Cobordism & Thom spectra

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• One would like to understand {smooth manifolds} / diffeomorphism. Too hard. Rather, impossible! The gp isomorphism problem (ie are two finitely presented gps isomorphic?) is undecidable, and ~~any~~ a manifold of dim ≥ 4 can have any fin. presented gp as π_1.

Getaway: Mod out by a coarser eq. relation

<u>Def</u> (Thom, 54'): Let M_1, M_2 be two oriented, n-dim manifolds. M_1, M_2 are <u>cobordant</u> if there is a $(n+1)$-dim oriented mnfd W st

$$\partial W \cong M_1 \amalg -M_2 \qquad , \quad -M_2 \equiv \text{opposite orientation}$$

Cobordism is an equivalence relation. Set

$$\Omega_n^{SO} := \frac{\{\text{oriented } n\text{-mnflds}\}}{\text{cobordism}}.$$

<u>Lemma</u>: Ω_n^{SO} is an abelian gp. Even more, $\Omega_*^{SO} := \bigoplus_{n\geq 0} \Omega_n^{SO}$ is a graded commutative ring.

Here:

- Addition is given by $\amalg$ (disjoint union of mnfds)
- Product is given by $\times$ (cartesian product)
- Inverse of $[M]$ is $[-M]$
- 0 is given by $\emptyset$
- 1 is given by $*$ $\equiv$ one-pt space.

~~Propo~~ Let us calculate a few low-dim cob. groups.

<u>Proposition</u>: $\Omega_0^{SO} \cong \mathbb{Z}$, $\Omega_1^{SO} \cong 0$, $\Omega_2^{SO} \cong 0$, $\Omega_3^{SO} \cong 0$.

<u>Pf</u>: (Ω_0^{SO}). An orientation of a 0-dim mnfd M is a map $M \to \{\pm 1\}$. So there are two connected, zero-dim, oriented mnfds $*_+$ and $*_-$. The oriented unit interval is a cob. betw. $*_+$ and $*_-$, so $-*_+ = *_-$ in Ω_0^{SO}. As the rest are disjoint unions, we conclude.

(Ω_1^{SO}): $\Omega_n^{SO} \cong 0$ means that all n-mnfds are cobordant, in particular cobordant to the empty n-mnfd, ie, $\Omega_n^{SO} \cong 0 \Leftrightarrow$ ~~all~~ any n-mnfd is the boundary of a $(n+1)$-mnfd. Obviously S^1 is the only compact 1-mnfd and $S^1 \cong \partial D^2$. So $[S^1] = 0$. Since the rest of 1-mnfds are $\amalg S^1$, we are done.

(Ω_2^{SO}): Any oriented, compact 2-mnfd is Σ_g, which is the boundary of a genus g handlebody.

(Ω_3^{SO}): Recall that a surgery of index k on a m-mnfd M along an embedding $i: S^{k-1} \times D^{m+1-k} \hookrightarrow \mathring{M}$ is

$$M' := \left(M - \overset{\circ}{\widehat{i(S^{k-1} \times D^{m+1-k})}}\right) \cup_{i'} \left(D^k \times S^{m-k}\right) \quad , \quad i' = i_{|\partial(\;)}$$

~~The Lickorish Wallace thm says that any 3-mnfd M arises from surgery along a framed link on S^3.~~

On the other hand, if W is an n-mnfd, attaching a k-handle along an embedding $i: S^{k-1} \times D^{n-k} \hookrightarrow \partial W$ is

$$W' := W \cup_i (D^k \times D^{n-k})$$

The key observation is that then $\partial W'$ is obtained from ∂W by a surgery of index k.

The Lickorish - Wallace thm says that any 3-mnfd M arises from surgery (along a framed knot) on S^3. Then the previous observation concludes: any 3-mnfd M ~~is the boundary~~ obtained from surgery on a p-comp. link is the boundary of a 4-disc with p 2-handles attached.

$\square$

<u>Remark</u> ($\Omega_4^{SO} \neq 0$) : Just as the knot signature defines a surjective gp hom $\sigma : \mathcal{C} \twoheadrightarrow \mathbb{Z}$ (showing that $\mathcal{C}$ = knot concordance gp $\neq 0$), we can define another surj gp hom $\sigma : \Omega_4^{SO} \twoheadrightarrow \mathbb{Z}$.

