References:
- Hatcher's AT §4.G
- May, Ponto's More Concise
- Martin Frankland, "Homotopy pullbacks"

HOMOTOPY (CO)LIMITS

• The notion of limit/colimit is not suitable if one searches a homotopy invariant construction, eg

$$\begin{tikzcd} D^n \arrow[d, "\wr"] & S^{n-1} \arrow[l] \arrow[r] \arrow[d, "\wr"] & D^n \arrow[d, "\wr"] \\ * & S^{n-1} \arrow[l] \arrow[r] & * \end{tikzcd}$$

commutes but colim (upper row) $= D^n \cup_{S^{n-1}} D^n \cong S^n$ whereas colim (lower row) $\cong *$ whereas vertical maps are htpy equivalences.

<u>Slogan</u>. Replace the maps by fibrations / cofibrations and later take limit / colimit.

HOMOTOPY PULLBACKS

• Recall that given a map $f: X \to Y$, it factors as $X \xrightarrow{\simeq} P_f \xrightarrow{\text{fibration}} Y$, where $P_f := X \times_Y Y^I$ and the fibration is $p: P_f \to Y$, $(x, \sigma) \mapsto \sigma(1)$.

<u>Definition</u>. Given a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$, its <u>homotopy pullback</u> is defined as

$$X \times_Z^h Y := P_f \times_Z P_g = \{ (x, \sigma, y, \sigma') \in X \times Z^I \times Y \times Z^I \;:\; \sigma(0) = f(x),\ \sigma'(0) = g(y),\ \sigma(1) = \sigma'(1) \}$$

$$\cong \{ (x, \gamma, y) \in X \times Z^I \times Y \;:\; \gamma(0) = f(x),\ \gamma(1) = g(y) \} \qquad (\sigma, \sigma' \leftrightarrow \gamma = \sigma * (\sigma')^{-1})$$

It comes with projection maps and a diagram

$$\begin{tikzcd} X \times_Z^h Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"'] \arrow[dr, phantom, "\circlearrowleft_{ho}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

commuting up to (canonical) homotopy $H: (X \times_Z^h Y) \times I \to Z$, $H(x, \gamma, y, t) = \gamma(t)$.

<u>Example</u>. Let $f: X \to Y$. The pullback of $* \xrightarrow{y_0} Y \xleftarrow{f} X$ is $f^{-1}(y_0)$, the fibre of f. The homotopy pullback is

$$* \times_Y^h X = \{ (\gamma, x) \in Y^I \times X : \gamma(0) = y_0,\ \gamma(1) = f(x) \} = \mathrm{hofib}_{y_0}(f),$$

the homotopy fibre of f.

• A diagram commuting up to htpy is called a homotopy commutative diagram.

Proposition (non-universal property): Given a homotopy commutative diagram

$$\begin{tikzcd} W \arrow[r, "\varphi_Y"] \arrow[d, "\varphi_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

there exists a map $\varphi : W \to X \times_Z^h Y$ (not necessarily unique) such that the following diagram is htpy comm

$$\begin{tikzcd} W \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\varphi_Y"] \arrow[ddr, bend right, "\varphi_X"'] & & \\ & X \times_Z^h Y \arrow[r, "pr_X"] \arrow[d, "pr_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

and

$$pr_X \circ \varphi = \varphi_X$$
$$pr_Y \circ \varphi = \varphi_Y$$

In other words, the map

$$[W, X \times_Z^h Y] \longrightarrow [W, X] \times_{[W,Z]} [W, Y]$$

is surjective.

- The next prop says that it sufficies to change one of the maps of a pullback to compute the htpy pullback. In particular if one of the maps is a fibration then we can just take pullback.

Proposition: The maps $P_f \times_Z Y \xrightarrow{\simeq} X \times_Z^h Y$ and $X \times_Z P_g \xrightarrow{\simeq} X$ are homotopy equivalences. In particular, if either f or g is a fibration, then $X \times_Z Y \xrightarrow{\simeq} X \times_Z^h Y$ is a htpy equivalence.

Proposition: If either f or g is a Serre fibration, then $X \times_Z Y \to X \times_Z^h Y$ is a whe.

Definition: A <u>homotopy pullback square</u> or <u>homotopy Cartesian square</u> is a htpy comm diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_{ho}"] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

if there is a htpy equivalence $\varphi : W \xrightarrow{\simeq} X \times_Z^h Y$ s.t. $\varphi_X = pr_X \circ \varphi$, $\varphi_Y = pr_Y \circ \varphi$.

