A note on Drinfeld associators

Let k be a field of char 0, and let $k\langle\langle X,Y\rangle\rangle$ be the ring of formal power series in two non-comm variables X, Y.

• Observe that $k\langle\langle X,Y\rangle\rangle$ is a (cocommutative) bialgebra with

$$\Delta(X) = X\otimes 1 + 1\otimes X \quad , \quad \Delta(Y) = Y\otimes 1 + 1\otimes Y.$$

and $\varepsilon(X) = 0$ $\qquad \varepsilon(Y) = 0$

(in fact $k\langle\langle X,Y\rangle\rangle$ is a complete Hopf algebra $\longrightarrow$ Hopf alg str comes from $\mathcal{U}(\mathcal{L}(x_1,x_2)) = k\langle x_1,x_2\rangle$ after completion

Def: Let A be a bialgebra. We say that $x\in A$ is

1) primitive if $\Delta(x) = x\otimes 1 + 1\otimes x$,
2) group-like if $\Delta(x) = x\otimes x$ and x is invertible

The set of primitive elem'ts $\mathcal{P}(A)$ is a linear subspace, and it is closed under the commutator $[x,y) = xy - yx$, so it forms a Lie algebra.

Lemma (Chamoton). For the graded completion $\hat{A}$ of a connected* graded bialgebra A, the functions exp and log, defined by the usual power series, establish a bijection

$$\mathcal{P}(\hat{A}) \;=\!=\; \mathcal{G}(\hat{A}) \leftarrow \text{gp like}$$

$$x \longmapsto \exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\log y = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(y')^k \longleftarrow y = 1 + y'$$

*connected: the unit $\eta: k\to A$ induces iso $\eta: k \xrightarrow{\simeq} A_0$, $A = \bigoplus A_i$

Lemma. The Lie alg of $\mathcal{P}(k\langle\langle X,Y\rangle\rangle)$ is exactly the degree-completion of the free Lie algebra $L(X,Y)$ on X,Y (which is a graded Lie alg w/ $\deg X = \deg Y = 1$).

~~Theorem (Furusho). Let $\varphi \in k\langle\langle X,Y\rangle\rangle$ be a gp-like elmt satisfying the pentagon equation~~

Theorem (Furusho). Let $\varphi \in k\langle\langle X,Y\rangle\rangle$ be a gp-like elmt satisfying the pentagon equation

$$\varphi(t_{12}, t_{23}+t_{24})\,\varphi(t_{13}+t_{23}, t_{34}) = \varphi(\quad)\,\varphi(\quad)\,\varphi(\quad)$$

in the deg-completion $\hat{U}(t_4)$ of the Drinfeld-Kohno algebra.

Then there exists an elmt $\mu \in \bar{k}$ (alg closure of k) st (φ,μ) satisfies the two hexagon equation

$$\exp\left(\frac{\mu(t_{13}+t_{23})}{2}\right) = \cdots$$

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In fact, if $c_2(\varphi)$ denotes the coef of the monomial XY of φ, then $\mu^2 = 24\cdot c_2(\varphi)$, ie $\mu = \pm\sqrt{24\,c_2(\varphi)}$.

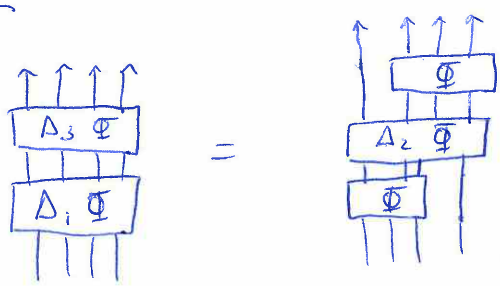
General associators

Def. An associator is an element $\Phi \in \mathcal{A}(\uparrow\uparrow\uparrow)$ satisfying

