Classification of Riemann surfaces

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§1 : Introduction

I would like to adress the following

<u>Question</u> : How many Riemann surfaces are there out there (up to biholomorphism)?

• As a first approximation, we can compare Riemann surfaces with their underlying topological surface via the forgetful functor

$$\text{Riemann surfaces} \longrightarrow \text{Topological surfaces}.$$

We understand topological surfaces rather well, especially the closed (compact, w/o boundary) ones:

<u>Theorem</u> (Brahana, 1922). Any connected closed top surface is homeomorphic to one of the following:

1) The sphere $S^2 (\cong \mathbb{CP}^1)$
2) A connected sum of tori $\Sigma_g = \mathbb{T} \# \overset{g}{\cdots} \# \mathbb{T}$
3) A connected sum of projective planes $M_p = \mathbb{RP}^2 \# \overset{p}{\cdots} \# \mathbb{RP}^2$

(For non-compact top surfaces, the situation is more complicated).

- Let us consider, as analogy, the situation for smooth surfaces: there is an equivalence of categories

$$\begin{matrix}\text{Smooth} \\ \text{surfaces}\end{matrix} \xrightarrow{\ \simeq\ } \begin{matrix}\text{Topological} \\ \text{surfaces}\end{matrix} \qquad \left(\begin{matrix}\text{Radó 20s,} \\ \text{Munkres 60s}\end{matrix}\right)$$

stating that every top surface admits essentially a unique smooth structure. So a reasonable question is:

<u>Q</u>: Does every top surface admits a (unique?) Riemann surface structure?

- The first thing I would like to discuss is that there is an obstruction to admit a complex structure in terms of orientability:

<u>Definition</u>. A top. surface is called <u>non-orientable</u> if it admits a top embedding of a Möbius strip $Mob_2 \hookrightarrow X$,

$$Mob_2 := \frac{D^1 \times D^1}{(x,1) \sim (-x,-1)} .$$

<u>Examples</u>. The surfaces M_p are non-orientable, because $\mathbb{RP}^2 - \mathring{D}^2 \cong Mob_2$.

Lemma (Orientability criterion). Let X be a top surface with its unique smooth structure. Then X is orientable if and only if there is a continuous choice of orientation for T_pX for $p \in X$.

- In the lemma, "continuous choice" means that around any point there is a chart $\mathcal{U} \subset X \xrightarrow{\varphi} \overline{\mathcal{U}} \subset \mathbb{R}^2$ such that $\varphi_{*,p} : T_pX \xrightarrow{\cong} T_{\varphi(p)}\mathbb{R}^2 = \mathbb{R}^2$ is orientation-preserving for all $p \in \mathcal{U}$.

Proposition. Any Riemann surface is orientable.

Pf. Recall from Brown's talk that any Riemann surface has an almost complex structure, ie a real vector bundle map

$$J : TX \to TX \quad , \quad J^2 = -Id$$

(in fact this determines a Riemann surface structure). The almost-complex structure determines a canonical orientation on X with the property that for any $0 \neq v_p \in T_pX$, $(v_p, J_p v_p)$ is a positive basis for T_pX. The fact that J is smooth implies that the choice of orientation is continuous. $\square$

Corollary. None of the surfaces M_p admits a Riemann surface structure.

- What about the rest of closed surfaces, do they admit a (unique?) complex str.? What about non-compact?

For topological surfaces X (or non-pathological spaces in general), the situation is always $X \cong \widetilde{X}/G$ for some space $\widetilde{X}$ and a group G acting on $\widetilde{X}$, which is isomorphic to an algebraic invariant of X, namely its fundamental group.

§2 : Covering theory for Riemann surfaces

Warning. In what follows all top surfaces will be assumed to be path-connected (= connected).

Recall. If X is a top surface and $p \in X$, the fundamental group of X is

$$\pi_1(X,p) := \frac{\{\text{loops at } p\}}{\text{homotopy rel } \{0,1\}}$$

which is a group wrt concatenation of loops (not abelian in general). This is independent of the choice of basepoint (assuming X path-connected).

