthermore semisimple, then H-mod is a MFC.

All in all, we have a rough dictionary

A	A-mod
ss (prop)	ss
bialgebra (str)	monoidal ($\Bbbk$-extended)
antipode (prop)	rigid
quasi-triang (str)	braided
factorisable (prop)	factorisable
pivotal (str)	pivotal
ribbon (prop)	ribbon
fact. ribbon	MTC
ss fact ribbon	MFC

VII. THE DRINFELD CENTRE CONSTRUCTION & EXAMPLES

We have seen that (SS) factor. ribbon fd Hopf alg's produce MTC (MFC). I want to introduce a machinery to produce those. Typically, the ~~quantum~~ quantum groups come w/ a str of ribbon Hopf algebra which is induced by ~~the~~ this machinery.

Drinfeld double construction.

Input. A ~~f~~ Hopf algebra H of dim n

Output. A qt Hopf alg $D(H)$ of dim n^2.

Preliminary observations. 1) If H is a fd Hopf alg, then H^* is naturally equipped w/ a Hopf alg str, w/ str maps

$$\mu_{H^*} := \Delta_H^* : H^* \otimes H^* \cong (H \otimes H)^* \longrightarrow H^*$$

$$\Delta_{H^*} := \mu_H^* : H^* \longrightarrow (H \otimes H)^* \cong H^* \otimes H^*$$

$$S_{H^*} := S_H^* : H^* \longrightarrow H^*$$

$$\eta_{H^*} = \varepsilon_H^* : \mathbb{K} \cong \mathbb{K}^* \longrightarrow H^*$$

$$\varepsilon_{H^*} = \eta_H^* : H^* \longrightarrow \mathbb{K}^* \cong \mathbb{K}$$

2) If H is a Hopf alg, then

$$H^{op} := (H, \mu^{op}, \eta, \Delta, \varepsilon, S^{-1}) \quad , \quad H^{cop} := (H, \mu, \eta, \Delta^{op}, \varepsilon, S^{-1})$$

are Hopf algebras.

3) If ~~$\forall$~~ V a fd vs, there is a canonical iso

$$\mathrm{Hom}_{\mathbb{K}}(V,V) \cong \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, V\otimes V^*) \cong V\otimes V^*$$

$$(e \mapsto \omega(e)\cdot v) \longleftarrow\!\!\!| \quad v\otimes\omega$$

$$\mathrm{Id} \longmapsto F = \sum e_i \otimes e_i^*$$

(e_i) basis for V

(e_i^*) dual basis

for any basis F looks like this!!

Theorem (Drinfeld 89). Let H be a fd Hopf algebra. Then there exists a unique Hopf algebra structure on the vector space $D(H) := H\otimes H^*$ such that

1) The canonical embeddings

$$H \xhookrightarrow{i} D(H) \qquad , \qquad H^{*,\mathrm{cop}} \xhookrightarrow{j} D(H)$$

$$x \longmapsto x\otimes 1 \qquad\qquad \omega \longmapsto 1\otimes\omega$$

are Hopf algebra maps (embeddings)

2) The composite

$$H\otimes H^{*,\mathrm{cop}} \xrightarrow{i\otimes j} D(H)\otimes D(H) \xrightarrow{\mu_{D(H)}} D(H)$$

$$x\otimes y \longmapsto x\cdot y$$

is a linear isomorphism.

3) If $R := (i\otimes j)(F) = \sum (e_i\otimes 1)\otimes(1\otimes e_i^*)$, then the pair $(D(H), R)$ is a qt Hopf algebra.