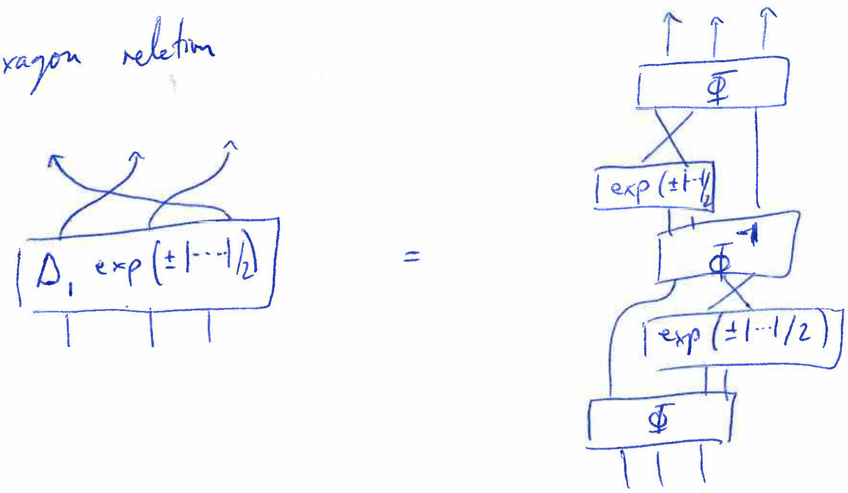
1) Φ group-like, ie $\Phi = \exp(\quad)$ (grp-like)

2) $\varepsilon_i(\Phi) = 1 \in \mathcal{A}(\uparrow\uparrow)$ $\forall\, i=1,2,3$. (strong invertibility)

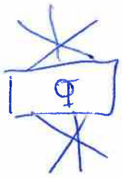
3) The pentagon relation

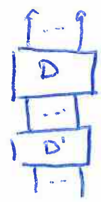


4) The hexagon relation



Lemma. For any associator $\Phi \in \mathcal{A}(\uparrow\uparrow\uparrow)$,

$\Phi^{-1} =$ 

Note: $\mathcal{A}(\uparrow_n)$ is an algebra w/ multiplication $D\cdot D' = D\circ D' =$ 

• By the lemma about primitive $\Longleftrightarrow$ gp like, we have that

$$\varphi = \exp(\text{elmt of } \hat{L}(X,Y)).$$

$$= \exp(\lambda \cdot [X,Y] + \text{stuff of deg} > 2)$$

$$= 1 + \lambda [X,Y] + \text{stuff of deg} > 2$$

$$= 1 + \lambda XY - \lambda YX + \cdots$$

$$\underline{\underline{c_2}} = \frac{\mu^2}{24}$$

So $\varphi = \exp\left(\frac{\mu^2}{24}[X,Y] + \text{stuff of deg} > 2\right)$. $(*)$

Upshot. Any φ gp like satisfying the pentagon eq is of the form $(*)$ where $\mu^2 \in k$. (and $\mu \in \bar{k}$).

In particular, we can take $\underline{\mu := 1}$, so

$$\varphi = \exp\left(\frac{1}{24}[X,Y] + \cdots\right).$$

We call these Drinfeld associators.

Note. φ_{KZ} has $\mu = 2\pi i$.

<u>Proposition</u>. If $\varphi \in k\langle\langle X, Y\rangle\rangle$ is a Drinfeld associator, then

$$\Phi := \varphi\left(\uparrow\!\text{-}\!\text{-}\!\uparrow\ \uparrow\ ,\ \uparrow\ \uparrow\!\text{-}\!\text{-}\!\uparrow\right)$$

is an associator.

<u>Q</u>. Does every associator arise in this way? Ie is every Φ $\varphi(\uparrow\uparrow\uparrow)$ for some Drinfeld associator φ?

(An associator like $\Phi = \varphi\left(\uparrow\!\text{-}\!\uparrow\ \uparrow\ ,\ \uparrow\ \uparrow\!\text{-}\!\text{-}\!\uparrow\right)$, which lives in the subalgebra $\mathcal{A}^h(\uparrow\uparrow\uparrow) \subset \mathcal{A}(\uparrow\uparrow\uparrow)$ of <u>horizontal chord diagrams</u>,

$$\mathcal{A}^h(\uparrow\uparrow\uparrow) = \hat{\mathcal{U}}(t_3)$$

is called <u>horizontal associator</u>)

<u>A</u>: <u>No</u>, there are non-horizontal associators in $\mathcal{A}(\uparrow\uparrow\uparrow)$, see Bar-Natan's "Associators and the Grothendieck-Teichmuller group".