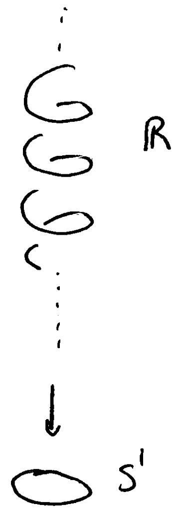
Eg. $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(\mathbb{T}) = \mathbb{Z} \oplus \mathbb{Z}$, $\pi_1(S^2) = 0$, $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$, ...
$[t \mapsto e^{2\pi i n t}] \leftrightarrow n$

- A (path-connected) surface X with $\pi_1(X) = 0$ is called simply-connected.

Definition. A covering map is a continuous map $p: Y \to X$ st every point $p \in X$ has a neighbourhood U st $p^{-1}(U) \cong \coprod_{i \in I} U_i$, $U_i \xrightarrow[\cong]{p} U$.

Eg:

$t \mapsto e^{2\pi i t}$



,

$$\mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$z \longmapsto z^n$$

$$\mathbb{C} \longrightarrow \mathbb{C}^*$$
$$z \longmapsto e^z$$

Definition. For a top surface X, write $\mathrm{Aut}(X) = \{\text{self-homeomorphisms } X \xrightarrow{\cong} X\}$.

An action $\mathcal{G} \curvearrowright X$ of a gp $\mathcal{G} \subseteq \mathrm{Aut}\, X$ is properly discontinuous if any point has a neighbourhood U s.t

$$g \neq g' \implies g(U) \cap g'(U) = \emptyset, \quad \forall g, g' \in \mathcal{G}.$$

Lemma: If $\mathcal{G} \curvearrowright X$ prop. discont then the projection $X \to X/\mathcal{G}$ is a covering map.

Definition. Let $p: Y \to X$ be a covering map. The group of deck transformations of the covering, $\mathrm{Aut}_X Y$, is the gp of self-homeo $Y \xrightarrow{\cong} Y$ covering p.

$$\begin{tikzcd} Y \arrow[rr, "\cong"] \arrow[dr, "p"'] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Fact: Every topological surface X has a unique (up to homeomorphism) simply-connected cover $\widetilde{X}$, which is called its universal cover. There is an isomorphism

$$\mathcal{G}(p) := \mathrm{Aut}_X \widetilde{X} \cong \pi_1(X)$$

and a homeomorphism

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "prj"'] \arrow[dr, "p"] & \\ \tilde{X}/\mathcal{G}(p) \arrow[rr, "\cong"] & & X \end{tikzcd}$$

• This is the general situation for top surfaces. Now we want to especialise to Riemann surfaces.

Remark. To avoid handling atlases, I will be using the following sheaf description of a Riemann surface: a topological surface X w/ a sheaf of functions $\mathcal{O}_X$ (this means that $\mathcal{O}_X(U) \subset \mathcal{C}(U) = \{$cont maps $U \to \mathbb{C}\}$ a $\mathbb{C}$-subalgebra, and

- $V \subset U \subset X$ and $f \in \mathcal{O}_X(U) \to f_{|V} \in \mathcal{O}_X(V)$
- $U = \bigcup_i U_i$ and $f_{|U_i} \in \mathcal{O}_X(U_i) \Rightarrow f \in \mathcal{O}_X(U)$)

such that every pt has a nbhd U st

$$(U, \mathcal{O}_U) \cong (\mathbb{C}, \mathcal{O}_{\mathbb{C}}^{hol}) \quad \text{as ringed spaces}.$$

Proposition. Let $\pi: Y \to X$ be a covering map where X is a Riemann surface. Then there exists a unique Riemann str. on Y st π is a local biholomorphism.

Pf. Existence: Define a (pre)sheaf on Y as the pullback of that of X:

$$\mathcal{O}_Y := \pi^* \mathcal{O}_X, \qquad \mathcal{O}_Y(V) := \mathcal{O}_X(\pi(V))$$

where we are taking into account that $\pi(V)$ is open as π is an open map (because any covering map is a local homeomorphism). The sheaf condition is easily verified.