<u>About the proof</u>. By 2) any elmt in $D(H)$ is a product $x\cdot\omega$ for $x\in H$ and $\omega\in H^*$ (this is really $(x\otimes 1)\cdot(1\otimes\omega)$, we identify x w/ $i(x)$ and ω w/ $j(\omega)$ here). It suffices to determine the multiplications $\omega\cdot x$; once we have this the rest of str maps follow from 1):

$$\Delta_{D(H)}(x\cdot\omega) = \Delta_{D(H)}(x)\cdot\Delta_{D(H)}(\omega) = \Delta_H(x)\cdot\Delta^{op}_{H^*}(\omega)$$

$$\varepsilon_{D(H)}(x\cdot\omega) = \varepsilon_{D(H)}(x)\cdot\varepsilon_{D(H)}(\omega) = \varepsilon_H(x)\cdot\varepsilon_{H^*}(\omega)$$

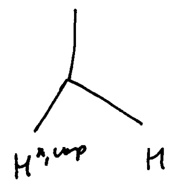
$$S_{D(H)}(x\cdot\omega) = S_{D(H)}(\omega)\cdot S_{D(H)}(x) = S^{-1}_{H^*}(\omega)\cdot S_H(x)$$

$$1_{D(H)} = 1_H\otimes 1_{H^*}\,.$$

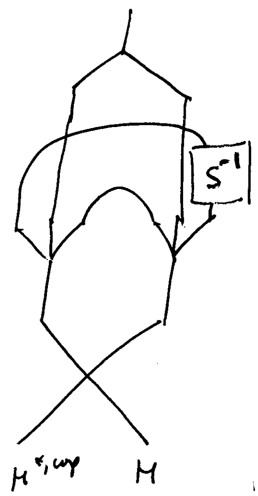
The product is seen to be given by

$$\boxed{(x\otimes\omega)\cdot(y\otimes\xi) = \sum_{(y),(\omega)} \langle y_{(1)}, S^{-1}(\omega_{(3)})\rangle\langle y_{(3)},\omega_{(1)}\rangle\,(x\cdot y_{(2)})\otimes(\omega_{(2)}\cdot\xi)}$$

or in pictures



$:=$



One then checks that this product is associative, that $\Delta_{D(H)}$ and $\varepsilon_{D(H)}$ are alg hom, and that R defines a universal R-matrix.

$\square$

Ex 1. $D(G)$ is precisely the Drinfeld double $D(\mathbb{K}[G])$.

Pf. First of all let us make explicit $\mathbb{K}[G]^*$. Note

$$\mathbb{K}[G]^* = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[G], \mathbb{K}) \cong \mathrm{Hom}_{Set}(G, \mathbb{K}) = \mathbb{K}(G)$$

that we found before. ~~Its~~ str maps are given by

$$\mu_{\mathbb{K}(G)} = \Delta^*_{\mathbb{K}[G]} : \mathbb{K}[G]^* \otimes \mathbb{K}[G]^* \to \mathbb{K}[G]^*$$

$$f_1 \otimes f_2 \longmapsto \Delta^*_{\mathbb{K}[G]}(f_1 \otimes f_2) = (f_1 \otimes f_2)\Delta_{\mathbb{K}[G]}$$

so $(f_1 \cdot f_2)(g) = (f_1 \otimes f_2)\Delta_{\mathbb{K}[G]}(g) = (f_1 \otimes f_2)(g \otimes g) = f_1(g) f_2(g) \in \mathbb{K}$

precisely as we defined it.

$\Delta_{\mathbb{K}(G)} = \mu^*_{\mathbb{K}[G]}$ given by $\Delta_{\mathbb{K}(G)}(\varphi)(g \otimes h) = \varphi(gh)$,

