Slice knots and knot concordance

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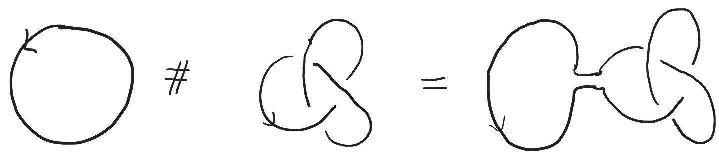
I would like to give an introduction to a classical topic in knot theory, namely the knot concordance group. Hopefully this will serve as a warm-up for the talks in the Winter Braids school.

1 The knot concordance group

Usually one studies knot theory having as study object the set $\mathcal{K}$ of isotopy classes of oriented (tame / PL / smooth) embeddings $K : S^1 \hookrightarrow S^3$. Having a set is okay, but having algebraic tools to manipulate that is always better. In $\mathcal{K}$ there is an operation called **connected sum** and denoted by #,



turning $(\mathcal{K}, \#)$ into a commutative monoid, with the unknot O as the unit element. It is commutative as one can slide one knot from the left-hand side of the page to the right-hand side along the other knot. It is important to note that there is not any non-trivial knot with an inverse in $\mathcal{K}$. For argue with the genus, the knot group or the Mazur swindle (if you dare!).

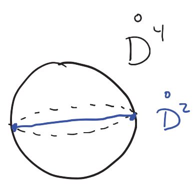
Now, can we solve the problem of not having inverses? In other words, can we turn $(\mathcal{K}, \#)$ into an abelian group? A general strategy might be to consider the *Grothendieck construction* (also known as the group completion) of a monoid. This is in fact the left-adjoint to the forgetful $U : \mathsf{Ab} \longrightarrow \mathsf{CMon}$. However this returns something strictly larger than the initial monoid (eg the group completion of the natural numbers is the integers), and $\mathcal{K}$ is already quite large! Instead of considering a group completion, it might be wiser to consider a quotient of $\mathcal{K}$ - hopefully with an abelian group structure.

Let us step back and recall the cobordisms we studied for Khovanov homology, but this time embedded in D^4.

Notation. Given an oriented knot K, denote $\overline{K}$ the mirror image of K (ie all crossings are reversed) and K^* the knot K with reverse orientation.

Definition. Two knots K_0, K_1 are **concordant** if there is a locally flat embedding $S^1 \times I \hookrightarrow S^3 \times I$ in such a way that the ends $S^1 \times \{i\}$ are embedded in $S^3 \times \{i\}$ as $K_i \times \{i\}$, for $i = 0, 1$.

In the given definition, *locally flat* means that every point in the interior image of the embedding has a neighbourhood which is isomorphic to the pair $(\mathring{D}^4, \mathring{D}^2)$ in the standard way.



Knot concordance is an equivalence relation (check!) and so

Definition. The **knot concordance group** is the quotient

$$\mathcal{C} := \frac{(\mathcal{K}, \#)}{\text{concordance}}.$$

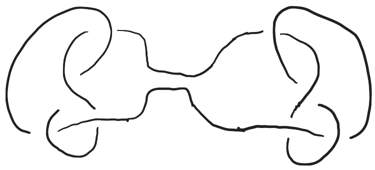
A priori this is just a set. Hold your horses for a minute and we will check this is really a group.

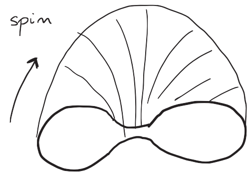
Definition. A knot is called **slice** if $[K] = [O]$ in $\mathcal{C}$

Suppose that K is slice. By capping the unknot component in the embedded cylinder, we obtain a locally flat disc. So

Proposition 1.1 *A knot K is slice if and only if it bounds a locally flat disc in D^4.*

This may seem an awkward condition but it is not hard to build plenty of slice knots actually. In fact if K is any knot then $K\#\overline{K^*}$ is always slice. For place $\overline{K^*}$ in a symmetric way with respect to K and spin the knot forming a surface of revolution in the upper part of D^4 (recall Artin's spin construction Roland explained 2 weeks ago!).