If M is a $2d$-dim oriented, connected mfd, Poincaré duality induces

$$H^d(M;\mathbb{R}) \otimes H^d(M;\mathbb{R}) \longrightarrow \mathbb{R}$$
$$[\alpha] \otimes [\beta] \longmapsto \langle \alpha \cup \beta, [M] \rangle$$

$[M]$ = fundamental class

which is non-deg, and symmetric if d even (skew-sym if d odd).

The <u>signature</u> of M, $\sigma(M) \in \mathbb{Z}$, is the signature of this nondeg, b.l. form.

<u>Fact</u>. If M, M' cobordant, then $\sigma(M) = \sigma(M')$.

This gives a gp hom $\sigma : \Omega_4^{SO} \to \mathbb{Z}$ (gp hom by def for $\sqcup$).

This is non-trivial as $\sigma(\mathbb{CP}^2) = 1$. For $H^*(\mathbb{CP}^2; \mathbb{Z}) \cong \dfrac{\mathbb{Z}[x]}{(x^3)}$ so $x^2 \neq 0$ and hence the pairing is non-trivial.

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• Now the question is: how to compute the rest? And the ring structure of Ω_*^{SO}?

At least in modern language, Thom derived Ω_n^{SO} as

(1) The coefficient groups of some generalized reduced (co)homology theory

(2) The homotopy groups of some "object" MSO, $\Omega_n^{SO} \cong \pi_n(MSO)$.

Both (1) and (2) are closely related. The kind of object MSO is goes under the name of spectrum.

<u>Def</u>: A spectrum E is a collection of topological spaces $(E_n)_{n\geq 0}$ together with structure maps $\sigma_n : \Sigma E_n \to E_{n+1}$. If the corresponding maps $\tilde{\sigma}_n : E_n \to \Omega E_{n+1}$ (by the $\Sigma \dashv \Omega$) are weak htpy equivalences, then E is called an <u>Ω-spectrum</u>

- Ω-spectra is closely related to generalized reduced cohomology theories: if E is an Ω-spectrum ~~then~~ and X is a space then

$$E^n(X) : \begin{cases} [X, E_n] & , n \geq 0 \\ [\Sigma^{-n} X, E_0] & , n < 0 \end{cases} \quad , n \in \mathbb{Z}$$

defines a gen. red. coh theory. Even more, the Brown representability thm implies that on CW-complexes all red gen coh th's arise in this way.

~~More generally, given a spectrum E, it gives rise to both a red. gen. homology theory and cohomology theory.~~

Just as for spaces, one can consider homotopy groups of spectra:

<u>Def</u>: Given a spectrum E and $k \in \mathbb{Z}$,

$$\pi_k(E) := \operatorname{colim}_n \pi_{n+k}(E_n)$$

where the colimit is taken over the composite $\pi_{n+k}(E_n) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma E_n) \xrightarrow{(\sigma_n)_*} \pi_{n+k+1}(E_{n+1})$.

(for $k<0$, the sequence starts as soon as $n+k \geq 0$).

$$\left(\text{Recall } \operatorname{colim}_n \left(G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to \cdots \right) \cong \frac{\bigoplus G_i}{g_{i+1} \sim f_i(g_i)} \right).$$

G_i abelians

More generally, given a spectrum E, it gives rise both to a red. gen. homology and cohomology theories, as follows:

$$E_k(X) := \pi_k(E \wedge X)$$

, where $(E \wedge X)_n = E_n \wedge X$ with $\sigma_n \wedge \mathrm{Id}$ as str. maps.

$$E^k(X) := \operatorname{colim}_n [\Sigma^n X, E_{n+k}] \quad (\text{as soon as } n+k \geq 0)$$

finite CW-complex

taken over the composite

$$[\Sigma^n X, E_{n+k}] \xrightarrow{\Sigma} [\Sigma^{n+1} X, \Sigma E_{n+k}] \xrightarrow{(\sigma_{n+k})_*} [\Sigma^{n+1} X, \Sigma E_{n+k+1}]$$

Observe that

$$E_k(S^0) = \pi_k(E \wedge S^0) = \pi_k(E),$$

$$E^k(S^0) = \operatorname{colim}_n \pi_n(E_{n+k}) = \operatorname{colim}_n \pi_{n-k}(E_n) \cong \pi_{-k}(E)$$

and these gps are called the coefficients of the generalised (co)homology theories.