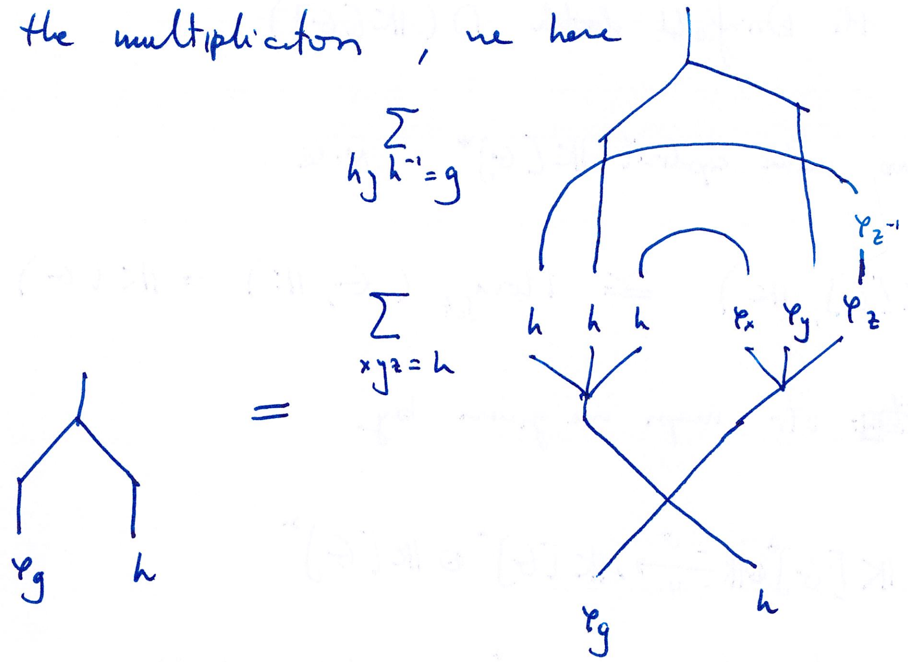
or in terms of the basis, $\Delta_{\mathbb{K}(G)}(\varphi_g) = \sum_{xy=g} \varphi_x \otimes \varphi_y = \sum_y \varphi_{gy^{-1}} \otimes \varphi_y$

Note this means $\Delta^{(3)}_{\mathbb{K}(G)}(\varphi_g) = \sum_{xyz=g} \varphi_x \otimes \varphi_y \otimes \varphi_z$.

The counit $\varepsilon_{\mathbb{K}(G)}(\varphi) = \varphi(1)$

The antipode $S_{\mathbb{K}(G)}(\varphi) = \varphi(-^{-1})$, ie $S(\varphi_g) = \varphi_{g^{-1}}$

For the multiplication, we have



$\varphi_{z^{-1}}(h) \neq 0$ only when $h = z^{-1}$

$\varphi_x(h) \neq 0$ only when $h = z^{-1}$

So we obtain

$$\varphi_g \cdot h = \sum_{hyh^{-1}=g} h \cdot \varphi_y \overset{\uparrow}{=} h \cdot \varphi_{h^{-1}gh}$$

$hyh^{-1}=g \Leftrightarrow y = h^{-1}gh$

precisely the formula we gave before. The rest of str maps are computed similarly, e.g.

$$\Delta_{D(G)}(g \cdot \varphi_h) = \Delta_{\mathbb{K}[G]}(g) \cdot \Delta^{op}_{K(G)}(\varphi_h) = (g \otimes g) \cdot \left(\sum_y \varphi_y \otimes \varphi_{gy^{-1}} \right)$$

$$= \sum_y g\varphi_y \otimes g\varphi_{gy^{-1}} \quad ,$$ the formula we obtained.

The universal R-matrix is then basis $\otimes$ dual basis,

$$R = \sum_g g \otimes \varphi_g$$, coming from the canonical element.

Because H is semisimple, we will see later that taking $\kappa = 1 \in D(G)$ ~~will~~ automatically produce a ribbon structure. It is also fact (we'll see later)

Ex 2. Assume r odd, ie $r = r'$. $U_q sl_2$ (and in general the quantum groups $U_q \mathfrak{g}$), is not a Drinfeld double "on the nose", but is is the quotient of the Drinfeld double of some Hopf subalgebra. More precisely, let

$$B_q := \mathrm{span}_{\mathbb{C}} (E^i K^j)_{i,j = 0 \dots, r'-1} .$$

It is immediate to see that $B_q \subset U_q sl_2$ is a Hopf subalg,

$$B_q = \mathbb{C}\langle E, K\rangle \Big/ \underbrace{KE = q^2 EK, \ E^{r'} = 0, \ K^{r'} = 1}_{(A)} .$$