Now take $q \in Y$ and let $p := \pi(q)$. Let $\mathcal{U}$ be an evenly covered open nbhd of p and take $\tilde{\mathcal{U}}$ the component of $\pi^{-1}(\mathcal{U})$ containing q. Then we have $\pi : \tilde{\mathcal{U}} \xrightarrow{\cong} \mathcal{U}$ and hence

$$(\tilde{\mathcal{U}}, \mathcal{O}_{\tilde{\mathcal{U}}}) \xrightarrow[\cong]{\pi} (\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \cong (\mathbb{C}, \mathcal{O}_{\mathbb{C}}^{hol})$$

<u>Uniqueness</u>: Suppose that there are two Riemann structures $(Y, \mathcal{O}_Y^1), (Y, \mathcal{O}_Y^2)$ such that $\pi : (Y, \mathcal{O}_Y^i) \to (X, \mathcal{O}_X)$ is a local biholomorphism. Then we have $\mathcal{O}_Y^1 = \mathcal{O}_Y^2$, ie,

$$\mathcal{O}_Y^1(V) = \mathcal{O}_Y^2(V) \subset \mathcal{C}(V) \qquad \forall V \text{ open in } Y.$$

For such $V \subset Y$, let $\pi(V) = \bigcup \mathcal{U}_i$ with $\mathcal{U}_i$ evenly covered. Let $\tilde{\mathcal{U}}_i$ be one lift of $\pi^{-1}(\mathcal{U}_i)$ in V, so that $V = \bigcup \tilde{\mathcal{U}}_i$ with $\pi : \tilde{\mathcal{U}}_i \xrightarrow{\cong} \mathcal{U}_i$. Then

$$\mathcal{O}_Y^1(\tilde{\mathcal{U}}_i) \xrightarrow[\cong]{\pi} \mathcal{O}_X(\mathcal{U}) \xleftarrow[\cong]{\pi} \mathcal{O}_Y^2(\tilde{\mathcal{U}}_i)$$

so $\mathcal{O}_Y^1(\tilde{\mathcal{U}}_i) = \mathcal{O}_Y^2(\tilde{\mathcal{U}})$. We conclude by the sheaf condition. $\square$

Corollary. If X is a Riemann surface, so is its universal cover $\tilde{X}$.

Proposition. Let X be a Riemann surface and let $G \subset \mathrm{Aut}_{hol}(X)$ be a group acting properly discontinuously through biholomorphisms. Then there exists a unique Riemann surface structure on X/G such that $X \to X/G$ is a local biholomorphism (and a covering map).

Pf Similar to the previous one; now use $\mathcal{O}_{X/G}(U) := \mathcal{O}_X(\pi^{-1}(U))$. □

• The previous results show that any Riemann surface is $\tilde{X}/G$ where $\tilde{X}$ is a simply-connected Riemann surface and $G \subset \mathrm{Aut}_{hol}(\tilde{X})$ acts prop. disc. So roughly we need to answer two questions to have a classification.

a) What Riemann surfaces $\tilde{X}$ are simply-connected,

b) How many subgroups $G \subset \mathrm{Aut}_{hol} \tilde{X}$ acts prop disc.

• The answer to the first question is answered by

Theorem (Uniformisation, Poincaré - Koebe 1907). Any simply-connected Riemann surface is biholomorphic to exactly one of the following:

1) $\mathbb{CP}^1 \cong \mathbb{C}_\infty$

2) $\mathbb{C}$

3) $\mathring{\mathbb{D}}^2 \cong \mathbb{H}$

<u>Remark</u>. The plane and the open disc are homeomorphic but not biholomorphic, because any $\overset{hol}{\text{map}}$ $\mathbb{C} \to \mathring{D}^2 \hookrightarrow \mathbb{C}$ would be bounded and hence constant by the Liouville theorem.