Associators up to deg 2

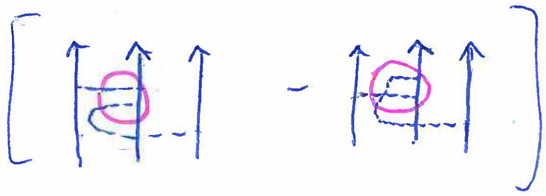
Lie series

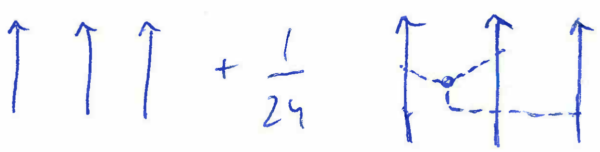
Let φ be a Drinfeld associator, $\varphi(X,Y) = \exp\left[\frac{1}{24}[X,Y] + \deg>2\right] =$

$= 1 + \frac{1}{24}[X,Y] + (\deg>2)$. Then

$\Phi_\varphi = \varphi($  $) =$

$=$  $+ \deg>2$

$=$ ” $+ \frac{1}{24}$  $+ \deg>2$

$\overset{STU}{=}$  $+(\deg>2)$ $(*)$

Now I claim that this also happens if Φ is a general associator, non necessarily horizontal.

<u>Lemme</u>. Any associator $\Phi \in \mathcal{A}(\uparrow\uparrow\uparrow)$ is of the form $(*)$

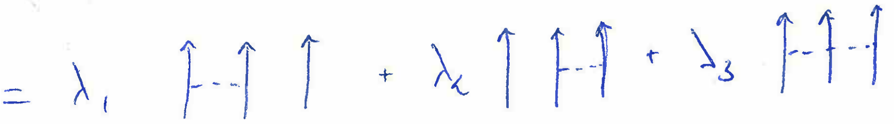
Pf. Since $\mathcal{A}(\uparrow\uparrow\uparrow)$ is a complete Hopf algebra, the lemma from the beginning still holds and any associator Φ is

$$\Phi = \exp[\text{ primitive elmt }]$$

The subspace $\mathcal{P}(\uparrow\uparrow\uparrow) \subset \mathcal{A}(\uparrow\uparrow\uparrow)$ of primitive elmts is spanned by connected Jacobi diagrams. ~~and since~~ Let us put

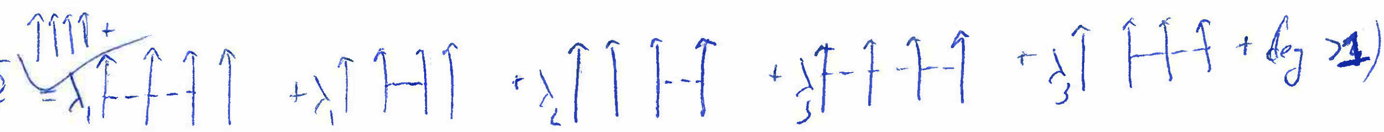
$$\text{primitive elmt} = \text{deg } 1 + \text{deg } 2 + \text{deg} > 2$$

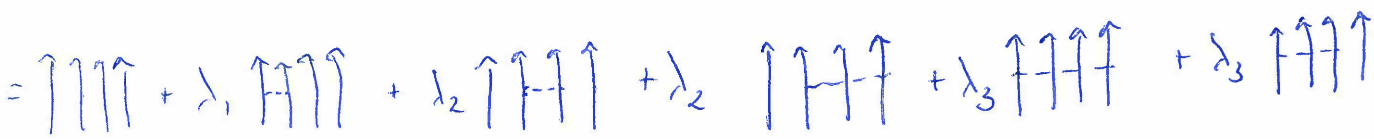
deg 1 part means that one can only have one chord.