Consider B_q^*, w/ the linear forms $\alpha, \beta : B_q \to \mathbb{K}$ def on basis elmts as

$$\alpha (E^m K^n) = \delta_{m,0} \cdot q^{2n} , \qquad \beta (E^m K^n) = \delta_{m,1} .$$

As an algebra we have

$$B_q^* = \mathbb{C}\langle \alpha, \beta\rangle \Big/ \underbrace{\alpha \beta = q^{-2} \beta \alpha , \ \alpha^r = 1, \ \eta^r = 0}_{(B)} .$$

and as an algebra

$$D(B_q) = \mathbb{C}\langle E, K, \alpha, \beta\rangle \Big/ \begin{array}{l} (A), (B), \ K\alpha = \alpha K, \ K\beta = q^{-2} \beta K \\ E\alpha = q^{-2} \alpha E, \ E\beta = -q^{-2} (1 - \beta E - \alpha K) . \end{array}$$

Fact.

$$\chi : D(B_q) \longrightarrow u_q sl_2$$
$$\beta^i \alpha^j E^k K^\ell \longmapsto \left(\frac{q-q^{-1}}{q^2}\right)^i q^{2(i+j)k - i(i-1)} F^i E^k K^{i+j+\ell}$$

$i, j, k, \ell = 0, \dots, r-1$, defines a surjective Hopf alg hom.

Upshot. $u_q sl_2$ is a quotient of $D(B_q)$, and

$$R := (\chi \otimes \chi)\, R_{D(B_q)}$$

is a universal R-matrix for $u_q sl_2$. This is exactly that we gave before (see the computation in Kassel IX.7).

What I would like to explain next is how properties we have seen, ss, factorisability, behave under the Drinfeld double. We will then be able to say things about our two running examples. We start by semisimplicity.

Theorem (Larson-Radford 88). Let char $\mathbb{k} = 0$, and let H be a f d Hopf algebra. Then TFAE:

(1) H is semisimple

(2) H^* is semisimple

(3) $S^2 = \mathrm{Id}$.

<u>Corollary</u>. Let char $\mathbb{K} = 0$, and H a fd Hopf algebra. Then

$$H \text{ semisimple} \iff D(H) \text{ semisimple}.$$

<u>Pf</u>. We saw before that a criterion for ss (generalized Maschke's thm) was that $\varepsilon(\mathcal{I}_\ell(H)) \neq 0$. One can show that integrals in $D(H)$ are of the form $t \otimes T$. So

$$\begin{aligned} H \text{ ss} \iff H \ \& \ H^* \text{ ss} &\iff \varepsilon_H(\mathcal{I}_\ell(H)) \neq 0 \ \& \ \varepsilon_{H^*}(\mathcal{I}(H^*)) \neq 0 \\ &\iff \varepsilon_{DH}(t \otimes T) = \varepsilon_H(t) \cdot \varepsilon_{H^*}(T) \neq 0 \\ &\iff D(H) \text{ ss} \end{aligned}$$

□

<u>Corollary</u>. $D(G)$ is semisimple.

Let us now turn to factorisability. We have the following:

<u>Proposition</u>. For any fd Hopf alg H, we have that $D(H)$ is factorisable.

<u>Pf</u>. We have to check that the Drinfeld map

$$\begin{aligned} \Phi : D(H)^* &\longrightarrow D(H) \\ \omega &\longmapsto (\mathrm{id} \otimes \omega)(Q) \end{aligned}$$

is an iso.

$R = \sum e_i \otimes e_i^*$, so

$$Q = R_{21} R = \sum_{ij} e_i^* e_j \otimes \underbrace{e_i e_j^*}_{\text{standard basis for } H \otimes H^*}$$

By our characterisation of factorisability in terms of $Q =$ basis $\otimes$ basis, it suffices to see that $(e_i^* e_j)_{ij}$ forms a basis for $H \otimes H^*$ as well.