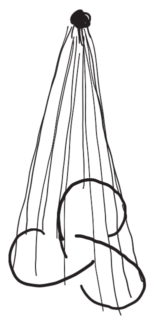




Lemma 1.2 *$\mathcal{C}$ is indeed a group.*

Proof. The connected sum of knots is well-defined in concordance classes by cutting open the embedded cylinders and glueing them together. The inverse of the class of a knot $[K]$ is given by $[\overline{K^*}]$ as $[K]\#[\overline{K^*}] = [K\#\overline{K^*}] = [O]$. $\square$

Remark 1.3 The locally flat assumption for the embedding is essential! Otherwise any knot would be slice. Given any non-trivial knot K, let $C(K) := K \times I / K \times 0 \subset S^3 \times I$ be the cone of K. This is topologically a disc but it is not flat around the cusp: any ball D^4 around the cusp will intersect $C(K)$ in a 2-disc that has K as boundary (!).



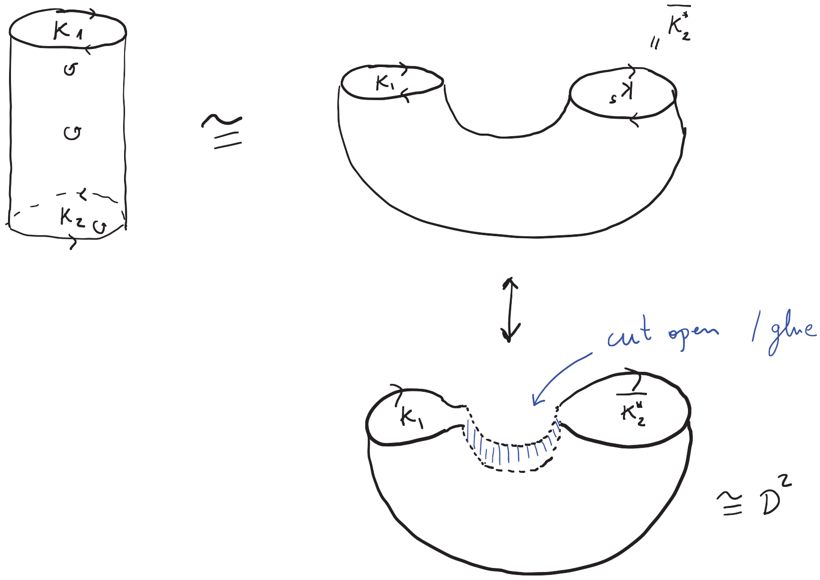
Remark 1.4 You may wonder: why do we care about locally flat discs in $S^3 \times I \subset S^4$ that bound knots rather than in S^3? It turns out that in S^3 the answer is quite boring: the unknot is the only knot that bounds a locally flat disc in S^3. This is the content of the so-called *unknotting lemma*.

There are a few ways to characterise concordant knots, one of them which I particularly like (the why later). Let me put them all together:

Theorem 1.5 *Two knots K_1, K_2 are concordant if and only if any of the following equivalent conditions holds (where "slice" means "bounds a locally flat disc in D^4"):*

1. *K_1 is cobordant to K_2 through a locally flat cylinder,*
2. *$K_1\#\overline{K_2^*}$ is slice,*
3. *There exist slice knots S_1, S_2 such that $K_1\#S_1 = K_2\#S_2$.*

Proof. The equivalence 1 $\iff$ 2 is clear by cutting open/glueing a strip to the cylinder/disc.



(2 ⇒ 3): Call $S := K_1\#\overline{K_2^*}$. Then we have $K_1\#(K_1\#\overline{K_2^*}) = S\#K_2$, and both S and $K_1\#\overline{K_2^*}$ are slice.

(3 ⇒ 2): If there exist slice knots S_1, S_2 such that $K_1\#S_1 = K_2\#S_2$, then by connected summing $\overline{K_1^*}$ we have $K_1\#S_1\#\overline{K_1^*} = K_2\#S_2\#\overline{K_1^*}$, that is $(K_1\#\overline{K_1^*})\#S_1 = K_2\#\overline{K_1^*}\#S_2$. The connected sum of slice knots is slice, and if $A\#B$ is slice and B is slice, so is A (for $A \sim A\#O \sim A\#B \sim O$, where $\sim$ denotes concordance). □

Remark 1.6 I like specially 3 in the theorem. If you have studied algebraic or topological K-theory (here the K does not stand for a knot!), then this should ring a bell. The key argument that makes 3 in the theorem work is that slice knots are closed under connected sum and that for any knot there is another one such that the connected sum is slice.

