Algebraic invariants of common topological spaces

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In the following:
- S^n is the n-sphere.
- Σ_g is the closed orientable surface of genus g, that is, $\Sigma_g = \mathbb{T} \# \overset{g}{\cdots} \# \mathbb{T}$.
- $\Sigma_{g,h}$ is the closed orientable surface of genus g and h boundary components.
- M_p is the closed non-orientable surface of genus p, that is, $M_p = \mathbb{RP}^2 \# \overset{p}{\cdots} \# \mathbb{RP}^2$.
- $M_{p,h}$ is the closed non-orientable surface of genus p and h boundary components.
- $\mathbb{RP}^n$ is the n-dimensional real projective space.
- $\mathbb{CP}^n$ is the $2n$-dimensional complex projective space.
- X_K is the knot complement of a knot $K \subset S^3$.
- A is an abelian group (that is, a $\mathbb{Z}-$module).
- $F(x_1, \dots, x_n)$ is the free group generated by elements $x_1, \dots, x_n$.

1 Homology

1. $H_k(S^n; A) = \begin{cases} A, & k = 0, n \\ 0 & \text{else} \end{cases}, \qquad n > 0.$

   $H_k(S^0; A) = \begin{cases} A \oplus A, & k = 0 \\ 0 & \text{else} \end{cases}$

2. $H_k(\Sigma_g; A) = \begin{cases} A, & k = 0 \\ A^{2g}, & k = 1 \\ A, & k = 2 \\ 0, & k \geq 3. \end{cases}$

3. $H_k(M_p; A) = \begin{cases} A, & k = 0 \\ A^{p-1} \oplus A/2A, & k = 1 \\ 0, & k \geq 2. \end{cases}$

4. $H_k(\mathbb{CP}^n; A) = \begin{cases} A, & k = 0, 2, 4, \cdots, 2n \\ 0, & \text{else} \end{cases}$

5. $H_k(\mathbb{CP}^\infty; A) = \begin{cases} A, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$

6. $H_k(\mathbb{RP}^n; A) = \begin{cases} A, & (k=0) \text{ and } (k=n \text{ and odd}) \\ A/2A, & 0<k<n \text{ and } k \text{ odd} \\ {}_2A, & 0<k\le n \text{ and } k \text{ even} \\ 0, & \text{else} \end{cases}$

7. $H_k(\mathbb{RP}^\infty; A) = \begin{cases} A, & k=0 \\ A/2A, & k \text{ odd} \\ {}_2A, & k \text{ even and} > 0 \\ 0, & \text{else} \end{cases}$

8. $H_k(X_K; A) = \begin{cases} A, & k=0,1 \\ 0, & k\ge 2 \end{cases}$

2 Cohomology

1. $H^k(S^n; A) = \begin{cases} A, & k=0,n \\ 0 & \text{else} \end{cases} \quad (n>0),$

   $H^\bullet(S^n; \mathbb{Z}) = \mathbb{Z}[x]/(x^2)$, $|x| = n$.

   $H^k(S^0; A) = \begin{cases} A\oplus A, & k=0 \\ 0 & \text{else} \end{cases}$

2. $H^k(\Sigma_g; A) = \begin{cases} A, & k=0 \\ A^{2g}, & k=1 \\ A, & k=2 \\ 0, & k\ge 3. \end{cases}$

   Call 1 the generator of $H^0 = \mathbb{Z}$, $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ the generators of $H^1 = \mathbb{Z}^{2g}$, and σ the generator of $H^2 = \mathbb{Z}$. Then the cohomology ring $H^\bullet(\Sigma_g; \mathbb{Z})$ is given by

   $$\alpha_i\alpha_j = 0, \quad \beta_i\beta_j = 0, \quad \alpha_i\beta_j = \delta_{ij}\sigma.$$

   For $g = 1$, we have $H^\bullet(\mathbb{T}; \mathbb{Z}) = \mathbb{Z}[x,y]/(x^2, y^2)$ with anticommutative product, $xy = -yx$.