For simply note:

$$S(e_i^* e_j) = S(e_j) S(e_i^*) = \underbrace{\underset{\substack{\uparrow \\ \text{basis} \\ \text{for } H}}{S(e_j)} \otimes \underset{\substack{\uparrow \\ \text{basis} \\ \text{for } H^*}}{S(e_i^*)}}_{\text{basis for } H \otimes H^*},$$

and since S is an isomorphism, we conclude. $\square$

Corollary. $D(G)$ is factorisable.

So: the upshot is that if H is a fd Hopf algebra, then $D(H)$ is a factorisable ~~...~~ quasi-triangular Hopf algebra. If H is ss, so is $D(H)$.

In particular, $D(H)$ is always a factorisable braided FTC.

Q. Does $D(H)$ always admit a ribbon structure?

A. No. Kauffman and Radford (93) give necessary & sufficient conditions for this.

Ex. let $n \geq 1$ and ~~three~~ ξ a primitive n-th rt of 1. let

$$T_n = \mathbb{C}\langle a, x\rangle \Big/ a^n = 1, \quad x^n = 0, \quad xa = \xi \cdot ax$$

the Taft algebra, a Hopf alg w/

$$\Delta(a) = a \otimes a$$
$$\Delta(x) = x \otimes 1 + a \otimes x$$
$$\varepsilon(a) = 1$$
$$\varepsilon(x) = 0$$
$$S(a) = a^{-1}$$
$$S(x) = -a^{-1}x \quad ,$$

a Hopf alg of dim n^2.

It is a non-ss Hopf alg, and S has order $2n$.

One can show that (Kauff-Radf Prop 7):

$$D(T_n) \text{ has a ribbon elmt} \iff n \text{ odd}.$$

However, if H is ss, then the answer is always affirmative!

Theorem let (H, R) be a q.t. ss fd Hopf algebra, and assume char $\mathbb{K} = 0$. Then the choice of pivt $\kappa := 1$ (equivalently $v := u$ the Drinfeld elmt) makes $(H, R, \kappa = 1)$ a ribbon Hopf algebra.

About the pf. By Larson-Radford's thm, we have ~~that~~ that once that H^* is ss as well and moreover $S^2 = \mathrm{Id}$. Since ss $\Rightarrow$ unimodularity, we have that H and H^* are unimodular.

Now [Kauff-Radf, Prop 4] tells us that $K=1$ (ie $v=u$) is a ribbon element. $\square$

Corollary. Let H be a fd ss Hopf algebra, and char $\mathbb{K}=0$. Then $D(H)$ is a ss factorisable ribbon Hopf algebra, w/ $K=1$.

In particular $D(H)$-mod is a MFC.

Ex 1. The previous corollary explains why the choice of $K=1$ in $D(G)$ made $D(G)$ into a ribbon Hopf algebra. In particular $D(G)$-mod is a MFC.

Ex 2. $u_q sl_2$ was shown to be a ribbon Hopf algebra, and when r odd this was also factorisable, ie $u_q sl_2$-mod is a MTC.

I would like to finish these notes by giving a description of $D(G)$-mod and $u_q sl_2$-mod (for the latter, at least for its simple modules).

Ex 1. Description of D(G) - mod

D(G) is a ss category, so its objects are fully determined by simple ones. Here it is a description of those:

$$\mathrm{Conj}(G) := \text{set of conj classes of } G$$

$$= \{ \underset{C_1}{\underset{\shortparallel}{[g_1]}} = [e], \underset{C_2}{\underset{\shortparallel}{[g_2]}}, \dots, \underset{C_r}{\underset{\shortparallel}{[g_r]}} \}, \qquad g_i \text{ representatives}$$

For $g \in G$, $Z(g) := \{ h \in G : hg = gh \}$ the centraliser of g.

Note. The cardinality of Conj(G) is exactly the cardinality of the set of simple $\mathbb{K}[G]$-modules.

For the representatives $g_1 \cdots g_r$, let $Z_i = Z(g_i)$.