• (Construction of the oriented Thom spectrum MSO): Let $\xi \to B$ be a vector bundle over some top space B (I use ξ insted of $E \to B$ to avoid the clash of notation with spectra). Consider the disc bundle $D(\xi)$, ie vectors of norm ≤ 1 wrt some metric, and $S(\xi)$ the sphere bundle ie vectors of norm $= 1$.

Then the Thom space of ξ is $Th(\xi) := D(\xi)/S(\xi) \to B$.

If B is compact, then $Th(\xi) \cong$ one-pt compactification of ξ.

Being interested in Thom spaces, the most interesting vector bundle to take Th is the universal bundle $\gamma_n = ESO(n) \to BSO(n)$ over the classifying space $BSO(n)$ for n-dim oriented vector bundles. In particular, for the inclusion $Bi : BSO(n) \to BSO(n+1)$ we have that

$(Bi)^* \gamma_{n+1} \cong \gamma_n \oplus \underline{1}$, $\underline{1}$ = trivial 1-dim VS $/BSO(n)$.

So there is a map $\gamma_n \oplus \underline{1} \to \gamma_{n+1}$ (covering Bi).

In general, we have that

$$\mathrm{Th}(\underset{\substack{\downarrow\\ B}}{\xi} \times \underset{\substack{\downarrow\\ B'}}{\xi'}) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\xi').$$

Since for any bundle $\xi \to B$, the bundle $\xi \oplus \underline{m} \to B$ can be viewed as the bundle $\xi \times \underline{m} \to B \times * \cong B$. Of course $\mathrm{Th}(\underline{m} \to *) \cong$ $\cong S^m$. The upshot is that $\gamma_n \oplus \underline{1} \to \gamma_{n+1}$ induces

$$\begin{array}{l} \mathrm{Th}(\gamma_n \oplus \underline{1}) \longrightarrow \mathrm{Th}(\gamma_{n+1}) \\ \quad \| \\ \mathrm{Th}(\gamma_n \times \underline{1}) \\ \quad \| \\ S^1 \wedge \mathrm{Th}(\gamma_n) \\ \quad \| \\ \Sigma\, \mathrm{Th}(\gamma_n) \end{array}$$

The upshot is that this defines a spectrum: set

$$MSO_n := \mathrm{Th}(\gamma_n) \qquad \text{with str. maps as above.}$$

This is called the <u>oriented Thom spectrum</u>.

<u>Theorem</u> (Thom). There are gp isomorphisms

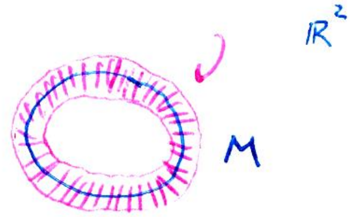
$$\Omega_n^{SO} \cong \pi_n(MSO)$$

(even more, a ring isomorphism $\Omega_*^{SO} \cong \pi_*(MSO)$)

I would like to explain at least how to define the direct map

$$\Omega_n^{SO} \longrightarrow \pi_n(MSO).$$

This is called the <u>Pontryagin - Thom construction</u>. Let M be a compact n-dim mnfd. By the Whitney embedding thm, we can embed $M \hookrightarrow \mathbb{R}^{n+k}$ for some $k \geq 0$. Consider ν the k-dim normal bundle of M in $\mathbb{R}^{n+k}$, satisfying $TM \oplus \nu \cong \underline{n+k}$. The tubular nbhd thm gives an embedding $\nu \hookrightarrow \mathbb{R}^{n+k}$ such that the restriction to the zero-section is the embedding $M \hookrightarrow \mathbb{R}^{n+k}$ (of course in the embedding $\nu \hookrightarrow \mathbb{R}^{n+k}$ we think of ν as a diffeomorphic subspace of the original normal bundle which is a tubular nbhd of M).



Now consider the <u>Pontryagin - Thom collapse map</u>, which collapses the complement of the image of ν in $\mathbb{R}^{n+k}$ to a single point. This map naturally extends to $S^{n+k} = \mathbb{R}^{n+k} \cup \infty$ by sending ∞ to the collapsed pt.

$$S^{n+k} = \mathbb{R}^{n+k} \cup \infty \longrightarrow \frac{\mathbb{R}^{n+k}}{\mathbb{R}^{n+k} - \operatorname{Im} \nu} \cong Th(\nu) \longrightarrow Th(\gamma_k)$$

The last map is induced by the canonical bundle map $\nu \to \gamma_k$ covering the classifying map $M \to BSO(k)$ for ν.