• We will not prove this here, as none of the existing proofs is elemental and showing this would require one entire lecture. For the curious reader, a readable sketch of the proof in terms of Green functions can be found in $\underline{\text{XVI}}$.6 of "Complex Analysis" by T.W. Gamelin.

• The uniformisation theorem leads to the following

<u>Theorem</u> (Classification of Riemann surfaces): Any connected Riemann surface X is biholomorphic to $M/\mathcal{G}$ where $M = \mathbb{C}_\infty$, $\mathbb{C}$ or $\mathbb{H}$ and $\mathcal{G} \subset \mathrm{Aut}_{hol} M$ acting prop. disc. Furthermore $\pi_1(X) \cong \mathcal{G}$.

Besides, two groups $\mathcal{G}, \mathcal{G}' \subset \mathrm{Aut}_{hol}(\widetilde{X})$ acting prop disc. define biholomorphic Riemann surfaces if and only if they are conjugated in $\mathrm{Aut}_{hol} \widetilde{X}$.

<u>Pf</u> Let $\widetilde{X}$ be the universal cover of X. By the corollary, $\widetilde{X}$ is a simply-connected Riemann surface, so by the uniformisation theorem $\widetilde{X} = \mathbb{C}_\infty$, $\mathbb{C}$ or $\mathbb{H}$. If

$$\mathcal{G} := \mathrm{Aut}_{hol\, X} \widetilde{X} \subset \mathrm{Aut}_{hol} \widetilde{X} \qquad (\mathcal{G} \cong \pi_1(X))$$

then by the proposition $\widetilde{X}/\mathcal{G}$ is a Riemann surface and by the fact

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "proj"'] \arrow[dr, "\pi"] & \\ \widetilde{X}/G \arrow[rr, "\cong"] & & X \end{tikzcd}$$

where the homeomorphism is actually a biholomorphism by the uniqueness of the Riemann structure on $\widetilde{X}/G$. $\square$

- The upshot is that we can divide Riemann surfaces into three classes, depending on its universal cover:

ELLIPTIC	PARABOLIC	HYPERBOLIC
$\mathbb{C}_\infty$,	$\mathbb{C}$,	$\mathbb{H}$.

- So, according to the classification theorem, the study of Riemann surfaces amounts to the study of the prop disc subgroups of $\mathrm{Aut}_{hol}(M)$ (up to conjugacy) for $M = \mathbb{C}_\infty$, $\mathbb{C}$ or $\mathbb{H}$.

§ 3. ELLIPTIC CASE

Lemma. The biholomorphisms $\mathbb{C}_\infty \to \mathbb{C}_\infty$ are precisely the Möbius transformations

$$f(z) = \frac{az+b}{cz+d} \quad , \quad ad-cb \neq 0.$$

(that is, the biholomorphisms $\mathbb{CP}^1 \to \mathbb{CP}^1$ are precisely the homographies). That is,

$$\mathrm{Aut}_{hol}(\mathbb{C}_\infty) \cong \underset{\substack{\uparrow \\ \text{projective general linear gp}}}{PGL_2(\mathbb{C})} := GL_2(\mathbb{C}) / \mathbb{C}^* .$$

Pf. Recall that $\mathbb{CP}^1 \cong \mathbb{C}_\infty$, $[z:w] \leftrightarrow z/w$. Then we have the following diagram, where we already saw the first row:

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{ratios } \frac{p(z,w)}{q(z,w)} \\ \text{of hom. poly. of} \\ \text{same degree}\end{array}\right\} \arrow[r, equal] \arrow[d, "\cong"'] & \left\{\begin{array}{c}\text{meromorphic} \\ \text{functions on} \\ \mathbb{CP}^1\end{array}\right\} \arrow[r, equal] \arrow[d, "\cong"'] & \left\{\begin{array}{c}\text{holomorphic} \\ \text{maps} \\ \mathbb{CP}^1 \to \mathbb{C}_\infty \\ \text{not id } \infty\end{array}\right\} \arrow[d, "\cong"'] \\
\left\{\begin{array}{c}\text{ratios } p(z)/q(z) \\ \text{of polynomials}\end{array}\right\} \arrow[r, equal] & \left\{\begin{array}{c}\text{meromorphic} \\ \text{functions on} \\ \mathbb{C}_\infty\end{array}\right\} \arrow[r, equal] & \left\{\begin{array}{c}\text{holomorphic} \\ \text{maps } \mathbb{C}_\infty \to \mathbb{C}_\infty \\ \text{not id } \infty\end{array}\right\}
\end{tikzcd}$$