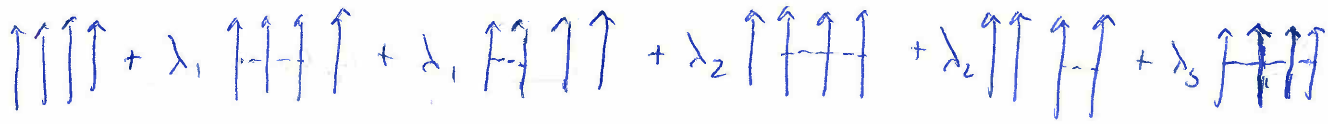
So deg 1 =  , so

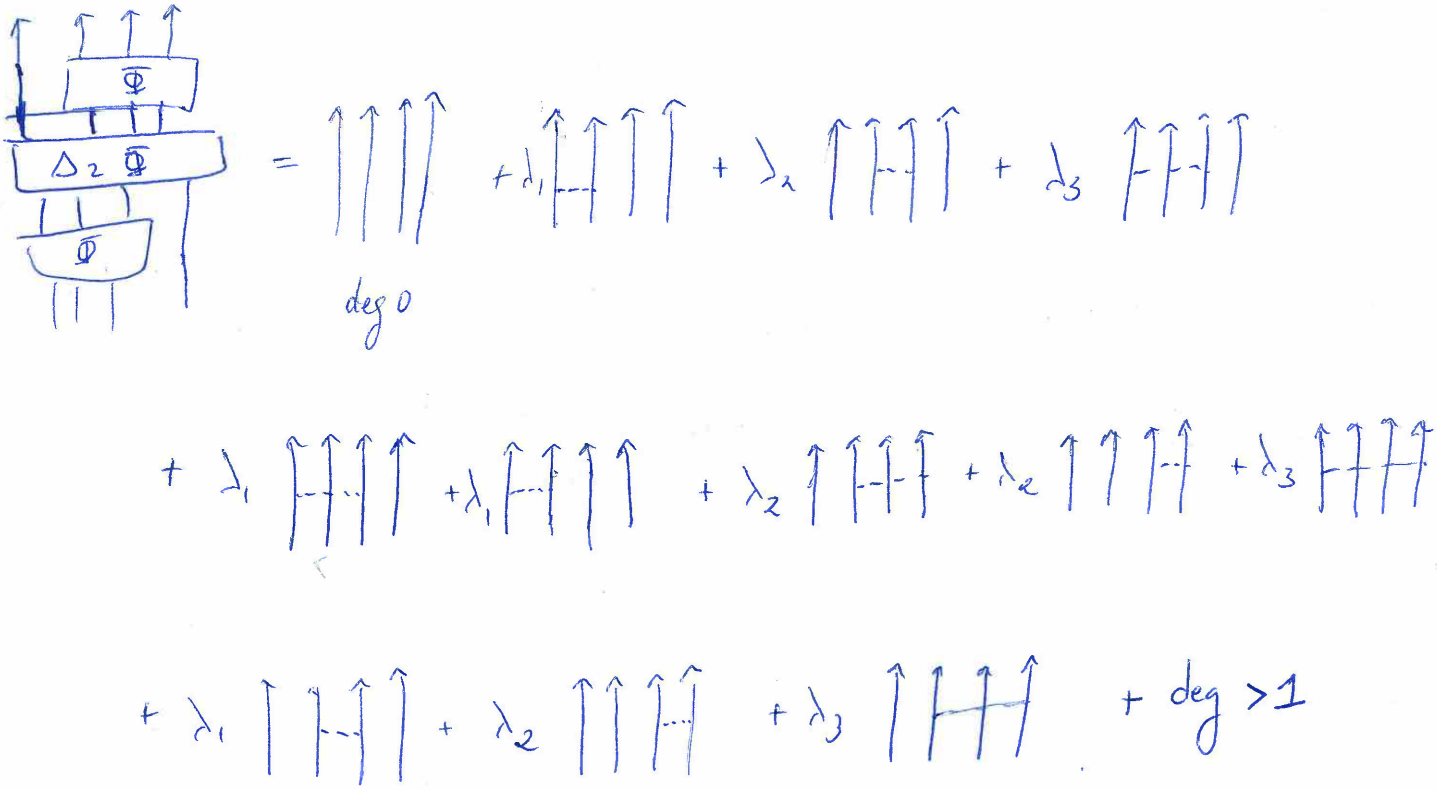
the deg 1 of Φ is $\lambda_1(\quad) + \lambda_2(\quad) + \lambda_3(\quad)$.

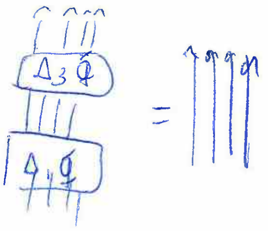
~~The pentagon relation implies~~
I want to see that the pentagon rel implies that $\lambda_i = 0$.