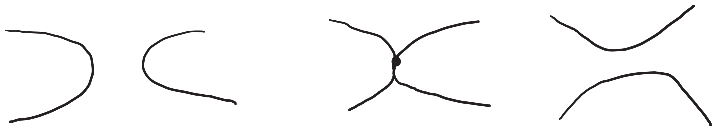
In topological K-theory, one looks at the commutative monoid $(\mathrm{Vect}^\bullet_{\mathbb{C}}(X), \oplus)$ of complex vector bundles over a compact Hausdorff space X, with respect to the direct sum. If $\underline{n} \longrightarrow X$ denotes the n-dimensional trivial vector bundle over X, one defines (reduced) K-theory as the quotient of the monoid $\mathrm{Vect}^\bullet_{\mathbb{C}}(X)$ under the equivalence relation $E_1 \sim E_2$ iff there exist n, m such that $E_1 \oplus \underline{n} \cong E_2 \oplus \underline{m}$. And the same argument works because $\underline{n} \oplus \underline{m} \cong \underline{n+m}$ and for any vector bundle $E \longrightarrow X$ there exists another one E' such that $E \oplus E' \cong \underline{k}$ for some k. Literally the same.

In algebraic K-theory, one considers the commutative monoid $(\mathrm{Proj}(R), \otimes)$ of finitely-generated projective modules over some commutative ring R, with respect to the direct sum. If $F_n := R^n$ denotes the rank-n free R-module, then one defines the reduced K-theory of R as the quotient of $\mathrm{Proj}(R)$ modulo the equivalence relation $P_1 \sim P_2$ iff there exist n, m such that $P_1 \oplus F_n \cong P_2 \oplus F_m$. And once more the same argument works because $F_n \oplus F_m \cong F_{n+m}$ and for any projective module P there is another one Q such that $P \oplus Q \cong F_k$ for some k.

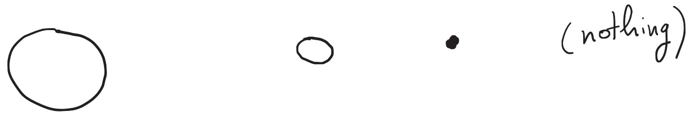
2 Visualising knot concordance

How to see that two knots are concordant? We can view the cylinder embedding as a movie along I, that is through a sequence of maps $K_t : S^1 \hookrightarrow S^3$ interpolating K_0 and K_1. But watch out! The images of these maps may self-intersect in our standard projection $pr_1 : S^3 \times I \longrightarrow S^3$! Luckily these self-intersections only occur in a very mild way: recall from Roland's talk two weeks ago that Morse theory tells us what are the elementary pieces that we can find in such an embedding cylinder in D^4:

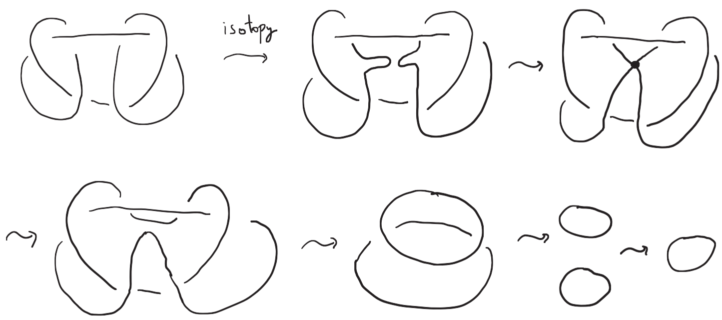
The movie around a saddle point goes like this (locally):



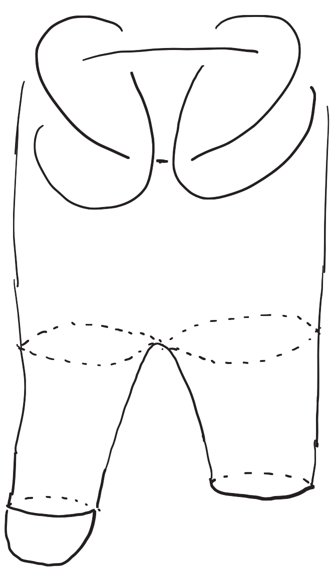
And the movie around a maximum/minimum goes like this (locally):



Example 2.1 The following know is slice, ie it is concordant to the unknot:



A nice schematic of the cylinder is the following:

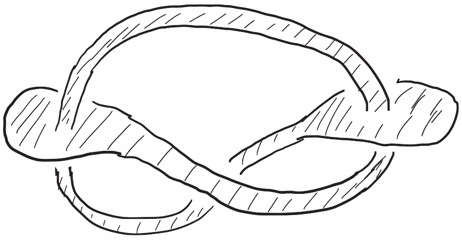


3 Ribbon knots

You may have heard of the slice-ribbon conjecture. Let's see what that is about.