$$p(z)/q(z) \longmapsto f(z) = \begin{cases} p(z)/q(z) & \text{if } z \text{ not a pole} \\ \infty & \text{if } z \text{ a pole} \end{cases}.$$

So we are left to see which of these are bijections: we claim that $\deg p \leq 1$: for if p has different roots, it is not injective by the fundamental thm of algebra; and if it has a root of order say n, then the local mapping theorem would say that locally it is n-to-one, so it couldn't be injective either. The same applies to q arguing w/ $\frac{1}{f} = q(z)/p(z)$. So $f(z) = \dfrac{az+b}{cz+d}$, and to be a bijection num and den cannot be proportional, ie $ad - cb \neq 0$, as otherwise f would be constant.

Fact. Any Möbius transformation $\neq \mathrm{Id}$ has exactly one or two fixed points.

- A group that contains an elemt $\neq \mathrm{Id}$ w/ fixed point cannot act prop. disc, so

Corollary. $\mathbb{C}_\infty$ is the only elliptic Riemann surface.

§4. PARABOLLIC CASE

Lemma. The biholomorphisms $\mathbb{C} \to \mathbb{C}$ are precisely the affine transformations

$$f(z) = az + b \quad , \quad 0 \neq a, b \in \mathbb{C}$$

That is,

$$\mathrm{Aut}_{hol}\, \mathbb{C} \cong \mathrm{Par}_2 := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : 0 \neq a, b \in \mathbb{C} \right\}.$$

"parabolic group".

Pf. Suppose $f: \mathbb{C} \to \mathbb{C}$ biholomph. Then it extends to a biholomorphm $\hat{f}: \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined as $\hat{f}(\infty) := \infty$ and $\hat{f}(z) = f(z)$ for $z \neq \infty$. By the elliptic case, $\hat{f}(z) = \dfrac{az+b}{cz+d}$. But then $c = 0$ (if $c \neq 0$ then we would have $\hat{f}(-d/c) = \infty$ and $\hat{f}(\infty) = a/c$). So $f(z) = \frac{a}{d} z + \frac{b}{d}$. $\square$

- We now have to look for the subgroups of $\mathrm{Aut}_{hol}\, \mathbb{C}$ whose action on $\mathbb{C}$ is prop disc. The answer is given by the following

Proposition. Any subgroup $\mathcal{G} \subset \mathrm{Aut}_{hol}(\mathbb{C})$ of properly discontinuous biholomorphisms is of one of the following types:

1) $\mathcal{G} = 0$,

2) $\mathcal{G} \cong \mathbb{Z}$ generated by a translation w.r.t a non zero vector

3) $\mathcal{G} \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by two translations w.r.t two $\mathbb{R}$-lin. indep. vectors.

Pf. Any affine transformation $f(z) = az + b$ w/ $a \neq 1$ has fixed points (namely $z = \frac{b}{1-a}$), so for $\mathcal{G}$ to act prop disc. it must be formed by translations, $f(z) = z + b$ (and in particular $\mathcal{G}$ is abelian). Representing each translation by its associated vector in $\mathbb{C}$, we can view $\mathcal{G}$ as a discrete subgroup of $\mathbb{C}$ (discrete as otherwise it wouldn't act prop disc). We conclude by the following

Claim: Every discrete additive subgroup of $\mathbb{R}^n$ is of the form

$$L = \mathbb{Z} v_1 + \cdots + \mathbb{Z} v_r$$

for some l.i. vectors $v_1, \dots, v_r$; $r \leq n$.