$\Delta_1 \Phi =$ 

$\Delta_3 \Phi =$ 

$\Delta_2 \Phi =$ 

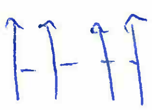




+ sum of the deg 1 parts of $\Delta_1 \Phi$ and $\Delta_3 \Phi$.

But now we have, looking at coef of  , $2\lambda_1 = \lambda_1$, ie $\lambda_1 = 0$.

Looking at  : $2\lambda_2 = \lambda_2 \Rightarrow \lambda_2 = 0$

Looking at  : $\lambda_3 = 2\lambda_3 \Rightarrow \lambda_3 = 0$.

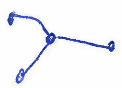
So Φ has no deg 1 part.

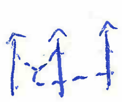
For the deg 2 part, we need to see how many connected Jacobi diag of deg 2 there are in $\uparrow\uparrow\uparrow$, ie, connected w/ 4 vertices.

So it has to be  placed in the strands of $\uparrow\uparrow\uparrow$ (all possibilities). However we have that $\varepsilon_i \Phi = \uparrow\uparrow$ $\forall i = 1,2,3$.

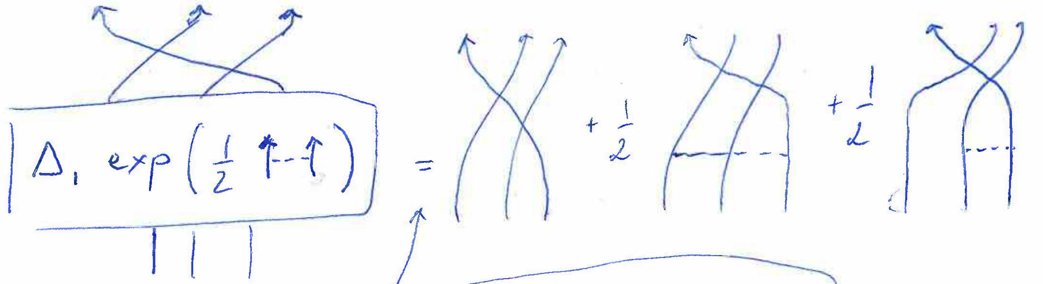
This means that $\varepsilon_i(\text{deg} > 0 \text{ part of } \Phi) = 0$ $\forall i$. But the Jacobi diagrams we are considering are connected, ie when applying ε_i either it entirely survives (in $\uparrow\uparrow$) or it vanishes. So the deg n part of $\varepsilon_i \Phi$ can only come from the deg n part of Φ. So since

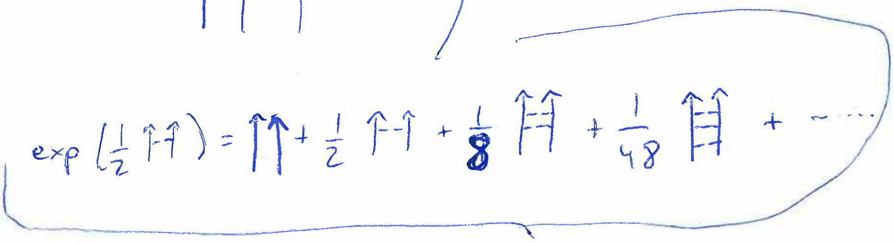
$\deg_2(\varepsilon_i \Phi) = 0 \Rightarrow \varepsilon_i(\text{conn Jacobi diag of deg 2 in } \Phi) = 0$.