Example: Since $PX \simeq *$ and for a H. fibration p, $p^{-1}(x) \simeq \mathrm{hofib}_x(f)$, we have that

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_{ho}"] & * \arrow[d, "x_0"] \\ * \arrow[r, "x_0"] & X \end{tikzcd}$$

is a htpy pullback square.

Lemma (Pasting). Let

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d, dash] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

be a htpy comm diagram, and suppose the right-hand side square is a htpy pullback. Then the left-hand square is a htpy pullback $\iff$ the outer rectangle is a htpy pullback.

Proposition. If

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner_{ho}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

is a htpy pullback square, then parallel homotopy fibres are htpy equivalent, ie
$\mathrm{hofib}(g') \simeq \mathrm{hofib}(g)$ and $\mathrm{hofib}(f') \simeq \mathrm{hofib}(f)$.

- Recall that the pullback of a fibration along a htpy eq is again a htpy eq. Since up to htpy every map is a fibration, the following should not be surprising

Proposition. A htpy pullback of a htpy equivalence is a htpy equivalence.

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner_{ho}"] & Y \arrow[d] \\ X \arrow[r, "f"] & Z \end{tikzcd} \qquad f \text{ htpy eq} \Rightarrow f' \text{ htpy eq}$$

Corollary: A htpy comm square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\wr"'] \arrow[dr, phantom, "\lrcorner_{ho}"] & Y \arrow[d, "\wr"] \\ X \arrow[r] & Z \end{tikzcd}$$

with vertical (or horizontal) htpy equivalences is a htpy pullback square.

*Theorem (Homotopy invariance of htpy pullbacks). Let

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\simeq"'] & C \arrow[d, "\simeq"'] & B \arrow[l] \arrow[d, "\simeq"'] \\ X \arrow[r] & Z & Y \arrow[l] \end{tikzcd}$$

be a comm diagram with the vertical arrows htpy equivalences. Then the induced map

$$\varphi : A \times^h_C B \xrightarrow{\simeq} X \times^h_Z Y$$

is a htpy equivalence. In particular we have the following htpy comm cube:

$$\begin{tikzcd}[row sep=small, column sep=small] & A \times^h_C B \arrow[rr] \arrow[dl] \arrow[dd, "\varphi", "\simeq"'] \arrow[ddrr, phantom, "\circlearrowleft h_0"] & & B \arrow[dl] \arrow[dd, "\simeq"] \\ A \arrow[rr, crossing over] \arrow[dd, "\simeq"'] & & C & \\ & X \times^h_Z Y \arrow[rr] \arrow[dl] \arrow[ddrr, phantom, "\circlearrowleft h_0"] & & Y \arrow[dl] \\ X \arrow[rr] & & Z \arrow[from=uu, crossing over, "\simeq"] & \end{tikzcd}$$

Lemma. There is a fibration sequence $\Omega Z \to X \times^h_Z Y \to X \times Y$.

Corollary: There is a les

$$\cdots \to \pi_{n+1}(Z) \to \pi_n(X \times^h_Z Y) \to \pi_n(X) \times \pi_n(Y) \to \pi_n(Z) \to \pi_{n-1}(X \times^h_Z Y) \to \cdots$$

Corollary (Weak htpy eq invariance of htpy pullbacks): If in the statement of the theorem, "htpy eq" is replaced by "whe", the theorem holds as well.

HOMOTOPY PUSHOUT

- Every single sentence from above can be dualised to give htpy pushouts instead of htpy pullbacks.

__Definition__. Let $B \xleftarrow{f} A \xrightarrow{g} C$ be a diagram. The __homotopy pushout__ is the space

$$B \cup_A^h C := M_f \cup_A M_g$$
$$\cong X \cup_A (A \times I) \cup_A Y$$

the double mapping cylinder.

__Prop__ (non-u. prop): $[B \cup_A^h C, W] \to [B, W] \times_{[A,W]} [C, W]$ is surjective.

__Proposition__: It is enough to replace one of the spaces by its mapping cylinder. More concretely, if either f or g is a cofibration, then $B \cup_A^h C \xrightarrow{\simeq} B \cup_A C$ is a htpy eq.

DIGGRESION : $\lim^1$ AND MILNOR SEQUENCE

- In some arguments involving (sequential) limits, an algebraic gadget called "Milnor's $\lim^1$" is needed. The name stands "the first right-derived functor of lim".