Pf of the claim: We can assume that the $\mathbb{R}$-lin subspace generated by L in $\mathbb{R}^n$ is n-dim (otherwise $L \subset \mathbb{R}^r$ for $r < n$ and the claim would follow for $n = r$).

Let $e_1 \dots e_n$ be n l.i. vectors in $\mathbb{R}^n$ and let $L' \subset L$ be the free ab gp generated by the e_i's. Now L is discrete and closed in $\mathbb{R}^n$, so $L/L' \subset \mathbb{R}^n/L'$

is discrete and closed as well, so finite as

$$\mathbb{R}^n/L' = \prod \mathbb{R}e_i/\mathbb{Z}e_i = S^1 \times \cdots \times S^1$$

is compact, ie L/L' is a finite group. Besides L is finite-generated as so is L' and L/L'; and L is also torsion-free as it is a subgp of $\mathbb{R}^n$. Therefore L is free and finite-generated, and since L/L' is a torsion group (because it is finite), rank L = rank L'.

□

- The upshot is that, according to the proposition, we have the following parabollic Riemann surfaces:

1) The plane $\mathbb{C} = \mathbb{C}/0$,

2) The cylinders $\mathbb{C}/\mathbb{Z}\alpha$, $\alpha \in \mathbb{C}^*$. The key observation here is that two subgroups $\mathbb{Z}\alpha, \mathbb{Z}\beta \subset \mathbb{C}$ are conjugated in $\mathrm{Aut}_{hol}\,\mathbb{C}$ by a dilatation $z \mapsto \frac{\alpha}{\beta} z$, and therefore all cylinders are biholomorphic to each other ("any cylinder has a unique complex structure").

By the way, a cylinder is biholomorphic to the punctured plane $\mathbb{C}^* = \mathbb{C} - 0$.

3) The tori $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, α, β $\mathbb{R}$-l.i.

The first thing we note is that any such torus is conjugated to one of the form $\mathbb{Z} \oplus \mathbb{Z}\tau$ w/ $\mathrm{Im}\,\tau > 0$. For if $\mathrm{Im}\left(\frac{\beta}{\alpha}\right) > 0$ then take $\tau := \beta/\alpha$ and conjugate by the dilatation $z \mapsto \alpha z$. Else take $\tau := -\beta/\alpha$ and conjugate w/ the dilatation $z \mapsto -\alpha z$. That is, every complex torus is biholomorphic to $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$ w/ $\tau\in\mathbb{H}$. Now the question we want to answer is when two complex tori are biholomorphic, ie when two such subgroups $\mathbb{Z}\oplus\mathbb{Z}\tau$, $\mathbb{Z}\oplus\mathbb{Z}\tau'$ are conjugated in $\mathrm{Aut}_{hol}(\mathbb{C})$.

- Recall that $SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) : \det = +1 \right\}$, and note that $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ as follows: for $A = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$A : \mathbb{H} \longrightarrow \mathbb{H}$$
$$\tau \longmapsto \frac{n_1\tau + n_2}{n_3\tau + n_4}$$

which is well-def as $\mathrm{Im}(A\tau) = \dfrac{(\det A)\cdot \mathrm{Im}\,\tau}{|n_3\tau + n_4|^2} > 0$.

<u>Theorem</u>: Two complex tori $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$, $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau'$ are biholomorphic if and only if there exists $A\in SL_2(\mathbb{Z})$ st $\tau' = A\tau$.