Passing to htpy classes gives an elmt of $\pi_{n+k}(MSO_k)$ and hence of $\pi_n(MSO)$.

We make a few choices in the construction:

Independent of the choice of tubular nbhd. Any two tub. nbhds are isotopic.

————— classifying map: Any two are homotopic, so they yield htpic maps $S^{n+k} \to Th(\gamma_k)$

————— embedding: If we had embedded $M \hookrightarrow \mathbb{R}^{n+k} \underset{\text{standard}}{\hookrightarrow} \mathbb{R}^{n+k+1}$ / instead

the same construction would have given an elmt in $\pi_{n+k+1}(MSO_{k+1})$. It can be checked that this new element is the image of the previous one under the composite

$$\pi_{n+k}(MSO_n) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma MSO_n) \xrightarrow{(\sigma_n)_*} \pi_{n+k+1}(MSO_{n+1})$$

So we can suppose that two embeddings are into the same $\mathbb{R}^{n+k}$ with $k \gg 0$ ($k \geq n+1$ will do). In this case a theorem by Wu says that such embeddings are isotopic, so they yield homotopic maps.

Independent of the choice of cobordism class of mnfd: By a form of the Whitney embedding thm, we can embed a bordism W between M_0 and M_1 into $\mathbb{R}^{n+k} \times I$ such that the intersections with $\mathbb{R}^{n+k} \times i$ are M_i. Hence we get a htpy between the corresponding maps $S^{n+k} \to Th(\gamma_k)$.

We finally remark that $\Omega_n^{SO} \to \pi_n(MSO)$ is a gp hom: given $M \amalg N$, we can embed in different regions. But then the resulting map $S^{n+k} \to Th(\gamma_k)$ is htpic to the composite $S^{n+k} \xrightarrow{\text{collapse equator}} S^{n+k} \vee S^{n+k} \xrightarrow{f \vee g} Th(\gamma_k)$ for f, g the corresponding maps for M and N individually. But that composite is precisely addition in π_{n+k}!

We briefly describe the inverse map $\pi_n(MSO) \to \Omega_n^{SO}$. The key ingredient is transversality. Consider a map $S^{n+k} \to MSO_k = Th(\gamma_k)$ for some $k \geq n+1$.

Note that $Th(\gamma_k) = \operatorname{colim} Th(\gamma_{k,p})$ where $\gamma_{k,p} \to Gr_k(\mathbb{R}^p)$ is the tautological bundle. By compactness, $S^{n+k} \to MSO_k$ factors through some $Th(\gamma_{k,p})$. We want to take a smooth representative $f: S^{n+k} \to Th(\gamma_{k,p})$ that is transversal to the zero-section $Gr_k(\mathbb{R}^p)$, but $Th(\gamma_{k,p})$ is not a mfd.

Getaway: consider $E = Th(\gamma_{k,p}) - \infty$ and choose a codimension 0 submfd $V \subset S^{n+k}$ with $V \subset f^{-1}(E)$ and $f^{-1}(Gr_k(\mathbb{R}^p)) \subset \mathring{V}$ with f transverse to $Gr_k(\mathbb{R}^p)$.

Now we apply the standard result of diff geometry saying that if $f: M_1 \to M_2$ is a smooth map and $N \subset M_2$ is a closed, transverse submfd, then $f^{-1}(N)$ is a submfd of M_1.

(Recall: $f \pitchfork N$ if $\forall x \in f^{-1}(N)$ $\operatorname{Im}(f_{*,x}) + T_{f(x)}N = T_{f(x)}M_2$.)

- The spectrum approach allows to compute $\Omega_*^{SO} \otimes \mathbb{Q}$ easily:

<u>Theorem</u> (Thom 54, Averbuh 59, Milnor 60, Novikov 60, Wall 60).

1) There is a ring isomorphism

$$\mathbb{Q}[y_4, y_8, y_{12}, \dots] \xrightarrow{\cong} \Omega_*^{SO} \otimes \mathbb{Q}$$

$$y_{4i} \longleftrightarrow [\mathbb{CP}^{2i}]$$

2) All torsion in Ω_*^{SO} is of order 2.

3) There is an isomorphism $\mathbb{Z}[z_4, z_8, z_{12}, \dots] \xrightarrow{\cong} \Omega_*^{SO} / \text{torsion}$

4) Two closed, oriented n-mnfds M, N are cobordant if and only if they have the same Stiefel-Whitney and Pontryagin classes.