3. $H^k(M_p; A) = \begin{cases} A, & k=0 \\ A^{p-1}\oplus {}_2A, & k=1 \\ A/2A, & k=2 \\ 0, & k\ge 3. \end{cases}$.

   For $p = 2$, we have $H^\bullet(\mathbb{K}, \mathbb{Z}/2) = \mathbb{Z}/2[x,y]/(x^3, y^2, x^2 - xy)$ for the Klein bottle.

4. $H^k(\mathbb{CP}^n; A) = \begin{cases} A, & k=0,2,\dots,2n \\ 0, & \text{else} \end{cases}$

   $H^\bullet(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}), \quad |x| = 2$

5. $H^k(\mathbb{CP}^\infty; A) = \begin{cases} A, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$

   $H^\bullet(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}[x], \quad |x| = 2$

6. $H^k(\mathbb{RP}^n; A) = \begin{cases} A, & (k=0) \text{ and } (k=n \text{ and odd}) \\ A/2A, & 0 < k \leq n \text{ and } k \text{ even} \\ {}_2A, & 0 < k < n \text{ and } k \text{ odd} \\ 0, & \text{else} \end{cases}$

   $H^k(\mathbb{RP}^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & k \leq n \\ 0, & k > n. \end{cases}$

   $H^\bullet(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^{n+1}), \quad |x| = 1.$

   $H^\bullet(\mathbb{RP}^{2n}; \mathbb{Z}) = \mathbb{Z}[y]/(2y, y^{n+1}), \quad |y| = 1.$

   $H^\bullet(\mathbb{RP}^{2n+1}; \mathbb{Z}) = \mathbb{Z}[y,z]/(2y, y^{n+1}, z^2, yz), \quad |y| = 1, |z| = 2n+1.$

7. $H^k(\mathbb{RP}^\infty; A) = \begin{cases} A, & k = 0 \\ A/2A, & k \text{ even and} > 0 \\ {}_2A, & k \text{ odd} \end{cases}$

   $H^k(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$ for all k.

   $H^\bullet(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x], \quad |x| = 1.$

   $H^\bullet(\mathbb{RP}^\infty; \mathbb{Z}) = \mathbb{Z}[y]/(2y), \quad |y| = 2.$

8. $H^k(X_K; A) = \begin{cases} A, & k = 0, 1 \\ 0, & k \geq 2 \end{cases}$

   $H^\bullet(X_K; \mathbb{Z}) = \mathbb{Z}[x]/(x^2)\ , |x| = 1.$

3 Homotopy groups

3.1 Fundamental group

1. $\pi_1(S^1) = \mathbb{Z}$.

2. $\pi_1(S^n) = 0$ for all $n \geq 2$.

3. $\pi_1(\Sigma_g) = \dfrac{F(a_1, b_1, \ldots, a_g, b_g)}{< a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} >}$.

   $\pi_1(\Sigma_g)^{\text{ab}} = \mathbb{Z}^{2g}$.

   $\pi_1(\mathbb{T}) = \mathbb{Z} \oplus \mathbb{Z}$.

4. $\pi_1(\Sigma_{g,h}) = F(a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_{h-1})$ (because $\Sigma_{g,h} \simeq \bigvee_{2g+h-1} S^1$).

5. $\pi_1(M_p) = \dfrac{F(a_1 \ldots, a_p)}{< a_1^2 \ldots a_p^2 >}$.

   $\pi_1(M_p)^{\text{ab}} = \mathbb{Z}^{p-1} \oplus \mathbb{Z}/2\mathbb{Z}$.

   $\pi_1(\mathbb{K}) = F(a,b)/ < aba^{-1}b >$.

6. $\pi_1(M_{p,h}) = F(a_1 \ldots, a_p, c_1, \ldots, c_{h-1})$ (because $M_{p,h} \simeq \bigvee_{g+h-1} S^1$).

7. $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$.

8. $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2\mathbb{Z}$.

9. $\pi_1(\mathbb{CP}^n) = 0$.

10. $\pi_1(\mathbb{CP}^\infty) = 0$.