Definition. A knot is **ribbon** if it bounds a disc $D^2 \subset S^3$ only with "ribbon singularities", that is, there is a map $h : D^2 \longrightarrow S^3$ which is an embedding except in a subset which is a disjoint union of arcs $A \subset h(D^2)$ for which $h^{-1}(A) \cong I \amalg I$, one of which is interior to the disc and the other has its boundary in the boundary of the disc.

In this case a picture is worth a thousand words. The knot shown is 6_1.



Proposition 3.1 *Every ribbon knot is slice. More particularly, a ribbon knot is a slice knot without local maxima.*

Proof. (Only the first part). Given the singular disc bounding a ribbon knot, we can push off a neighbourhood of the self-intersections into D^4 so that these disappear. □

The problem whether the converse is true is a long-standing problem.

Conjecture 3.2 (Fox) *Every slice knot is ribbon.*

The conjecture is these days believed to be false, but there are not counterexamples yet.

Exercise 3.3 For any knot K, we have that $K\#\overline{K^*}$ is ribbon (we already knew it was slice!).

4 Knot invariants and knot concordance

So far we've seen $\mathcal{C}$ as some simplification of $\mathcal{K}$. We know that $\mathcal{K}$ is vastly large. What about $\mathcal{C}$? Is it still large or we are modding out by something huge and $\mathcal{C}$ is trivial? I'll address the size question later in the next section.

One important drawback of $\mathcal{C}$ is that many of the invariants that we know and understand are not preserved under knot concordance. Eg $\Delta_O(t) = 1$ but $\Delta_{3_1\#\overline{3_1^*}}(t) = (t^{-1} + 1 + t)^2$. Same happens with the Jones polynomial, the genus, etc. Is there any knot invariant that is preserved under concordance?

Recall. Any nonsingular symmetric matrix with real coefficients diagonalises as diag (+1,..,+1, -1, ...-1) (this is called Sylvester's law of inertia). The signature of the matrix is the integer (# +1 's) $-$ (# -1's). Given a knot K with Seifert matrix V associated to a certain Seifert surface, $V + V^T$ is always a non-singular matrix (its determinant is the independent term of the Alexander polynomial, aka the determinant of the knot). The **signature** of K is defined as the signature of the $V + V^T$, and it is denoted $\sigma(K)$. This is always an even integer, as the Seifert surface has even dimension.

Lemma 4.1 $\sigma(\overline{K^*}) = -\sigma(K)$.

Proof. If the crossings are changed then the linking numbers will get a minus sign, hence the Seifert matrix for $\overline{K^*}$ will be the opposite of the Seifert matrix of K. So every +1 in the diagonal becomes a -1 and the other way around. □

Proposition 4.2 $\sigma(K\#K') = \sigma(K) + \sigma(K')$.

Proof. (Sketch). Consider Seifert surfaces for your knots in normal form. Then the connected sum of knots translates into the connected sum (along the boundary) of the Seifert surfaces. So a Seifert matrix for $K\#K'$ is given by a diagonal block matrix with the Seifert matrices of K and K' in the diagonal. □

For the following I will need the following property of slice knots. The proof requires some homology, so I will backbox this today. It can be found in Lickorish.

Fact. Any slice knot has a block Seifert matrix of the form

$$V = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}.$$

Theorem 4.3 *If K is slice, then $\sigma(K) = 0$.*

Proof. Write $L = A + B^T$ and $N = C + C^T$. Then

$$V + V^T = \begin{pmatrix} 0 & L \\ L^T & N \end{pmatrix}.$$

Since this matrix is nonsingular, so is L. Let

$$P = \begin{pmatrix} L^{-1} & 0 \\ CL^{-1} & -I \end{pmatrix}.$$

A direct computation shows that

$$P(V + V^T)P^T = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$$

and then

$$\sigma(K) = \sigma(V + V^T) = \sigma(P(V + V^T)P^T) = 0.$$

□

Corollary 4.4 *Concordant knots have the same signature. In particular, the signature induces a surjective group homomorphism*

$$\sigma : \mathcal{C} \longrightarrow 2\mathbb{Z}.$$

Corollary 4.5 *If $\sigma(K) \neq 0$, then $[K] \neq 0$ in $\mathcal{C}$ and furthermore it has infinite order.*

Example 4.6 The trefoil 3_1 has a Seifert matrix

$$V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Obviously the eigenvalues are -1 and -1 so $\sigma(3_1) = -2 \neq 0$. Hence $\mathcal{C}$ is non-trivial!