• (<u>Geometric interpretation of MSO</u>): As already mentioned, the spectrum MSO gives rise to a homology theory $MSO_n(X)$, called <u>oriented bordism</u>, and a cohomology theory $MSO^n(X)$, called <u>oriented cobordism</u>. The homology theory has a geometrical description: consider the set of maps $f: M \to X$ from an oriented n-dim mnfd M to X. Declare $f: M \to X$, $g: N \to X$ to be <u>cobordant</u> if $\exists\, H: W \to X$ from a compact oriented $(n+1)$-mnfd W s.t. H restricts to f and g under an orientation-preserv. diffeo $\partial W \cong M \amalg -N$.

Then $$MSO_n(X) \cong \frac{\{\text{maps } M \to X\}}{\text{cobordism}}.$$

• Geometric structures on mnflds So far we told the story for oriented manifolds. However we can also pay attention to unoriented manifolds, which yields a spectrum MO, or with more generally to manifolds with other (suitable) geometric structure, eg a 'stably normal complex structure' (an actual complex structure fails for odd-dim mnflds), which yields an spectrum MU. This stably normal cplx str is roughly the following: as seen before any embedding $M \hookrightarrow \mathbb{R}^{n+k}$ gives $TM \oplus \nu \cong \underline{n+k}$. Whereas ν depends on the embedding, the class $[\nu] \in \widetilde{KO}(M)$ does not as $[\nu]$ is the opposite of $[TM] \in \widetilde{KO}(M)$. This class $[\nu]$ is called the stable normal bundle of M.

A stably normal cplx str. is then a choice of complex str for $[\nu]$.

We can instead perhaps require a $Spin(n)$-structure on TM of a n-mnfd.

We will tackle all these structures from a common approach: just as real vb are classified by $BO(n)$, ie $Vect^n_{\mathbb{R}}(-) \cong [-, BO(n)]_*$, "stable vector bundles", ie elements in $\widetilde{KO}$, are classified by $BO = \operatorname{colim}_n BO(n)$,

$$\widetilde{KO}(-) = [-, BO]_*$$

So we will look at maps to BO.

Def: Let X be a space and $\xi : X \to BO$. Given a stable bundle $\nu : M \to BO$, an <u>X-structure</u> on it consists of an equivalence class of lift $g : M \to X$ st

$$\begin{tikzcd} & X \arrow[d, "\xi"] \\ M \arrow[ur, dashed, "g"] \arrow[r, "\nu"'] & BO \end{tikzcd}$$

commutes up to a fixed homotopy. Two lifts g_1, g_2 are said to be equivalent if they are htpic over BO, ie if there is an htpy $\xi g_1 \Rightarrow \xi g_2$ st the diagram of htpies

$$\begin{tikzcd} \xi g_1 \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \xi g_2 \arrow[dl, Rightarrow] \\ & \nu & \end{tikzcd}$$

commute up to htpy. At the level of vector bundles, an X-str. g induces a bundle isomorphism $g^* \xi = \nu$.

An <u>X-structure on a mnfd M</u> is an X-str on its stable normal bundle.

<u>Examples</u>:

1) $X = BO$, $\xi = \mathrm{Id}$: A BO-mnfd is just an (unoriented) mnfd.

2) $X = BSO$, ξ induced by the inclusions $SO(n) \hookrightarrow O(n)$. This yields an orientation of the stable normal bundle, which happens to be equivalent to an orientation of the tangent bundle, ie an orientation of M.

3) $X = BSpin$ (Recall. $Spin(n)$ is the unique connected 2-connected cover of $SO(n)$ (Lie gp). Then $BSpin = \mathrm{colim}\, BSpin(n)$ as usual), and ξ induced by $BSpin(n) \to BSO(n) \to BO(n)$ (appearing in the Whitehead tower of $BO(n)$). A $BSpin$-structure on M n-dim is equivalent to a $Spin(n)$-str on M.

4) $X = BU$, ξ induced by $BU(n) \to BO(2n)$ regarding any cplx vb as real.

Def. Define Ω_n^X to be the cobordism classes of closed n-manifolds with X-structure. More precisely, Ω_n^X is the quotient monoid of closed n-mnfds with X structure modulo the submonoid of manifolds of the form ∂W for W an $(n+1)$-dim mnfd with X-structure.