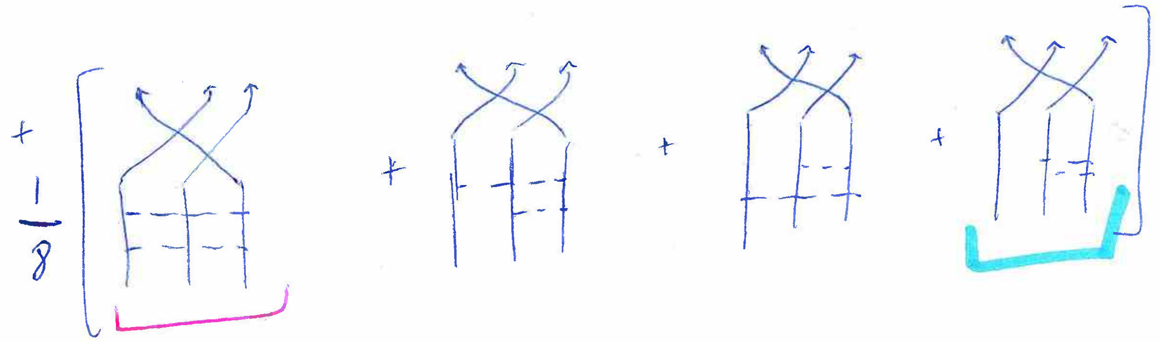
But for that  must have all its univalent vertices in the three strands of $\uparrow\uparrow\uparrow$, ie it must be $\lambda \cdot$ .

Ie $\Phi = \exp\left[\lambda \cdot\right.$  $\left. + \deg > 2\right] = \uparrow\uparrow\uparrow + \lambda$  $+ \deg > 2$

Now I claim that the hexagon equation implies that $\lambda = \frac{1}{24}$.