<u>Pf</u>. It is easy to see that two subgroups $\mathbb{Z}\oplus\mathbb{Z}\tau$, $\mathbb{Z}\oplus\mathbb{Z}\tau'$ are conjugated (in $\mathrm{Aut}_{hol}(\mathbb{C})$) if and only if $\exists\,\lambda\in\mathbb{C}^*$ st

$$\mathbb{Z}\oplus\mathbb{Z}\tau = \lambda\cdot(\mathbb{Z}\oplus\mathbb{Z}\tau') \subset \mathbb{C}.$$

ie $\mathbb{Z}\oplus\mathbb{Z}\tau = \mathbb{Z}\lambda\oplus\mathbb{Z}\lambda\tau'$. This means that $\lambda, \lambda\tau'$ can be expressed in the $\mathbb{Z}$-basis $(1,\tau)$, ie there exist $m_i \in \mathbb{Z}$, $i=1,\dots 4$ st

$$\left.\begin{array}{l} \lambda = m_4 + m_3 \tau \\ \lambda \tau' = m_2 + m_1 \tau \end{array}\right\}$$

ie τ' must be of the form $\tau' = \dfrac{m_1 \tau + m_2}{m_3 \tau + m_4}$. Furthermore $(\lambda, \lambda\tau')$ is a $\mathbb{Z}$-basis of $\mathbb{Z} \oplus \mathbb{Z}\tau$, so they must be l.i, ie

$$\det \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \pm 1 \in \mathbb{Z}^*$$

and in fact we must have $\det(\ ) = +1$ as $\tau' \in \mathbb{H}$. So $\tau' = A\tau$ for $A \in SL_2(\mathbb{Z})$ as required. $\square$

Corollary. The set of biholomorphism classes of complex tori is in bijection with

$$\mathbb{H}/SL_2(\mathbb{Z}) \quad \left(\underset{j\text{-function}}{\cong} \mathbb{C} \right)$$

and it is denoted $\mathcal{M}_1$ the moduli space of the torus.

- What we have here is a totally different structure to top surfaces. Now we have a continuum of non-biholomorphic complex structures on the torus, parametrised by $\mathcal{M}_1$.

§5. Hyperbolic case

- Essentially we have already seen that a Riemann surface is hyperbolic if and only if it is not biholomorphic to $\mathbb{C}_\infty$, $\mathbb{C}$, $\mathbb{C}^*$ or a torus. We briefly describe the situation now:

<u>Proposition</u>. The biholomorphisms $\mathbb{H} \to \mathbb{H}$ are precisely the linear fractional transformations

$$f(z) = \frac{az+b}{cz+d} \quad , \quad a,b,c,d \in \mathbb{R}, \; ad-bc = 1 .$$

That is,

$$\mathrm{Aut}_{hol}(\mathbb{H}) \cong PSL_2(\mathbb{R}) := SL_2(\mathbb{R}) / \pm I$$

↑ projective special linear group .

<u>Definition</u>. A subgroup $G \subset \mathrm{Aut}_{hol}(\mathbb{H})$ whose action is properly discontinuous is called a <u>Fuchsian group</u>. In the hyperbolic case, this is equivalent to requiring that G is a discrete subgroup of $PSL_2(\mathbb{R})$

- In this case giving a list is difficult for two reasons: one is that there are many conj classes of Fuchsian groups, and the other is the difficulty to express the underlying top surface if it is non-compact. Hence in this case we will content ourselves giving a finite list of examples:

Before we show

Theorem (Little Picard). Every holomorphic function $\mathbb{C} \to \mathbb{C}$ non constant omits at most one value.

Pf. If such function omits two, say $f: \mathbb{C} \to \mathbb{C} - \{p,q\}$, then as $\mathbb{C}-\{p,q\}$ is hyperbolic f lifts to its universal cover $\mathring{D}^2$, $\hat{f}: \mathbb{C} \to \mathring{D}^2$ holomorphic, which must be constant by the Liouville thm, so f is constant. $\square$

Examples. 1) Every closed surface of genus $g \geq 2$ is hyperbolic. In particular

$$\Sigma_g \cong \mathbb{H}/G \quad \text{where} \quad G \cong \pi_1(\Sigma_g) \cong \langle a_1, b_1, \dots, a_g b_g \mid [a_1,b_1]\cdots[a_g,b_g] = 1 \rangle.$$

2) $\mathring{D}^2 - 0$

3) More generally, if $X \subset \mathbb{C}$ is a domain (non-empty connected open subset) such that $\mathbb{C} - X$ has more than 2 pts, then X is hyperbolic: for it cannot be elliptic as it is non-compact; and it cannot be parabolic because otherwise its universal cover $\mathbb{C} \to X$ would contradict the Little Picard theorem.