Example. Ω_n^O = closed unoriented n-mnfds / cobordism

Note that here every non-zero elemt is of order 2, as $\partial(M \times I) = M \amalg M$.

We can compute the first Ω_n^O:

$\Omega_0^O \cong \mathbb{Z}/2 = \{\emptyset, *\}$, as $*$ is not the ∂ of a 1-dim mnfd (by the classification, they are either a circle or an interval)

$\Omega_1^O \cong 0$, just as in the oriented case as all 1-mnfd's are orientable

$\Omega_2^O \cong \mathbb{Z}/2 = \{\emptyset, \mathbb{RP}^2\}$: the oriented 2-mnfds are Σ_g and as seen before they are boundaries of handlebodies. An unoriented 2-mnfd is $M_g = \mathbb{RP}^2 \# \overset{g}{\cdots} \# \mathbb{RP}^2$.

Note $M_2 \cong$ Klein bottle and this is the boundary of the 3-mnfd

$D^2 \times I / (x,0) \sim (r(x),1)$ (r = reflexion) $\cong \mathbb{M} \times I$, $\mathbb{M}$ = Mobius strip.

We can see $\mathbb{RP}^2$ is not boundary of a 3-mnfd by an Euler characteristic argument: note $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$. But if a manifold M is a boundary, then $\chi(M)$ is even. For if $\partial N = M$, then:

- If $\dim N$ = even, then M odd-dim so by Poincaré duality $\chi(M) = 0$.
- If $\dim N$ = odd, let $S = N \cup_{\partial N} N$. This is a odd-dim mnfd so $\chi(S) = 0$. But on the other hand

$$\chi(S) = \chi(N) + \chi(N) - \chi(\partial N)$$

so $\chi(M) = \chi(\partial N) = 2 \cdot \chi(N)$ even.

In order to compute the rest of unoriented cobordism groups (Thom), we mimic the same approach as before. It turns out that we can define ~~a~~ Thom spectrum MX for any X-structure.

• (Construction of MX). Let $X \to BO$ be a map as before.

Define $X_n := X \times^h_{BO} BO(n)$ and write γ_n^X for the vector bundle classified by the projection $X_n \to BO(n)$. From the htpy commutative diagram

$$\begin{tikzcd}
X_n \arrow[r] \arrow[dr, dashed, "j_n"] \arrow[ddr, bend right] & BO(n) \arrow[dr, bend left] & \\
& X_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & BO(n+1) \arrow[d] \\
& X \arrow[r] & BO
\end{tikzcd}$$

we get maps $j_n : X_n \to X_{n+1}$ satisfying $j_n^* \gamma_{n+1}^X \cong \gamma_n^X \oplus \underline{1}$.

As before set $MX_n := Th(\gamma_n^X)$ with structure maps

$$\Sigma MX_n \cong Th(\gamma_n^X \oplus \underline{1}) \cong Th(j_n^* \gamma_{n+1}^X) \to Th(\gamma_{n+1}^X) = MX_{n+1}.$$

An adaptation of the previous thm for MSO shows that

<u>Theorem</u> (Pontryagin-Thom): For any X-structure, there are isomorphisms

$$\Omega_n^X \cong \pi_n(MX)$$

• In the cases of interest ($X = BO, BSO, BU$) the previous is actually a ring isomorphism (the case when MX is a ring spectrum, that happens if $X \to BO$ is a map of H-spaces).

We finish off by keeping record of the rings Ω_*^X for X as before.

Theorem (Thom). $\Omega_*^O \cong \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, \dots]$ a polynomial ring with one generator x_n in dim n for each $n \neq 2^p - 1$ for some $p \geq 0$.
In particular, $x_{2k} = [\mathbb{RP}^{2k}]$.

Moreover, two closed n-mnfd's are cobordant if and only if they have the same Stiefel-Whitney classes.

Theorem (Milnor, Novikov): $\Omega_*^U \cong \mathbb{Z}[x_2, x_4, x_6, x_8, \dots]$ a polynomial ring with one generator x_{2k} in dim $2k$ for every $k \geq 1$. If $k = p-1$ for some prime p, then $x_{2k} = [\mathbb{CP}^k]$.

Moreover, two stably complex mnfd's are bordant if and only if they have the same Chern classes.

References (by order of relevance)

- Switzer, "Algebraic Topology, Homology and Homotopy" (chapter 12)
- Kupers, "Oriented cobordism: calculation and application."
- Meier, "From Elliptic Genera to Topological Modular Forms.
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