Example 4.7 The figure-of-eight 4_1 has a Seifert matrix

$$V = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Obviously the eigenvalues are 1 and -1 so $\sigma(3_1) = 0$. Hence the signature does not rule out the possibility that 4_1 is slice.

I would like to finish off this part about knot invariants with the Alexander polynomial. It turns out that it captures part of the 4-dimensional properties of slice knots.

Theorem 4.8 (Fox-Milnor) *If K is a slice knot, then Alexander factors as*

$$\Delta_K(t) = f(t)f(t^{-1})$$

for some polynomial $f(t) \in \mathbb{Z}[t]$.

Proof. By the previous fact, K has a Seifert matrix

$$V = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$$

so the (normalised) Alexander polynomial will be

$$\det(t^{1/2}V - t^{-1/2}V^T) = \det\begin{pmatrix} 0 & t^{1/2}A - t^{-1/2}B^T \\ t^{1/2}B - t^{-1/2}A^T & ? \end{pmatrix} = \det(tA - B^T)\det(t^{-1}A - B^T).$$

□

Corollary 4.9 *If K is a slice knot, then its determinant $|\Delta_K(-1)| = n^2$ is a perfect square.*

Example 4.10 We have that $\Delta_{6_1}(t) = -2t + 5 - 2t^{-1} = (-2t+1)(-2t^{-1}+1)$, and likewise $\Delta_{6_1}(-1) = 9 = 3^2$.

Example 4.11 We have that $\Delta_{4_1}(t) = -t + 3 - t^{-1}$, so $\Delta_{4_1}(-1) = 5$ which is not a perfect square. Hence 4_1 is not slice.

Remark 4.12 One important question about $\mathcal{C}$ is to understand its torsion. Eg 4_1 is a torsion element as being nontrivial in $\mathcal{C}$ we have

$$2[4_1] = [4_1\#4_1] = [4_1\#\overline{4_1^*}] = 0$$

as the figure-of-eight knot satisfies $4_1 = \overline{4_1^*}$. This will happen for any negative amphichiral knot (ie $K = \overline{K^*}$). This is the only known torsion in $\mathcal{C}$ (!)

There is also some sufficient condition for a knot to be slice, but I will blackbox this as well:

Theorem 4.13 (Freedman '82) *If a knot has Alexander polynomial 1, then it is slice.*

5 The size of the concordance group

I would like to wrap up coming back to the question about the size of $\mathcal{C}$. So far we know that it is non-trivial and even more, it has infinite many elements. It turns out that it is really large.

Theorem 5.1 (Levine '69) *There exists a surjective group homomorphism*

$$\phi : \mathcal{C} \twoheadrightarrow \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}/4^\infty.$$

Here $\mathbb{Z}^\infty = \oplus_{i=1}^\infty \mathbb{Z}$. So this says $\mathcal{C}$ is infinite-generated as a group. In particular Livingston showed that $\operatorname{Ker}\phi$ has a subgroup isomorphic to $\mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty$.

I will not describe the map ϕ in detail but I will just say that $\phi([K])$ is some class associated to its Seifert matrix.

Remark 5.2 Everything said here has been done in the topological setting, taking locally flat embeddings of discs and cylinders. One may wonder if the story remains the same if one replaces "locally flat" by "smooth". The answer is that it doesn't (!). Smooth implies locally flat but the converse is not true.

One then has two knot concordance groups, let's say $\mathcal{C}_{\text{smooth}}$ and $\mathcal{C}_{\text{top}}$ which are not isomorphic. There is a obvious forgetful (surjective) homomorphism $\mathcal{C}_{\text{smooth}} \twoheadrightarrow \mathcal{C}_{\text{top}}$ that just forgets the smooth structure of the cylinders. In fact Endo showed that the kernel of this morphism is infinite-generated, and Hom that this kernel has one $\mathbb{Z}^\infty$ direct summand, using knot-Floer techniques.

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