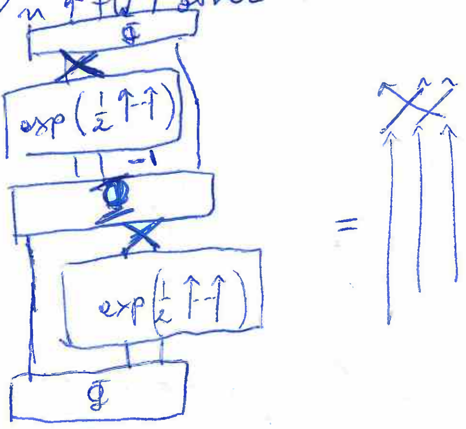




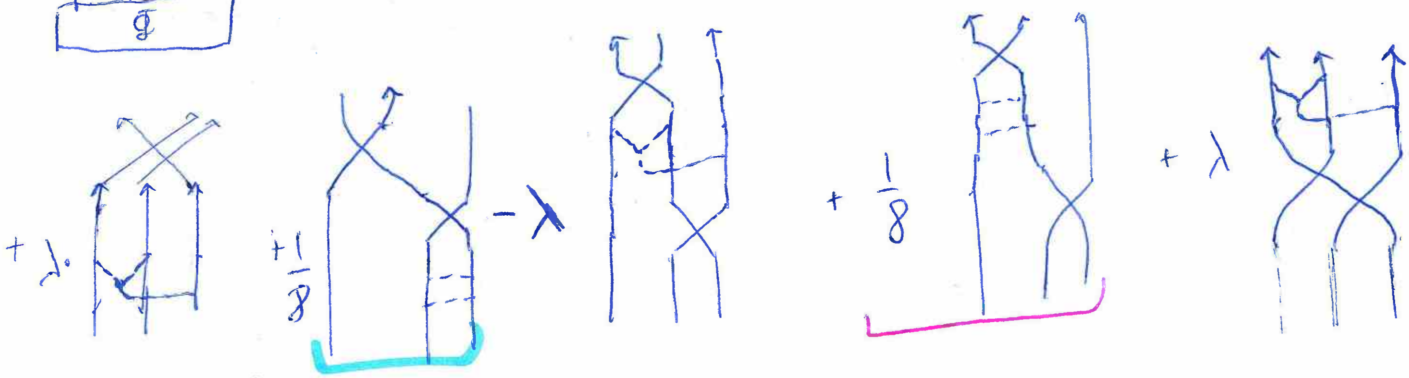


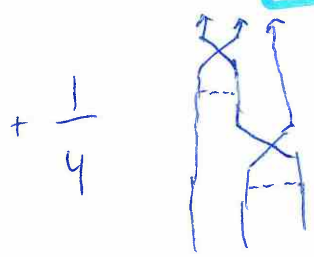
$+ \deg > 2.$

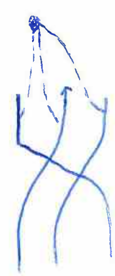
On the other hand

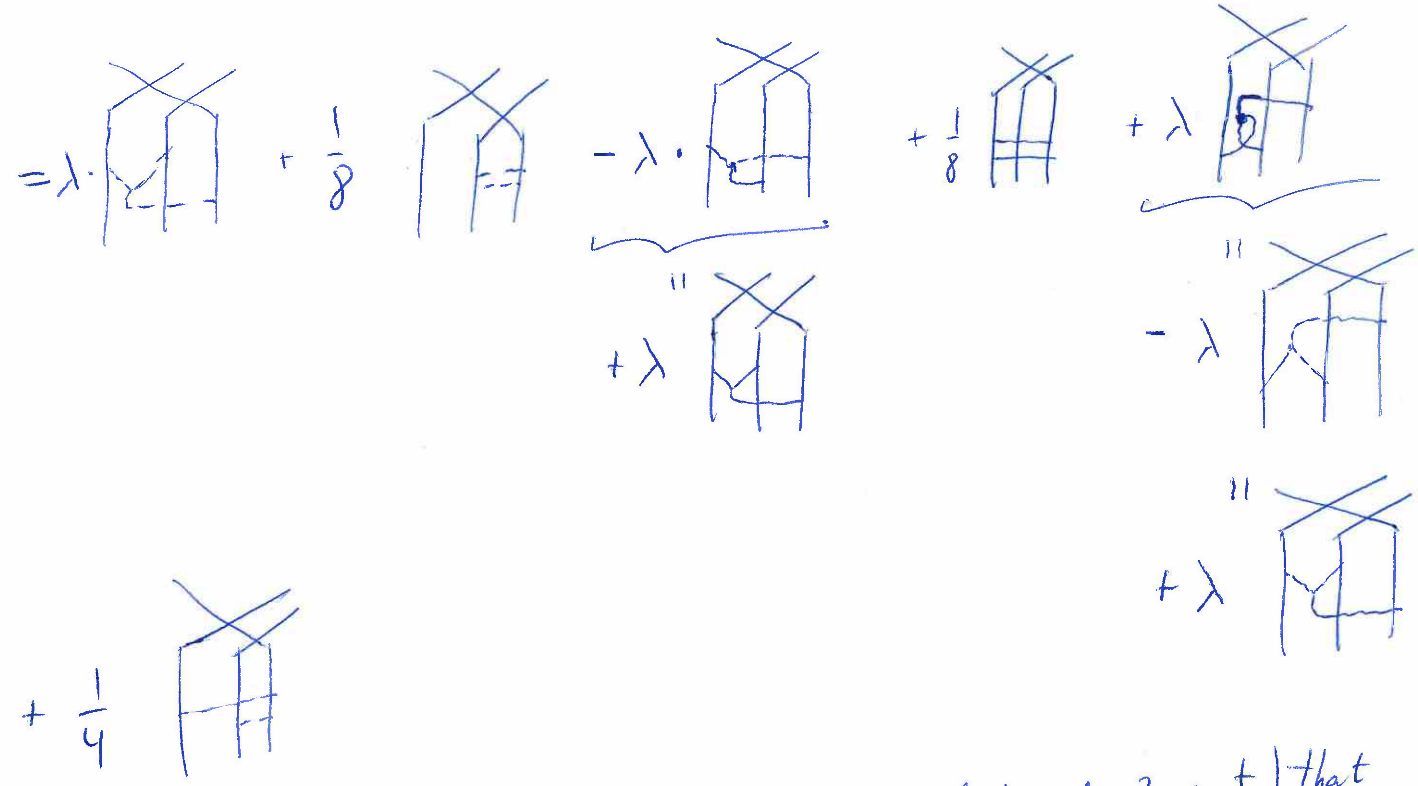


$+$ (Φ, Φ^{-1} don't have deg 1 so the only deg 1 ~~(no deg 1 because Φ, Φ^{-1} have not got)~~ comes from the exp, as in $\Delta_1 \exp\left(\frac{1}{2}\ \uparrow\text{-}\uparrow\right)$ so the same.)