4) Put $A(r,R) := \{ z \in \mathbb{C} : r < |z| < R \}$ for the annuli.

Two annuli $A(r,R)$, $A(r',R')$ are biholomorphic if and only if $R/r = R'/r'$.

§ 6. Teichmüller spaces and the moduli spaces $\mathcal{M}_g$

• For the case of the torus, we saw that the space parametrising bihol. classes of complex str. was $\mathcal{M}_1 := \mathbb{H}/SL_2(\mathbb{Z})$. I would like to generalise this to higher genus (for closed Riemann surfaces).

Definition. The Teichmüller space is the set (actually top space)

$$\mathcal{T}_g := \frac{\{\text{pairs } (X,\phi) : X \text{ Riemann surf } \& \ \phi : \Sigma_g \xrightarrow{\cong} X \text{ orient. pres. homeo}\}}{(X,\phi) \sim (X',\phi') \text{ if } \exists F : X \xrightarrow{\cong} X' \text{ biholo st } \phi \circ \phi'^{-1} \text{ homotopic to } F.}$$

• For the torus, there is a bijection

$$\mathbb{H}_1 \xrightarrow{\cong} \mathcal{T}_1 \qquad \text{w/ marking} \qquad \frac{\mathbb{R}\oplus\mathbb{R}}{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\cong} \frac{\mathbb{C}}{\mathbb{Z}\oplus\mathbb{Z}\tau}$$

$$\tau \longmapsto \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau \qquad\qquad (x,y) \longmapsto x + y\cdot\tau .$$

Definition. Let $g \geq 1$. The moduli space of the closed genus g surface is the set $\mathcal{M}_g$ of biholomorphism classes of Riemann surf str on Σ_g.

Proposition. There is a natural map

$$\mathcal{T}_g \longrightarrow \mathcal{M}_g$$

forgetting the marking inducing a bijection

$$\mathcal{T}_g / \mathrm{Mod}(\Sigma_g) \xrightarrow{\cong} \mathcal{M}_g$$

where $\mathrm{Mod}(\Sigma_g) := \pi_0 \mathrm{Homeo}^+(\Sigma_g)$ is the <u>mapping class group</u> of Σ_g

- For the torus case,

$$\mathrm{Mod}(\Sigma_1) \cong SL_2(\mathbb{Z})$$

$$f \longmapsto \big(f_* : \pi_1(\Sigma_1) = \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z} = \pi_1(\Sigma_1) \big)$$

- In general, the situation is that $\mathcal{T}_g \cong \mathbb{R}^{6g-6}$ (Riemann was aware that the different complex structures on Σ_g depended on $6g-6$ parameters), and $\mathcal{M}_g \cong \mathbb{R}^{6g-6}/\mathrm{Mod}(\Sigma_g)$ but this space is not a manifold (rather an orbifold from the geometric perspective). From the algebraic perspective, the moduli space $\mathcal{M}_g$ itself is an algebraic object called an "algebraic variety".

§ 7. Connection with Riemannian geometry

There is a similar, closely related story for Riemannian structures on a smooth surface. Namely, each of the simply-connected (Riemann) surfaces $\mathbb{C}_\infty = S^2$, $\mathbb{C}$ and $\mathbb{H}$ admits a Riemannian metric w/ constant Gaussian curvature equals to $+1$, 0 and -1, respectively. The groups of automorphisms acting prop disc. happen to act via isometries, so that the quotients M/G still carry a Riemannian metric w/ the same curvature. Therefore we obtain the following classification of Riemannian 2-manifolds:

ELLIPTIC	PARABOLIC	HYPERBOLIC
curvature $+1$	curvature $= 0$	curvature -1

- We see how the theory of complex analysis, algebraic topology and Riemannian geometry merge into this topic. Beautiful!