- Recall that if $A_\bullet = (\cdots \to A_3 \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0)$ is a inverse system of abelian groups, $\lim_i A_i$ is the kernel of the morphism $\partial : \prod A_n \to \prod A_n$, $(a_n)_{n \in \mathbb{N}} \mapsto (a_n - f_{n+1}(a_{n+1}))_{n \in \mathbb{N}}$

__Definition__: Given a tower as before, define $\lim^1 A_i := \operatorname{coker} \partial$, so that there is an exact sequence

$$0 \to \lim A_i \to \prod A_n \xrightarrow{\partial} \prod A_n \to \lim^1 A_i \to 0.$$

__Proposition__ (Properties)

1) If all f_i's are surjective, $\lim^1 A_i \cong 0$.
2) If all $f_i = 0$, then $\lim^1 A_i \cong 0$
3) If we replace the sequence $A_\bullet$ by any subsequence, then lim or $\lim^1$ do not change.

4) If $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a ses of inverse systems, the following seq. is exact:

$$0 \to \lim A_i \to \lim B_i \to \lim C_i \to \lim^1 A_i \to \lim^1 B_i \to \lim^1 C_i \to 0.$$

• Denote $Ab^{(\mathbb{N},\geq)}$ the cat of inv. systems of ab gps

Theorem: $Ab^{(\mathbb{N},\geq)}$ has enough injectives, and $\lim^1 : Ab^{(\mathbb{N},\geq)} \to Ab$ is the first right-derived functor of $\lim : Ab^{(\mathbb{N},\geq)} \to Ab$.

Proposition: If $A_\bullet$ satisfies the Mittag-Leffler condition ($\forall k\ \exists i \geq k$: if $j \geq i \geq k$ then $\mathrm{Im}(A_i \to A_k) = \mathrm{Im}(A_j \to A_k)$), then $\lim^1 A_i \cong 0$.

Theorem (Milnor exact sequence for cohomology): Let $h^\bullet$ be a reduced or unreduced cohomology theory and let $X = \mathrm{colim}(X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots)$, X_i subcomplex. Then there is a ses

$$0 \to \lim_i{}^1 h^{n-1}(X_i) \to h^n(X) \to \lim h^n(X_i) \to 0$$

which says that the failure of $h^n(X)$ to be isomorphic to $\lim h^n(X_i)$ is measured by $\lim^1$.

Remark: If $h_\bullet$ is a homology theory, $h_n(X) \cong \mathrm{colim}_i\, h_n(X_i)$ always holds.

Theorem (Milnor sequence for homotopy groups): Let $\dots \to X_3 \to X_2 \to X_1$ be a tower of fibrations (eg a Postnikov tower). There is a ses

$$0 \to \lim^1 \pi_{n+1}(X_i) \to \pi_n(\lim X_i) \to \lim \pi_n(X_i) \to 0$$

which says that the failure of $\pi_n(\lim X_i)$ to be isomorphic to $\lim \pi_n(X_i)$ is measured by $\lim^1$.

Remark: If $X = \mathrm{colim}(X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \dots)$, X_i subcomplex, then we always have

$$\mathrm{colim}_i\, \pi_n(X_i) \cong \pi_n(X).$$

(SEQUENTIAL) HOMOTOPY LIMITS

<u>Definition</u>: Let $\cdots \to X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1$ be a sequence of spaces. Its <u>homotopy limit</u> (or mapping microscope) is

$$\operatorname{holim}(X_\bullet) := \left\{ (x_i, \gamma_i) \in \prod_{i \geq 1} X_i \times X_i^I \;:\; \gamma_i(0) = f_{i+1}(x_{i+1}),\ \gamma_i(1) = x_i \right\}$$

$$\cong \left\{ (\gamma_i) \in \prod_{i \geq 1} X_i^I \;:\; \gamma_i(0) = f_{i+1}(\gamma_{i+1}(1)) \right\}$$

$$\cong \lim_n Y_n$$

where $Y_1 := X_1^I$ and Y_n arises from Y_{n-1} as the pullback

$$\begin{tikzcd}
Y_n \arrow[r] \arrow[d, "pr"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_{n-1} \arrow[d, "ev_0 \circ pr"] \\
X_n^I \arrow[r, "f_n \circ ev_1"] & X_{n-1}
\end{tikzcd}$$

<u>Proposition</u>: For any space Z, there is a ses

$$0 \to \lim^1 \pi_{n+1}(X_i) \to \pi_n(\operatorname{holim} X_i) \to \lim \pi_n(X_i) \to 0.$$

<u>Proposition</u>: If all maps f_i are fibrations, then the natural injection

$$\lim X_i \xrightarrow{\simeq} \operatorname{holim} X_i$$

is a htpy equivalence.