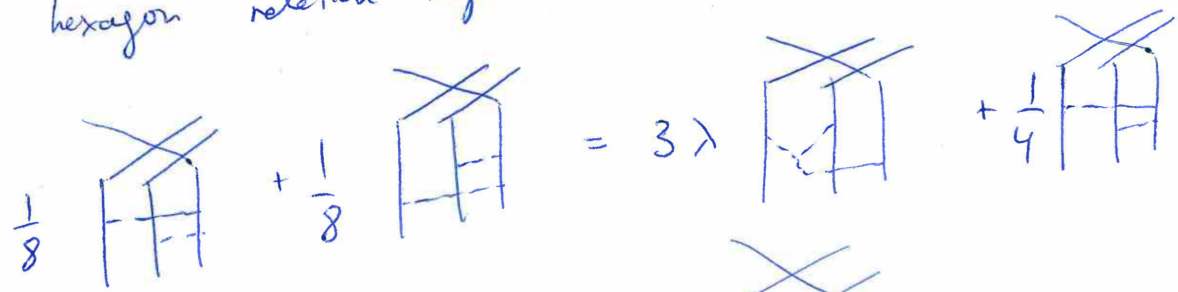


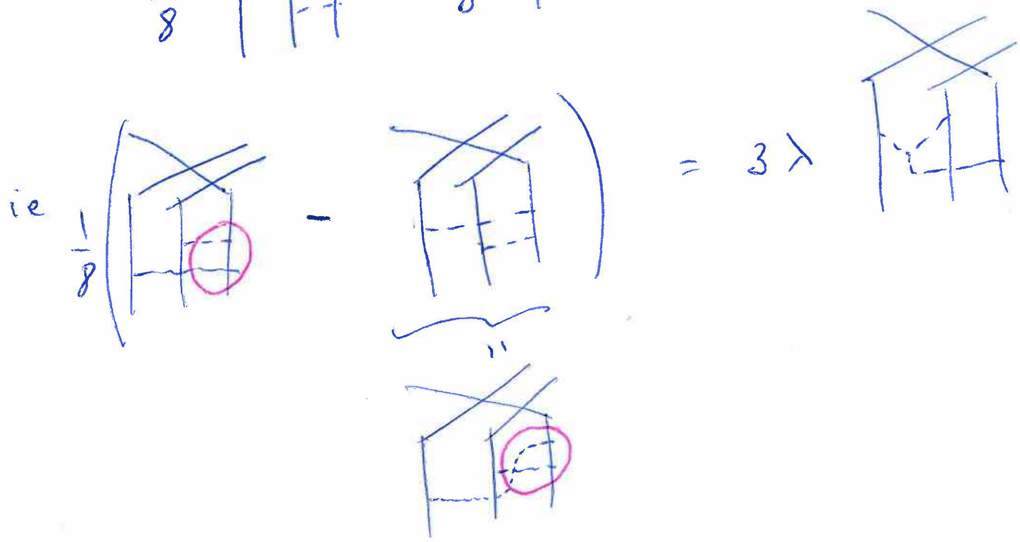


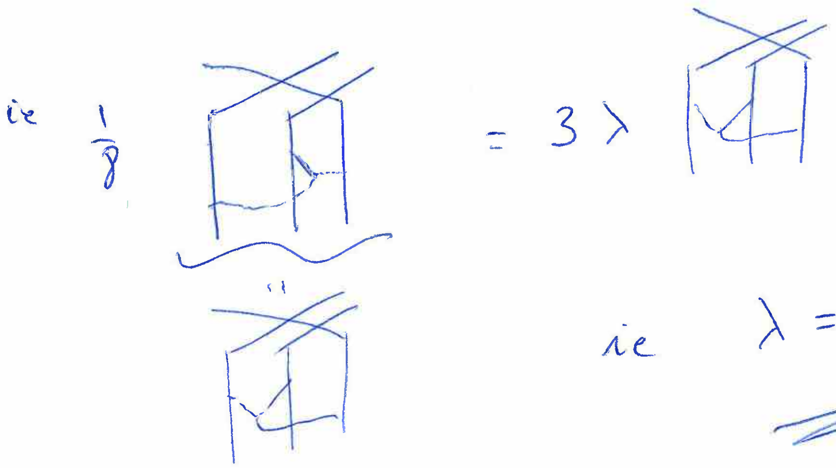




So the hexagon relation implies (looking at its deg 2 part) that







ie $\lambda = \dfrac{1}{24}$