If $s \in C_i$, by def $\exists\, a_s \in G$: $s = a_s g_i a_s^{-1}$ (there might exist multiple, but fix a a_s for every s). Eg a_{g_i} can be taken to be e.

For $k = 1 \dots r$, let (V_α^k) be a complete set of ~~iso~~ isomorphism classes of irr. representations of Z_k. Since these are subgps of G, we can lift them to G-modules by means of the induced represent.:

$$V_{K,\alpha} := \mathrm{Ind}_{Z_K}^{G}(V_\alpha^K) := \mathbb{C}G \otimes_{\mathbb{C}Z_K} V_\alpha^K$$

In general, if $H \subset G$ subgp and V is an H-module, the induced rep. is defined as

$$\mathrm{Ind}_H^G(V) = \mathbb{C}G \otimes_{\mathbb{C}H} V ,$$

this is a representation of dim $(\dim V) \cdot \frac{|G|}{|H|}$ $\leftarrow$ an integer by Lagrange's thm

In our case,

$$\dim V_{K,\alpha} = (\dim V_K^\alpha)\, \frac{|G|}{|Z_K|} = (\dim V_K^\alpha)\, \underbrace{|G/Z_K|}_{= C_K} = (\dim V_K^\alpha) \cdot |C_K|.$$

(The quotient G/Z_K $\cong$ as G-sets; and the isomorph $G/Z_K \cong C_K$ is just an instance of the orbit-stabiliser thm: $\mathrm{Orbit}(x) \cong G/\mathrm{stab}(x)$.)

In particular, this means that $V_{K,\alpha} = \mathrm{span}\,(a_s \otimes v)$ w/ $s \in C_K$

In this notation, $g \in G$ acts on $V_{K,\alpha}$ according to the formula

$$g \cdot (a_s \otimes v) := a_{gsg^{-1}} \otimes \underbrace{(a_{gsg^{-1}}^{-1} \cdot g \cdot a)}_{\text{Fact: } \in Z_K} \cdot v$$

This G-module $V_{K,\alpha}$ can be easily extended to $D(G)$: simply

$$\gamma_h \cdot (a_s \otimes v) := \begin{cases} 0, & h \neq s \\ a_s \otimes v, & h = s. \end{cases}$$

In other words, φ_h is precisely the projection

$$\varphi_h : V_{K,\alpha} \longrightarrow \underset{v \in V}{\mathrm{span}} \left(a_h \otimes v \right) .$$

The following thm gives a complete ~~description~~ description of $D(G)$-mod.

<u>Theorem</u>. Let G be a finite group.

1) The $D(G)$-modules $V_{K,\alpha}$ constructed above are simple.

2) $V_{K,\alpha} \cong V_{\ell,\beta} \Leftrightarrow K = \ell$, and $\alpha = \beta$, ie they are all non-iso

3) The collection $\left(V_{K,\alpha} \right)_{\substack{K = 1,\dots,|\mathrm{Conj}(G)| \\ \alpha \in Z_K^\vee}}$ exhausts all simple $D(G)$-modules.

<u>About the proof</u>. I will only comment on 3) because it uses some gen. result of rep theory: if A is a fd algebra, ~~and~~ then it has a finite number of irred. representations (V_i); and there is an isomorphism of A-modules

$$A / \mathrm{Jac}(A) \cong \bigoplus_i V_i^{\oplus n_i} \quad , \quad n_i = \dim V_i$$

In particular if A is ss, $A \cong \bigoplus_i V_i^{\oplus n_i}$ and conversely

$$\boxed{\dim A = \sum_{V \text{ irred rep}} (\dim V)^2}$$

Now we compute

$$\sum_{K,\alpha} (\dim V_{K,\alpha})^2 = \sum_{K} |C_K|^2 \underbrace{\sum_\alpha (\dim V_\alpha^K)^2}_{= |Z_K|}$$

$$= \sum_K |C_K|^2 \, |Z_K| = \sum_K \frac{|G|^2}{|Z_K|^{\cancel{2}}} \cdot \cancel{|Z_K|}$$

$$= |G| . \sum \frac{|G|}{|Z_K|} = |G|^2 = \dim D(G) .$$

$\square$

<u>Note</u>. One can actually give explicit formulas for the S-matrix of D(G)-mod, but formulas are ugly.