11. $\pi_1(X_K) = \dfrac{F(x_1, \ldots, x_n)}{< r_1, \ldots, r_n >}$, where n is the number of crossings of a diagram of K and r_i stands for the relation $x_k x_{i+1} = x_i x_k$ for a positive crossing and $x_k x_i = x_{i+1} x_k$ for a negative crossing.

3.2 Higher homotopy groups

1. $\pi_k(S^n) = \begin{cases} \mathbb{Z}, & k = n \\ 0, & k < n \\ \text{highly non-trivial}, & k > n \end{cases} \qquad (k, n > 0)$

$\pi_k(S^1) = \begin{cases} \mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$

2. $\pi_k(\mathbb{RP}^n) = \begin{cases} 0, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & k = 1 \\ \pi_k(S^n) & k > 1 \end{cases}$

3. $\pi_k(\mathbb{RP}^\infty) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$

4. $\pi_k(\mathbb{CP}^n) = \begin{cases} 0, & k = 0, 1 \\ \mathbb{Z}, & k = 2 \\ \pi_k(S^{2n+1}) & k > 2 \end{cases}$

5. $\pi_k(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z}, & k = 2 \\ 0, & k \neq 2 \end{cases}$

6. $\pi_k(\mathbb{T}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$

7. $\pi_k(\mathbb{K}) = \begin{cases} F(a, b)/ < aba^{-1}b >, & k = 1 \\ 0, & k \neq 1 \end{cases}$

8. $\pi_k(X_K) = \begin{cases} \pi_1(X_K), & k = 1 \\ 0, & k \neq 1 \end{cases}$

4 Euler characteristic

1. $\chi(S^n) = 1 + (-1)^n, \qquad n \geq 0.$
2. $\chi(\Sigma_g) = 2 - 2g, \qquad g \geq 0.$
3. $\chi(\Sigma_{g,h}) = 2 - 2g - h, \qquad g, h \geq 0.$
4. $\chi(M_p) = 2 - p, \qquad p \geq 0.$
5. $\chi(M_{p,h}) = 2 - p - h, \qquad p, h \geq 0.$
6. $\chi(\mathbb{CP}^n) = n + 1, \qquad n > 0.$
7. $\chi(\mathbb{RP}^n) = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}, \qquad n > 0.$
8. $\chi(X_K) = 0$

5 K-theory

In the following $p = 0, 1$.

1. $\widetilde{K}(S^n) = \begin{cases} \mathbb{Z}, & n \equiv 0 \mod 2 \\ 0, & n \equiv 1 \mod 2 \end{cases}$

   $K^\bullet(S^2) = \mathbb{Z}[x]/(x-1)^2$

   $\widetilde{KO}(S^n) = \begin{cases} \mathbb{Z}, & n \equiv 0, 4 \mod 8 \\ \mathbb{Z}/2, & n \equiv 1, 2 \mod 8 \\ 0, & n \equiv 3, 5, 6, 7 \mod 8 \end{cases}$

2. $\widetilde{K}^p(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}^n, & p = 0 \\ 0, & p = 1 \end{cases}$

   $K^\bullet(\mathbb{CP}^n) = \mathbb{Z}[x]/(x^{n+1}),$ where $x = L - 1$, L being the canonical line bundle over $\mathbb{CP}^1$.

3. $\widetilde{K}^p(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/(2^{\lfloor \frac{n}{2} \rfloor}), & p = 0 \\ 0, & p = 1, \quad n \text{ even}, \\ \mathbb{Z}, & p = 1, \quad n \text{ odd} \end{cases}$

   $\widetilde{KO}(\mathbb{RP}^n) = \mathbb{Z}/2^{\varphi(n)}, \quad \varphi(n) = \#\{s : 0 < s \leq n, s \equiv 0, 1, 2 \text{ or } 4 \mod 8\}$
   (eg $\varphi(n) = 2$ for $n = 2, 3$, $\varphi(n) = 3$ for $n = 4, 5, 6, 7$, $\varphi(n) = 4$ for $n = 8$, $\varphi(n) = 5$ for $n = 9$, $\varphi(n) = 6$ for $n = 10, 11, \dots$).