- There is an alternative description of this htpy limit (at least up to htpy equivalence), namely replacing inductively every map by a fibration and then take lim:

$$\begin{tikzcd}
\cdots \arrow[r] & X_4 \arrow[r, "f_4"] \arrow[d, "\wr"] \arrow[dr, dashed, "g_4"] & X_3 \arrow[r, "f_3"] \arrow[d, "\wr"] \arrow[dr, dashed, "g_3"] & X_2 \arrow[r, "f_2"] \arrow[d, "\wr"] & X_1 \arrow[d, equal] \\
\cdots \arrow[r] & X_4' := P_{g_4} \arrow[r, "f_4'"] & X_3' := P_{g_3} \arrow[r, "f_3'"] & X_2' := P_{f_2} \arrow[r, "f_2'"] & X_1'
\end{tikzcd}$$

Proposition. Both $\lim X_i'$ and $\operatorname{holim} X_i$ are htpy equivalent.

(SECUENTIAL) HOMOTOPY COLIMITS

Definition: Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots$ be a sequence. The homotopy colimit (or mapping telescope) of the sequence is

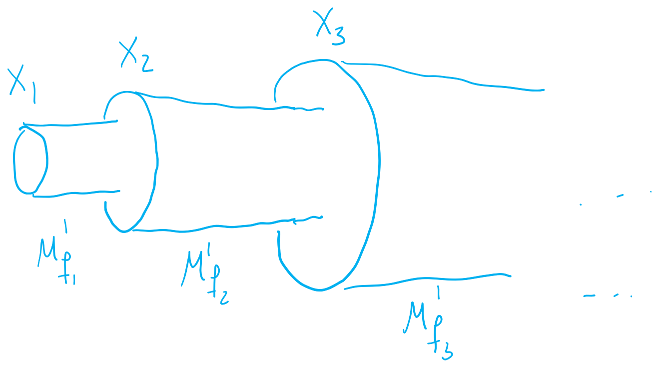
$$\operatorname{hocolim}(X_\bullet) := \coprod_{i \geq 1} M_{f_i} \Big/ (x_i, 1) \sim f_i(x_i)$$

$$\cong \operatorname{colim}_n Y_n$$

where if $M'_{f_i} := (X_i \times [i, i+1]) \cup_{X_i} X_{i+1}$ $(\cong M_{f_i})$, and $Y_1 := M'_{f_1}$, then Y_n arises from Y_{n-1} as the pushout

$$\begin{tikzcd} X_n \arrow[r, "i_n \circ \mathrm{incl}"] \arrow[d, "j"'] & Y_{n-1} \arrow[d] \\ M'_{f_n} \arrow[r, "i_n"] & Y_n \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

where $\mathrm{incl} : X_n \hookrightarrow M'_{f_{n-1}}$.



Proposition: Given $X_1 \to X_2 \to X_3 \to \cdots$ and a space Z, there is a ses

$$0 \to \lim{}^1 [\Sigma X_i, Z] \to [\operatorname{hocolim} X_i, Z] \to \lim [X_i, Z] \to 0$$

Proposition: If all maps f_i are cofibrations, the natural map

$$\operatorname{hocolim} X_i \xrightarrow{\simeq} \operatorname{colim} X_i$$

is a htpy equivalence.

- There is an alternative description of hocolim (up to htpy eq), namely replacing all the maps by cofibrations inductively and later take colim:

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, "f_1'"] & X_2 \arrow[r, "f_2"] & X_3 \arrow[r, "f_3"] & X_4 \arrow[r, "f_4"] & \cdots \\
X_1' \arrow[r, "f_1'"] & X_2' := M_{f_1} \arrow[u, "\wr"] \arrow[r, "f_2'"] \arrow[ur, dashed, "g_2"] & X_3' = M_{f_2'} \arrow[u, "\wr"] \arrow[r] \arrow[ur, dashed, "g_3"] & X_4' = M_{f_3'} \arrow[u, "\wr"] & \cdots
\end{tikzcd}$$

<u>Proposition</u>: $\operatorname{colim} X_i'$ deformation retracts onto the mapping telescope $\operatorname{hocolim} X_i$.