<u>Ex 2</u>. $u_q sl_2$ is a non-ss category. I will at least describe ~~...~~ all simple $u_q sl_2$-modules. I am going to define two diff. ~~classes~~ classes of moduls:

- $V_{\varepsilon,n}$, $\varepsilon = \pm 1$, $n \geq 0$, a rep. of dim $n+1$.
- V_λ , $\lambda \in \mathbb{C} - 0$, a rep of dim r'.

These are defined on generators as follows (the matrices denote the action by multp. on a certain basis):

$V_{\varepsilon, n}$:

$$E\cdot = \varepsilon \cdot \begin{pmatrix} 0 & [n] & 0 & \cdots & 0 \\ \vdots & 0 & [n-1] & \ddots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad F\cdot = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & & & \vdots \\ \vdots & [2] & \ddots & & \vdots \\ \vdots & 0 & \ddots & & \vdots \\ 0 & 0 & & [n] & 0 \end{pmatrix}$$

$$K\cdot = \varepsilon \cdot \begin{pmatrix} q^{n} & & & & 0 \\ & q^{n-2} & & & \\ & & \ddots & & \\ & & & q^{-n+2} & \\ 0 & & & & q^{-n} \end{pmatrix}$$

V_λ : let $\gamma_p = \dfrac{q^{-p}\lambda - q^{p}\lambda^{-1}}{q - q^{-1}}$, then

$$E\cdot = \begin{pmatrix} 0 & \gamma_0[1] & & & \\ \vdots & & \gamma_1[2] & & \\ \vdots & & & \ddots & \\ \vdots & & & & \gamma_{r'-2}[r'-1] \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

$$F\cdot = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ \vdots & & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}, \qquad K\cdot = \begin{pmatrix} \lambda & & & & \\ & \lambda q^{-2} & & & \\ & & \lambda q^{-4} & & \\ & & & \ddots & \\ & & & & \lambda q^{-2(r'-1)} \end{pmatrix}$$

Theorem: The following lists exhaust all simple $u_q\mathfrak{sl}_2$-modules, depending on r:

1) **r odd**: $V_{1,n}$ (dim $n+1$), $0 \leq n < r-1$, or $V_{q^{-1}}$ (dim r)

2) **r even and $r \equiv 0 \mod 4$**: $V_{\pm 1, n}$ (dim $n+1$), n even $0 \leq n < r'-1$

3) **r even and $r \not\equiv 0 \mod 4$**: $V_{1,n}$ (dim $n+1$) n even $0 \leq n < r'-1$

or $V_{-1,n}$ (dim $n+1$), n odd $0 < n < r'-1$

or $V_{-q^{-1}}$ (dim r')

(in particular, note that there are no simple $u_q\mathfrak{sl}_2$-modules of dim $> r'$).

References

- Schweigert, "Hopf algebras, quantum groups and topological field theories", lecture notes Summer term 2025
- Majid "Foundations of Quantum Group Theory".
- Etingof, Golberg, Hensel, Liu, Schwendner, Vaintrob, Yudovina "Introduction to representation theory".
- Beliakova - De Renzi, "Kerler-Lyubashenko functors on 4-dim 2-handlebodies"
- EGNO, "Tensor Categories"
- Kauffman & Radford, "A Necessary and Sufficient condition for a finite-dimensional Drinfeld Double to be a Ribbon Hopf algebra".
- Ulrich Pennig, "Modular Tensor Categories from Quantum Doubles of finite groups".
- Kassel, "Quantum Groups"
- Etingof - Schiffman, "Lectures on Quantum Groups"
- Shimizu: "Non-degeneracy conditions for braided finite tensor categories".