Principal bundles from a topologist's point of view

(Rough notes - Use at your own risk!)

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I would like to give a topological point of view to the theory of principal bundles. These are important structures used in Hamiltonian mechanics (eg Marsden - Weinsten - Meyer reduction), physics (eg Yang - Mills theory), differential geometry, etc. Excellent sources where to find a detailed exposition from this point of view are [Swi17, ch 11] and [tD08, ch 14] .

Warning. I am a topologist, so I do not know the derivative yet! So I will not be talking about connections, curvature, etc.

1 Classifying spaces

I will start from the beginning: given a topological group G, a *principal G-bundle* is the data of a continuous map $p : E \longrightarrow X$, where E is a right G-space and p is G-invariant, satisfying the local triviality condition that any point of X has an open neighbourhood U such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\cong} & U \times G \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array}$$

where the horizontal homeomorphism is G-equivariant (G acts by right multiplication on the second factor of $U \times G$).

Examples 1.1

1. The trivial bundle $X \times G \longrightarrow X$ is a G-bundle for any G.
2. The Hopf fibration $S^3 \longrightarrow S^2$ is a principal S^1-bundle. This map can be described as $S^3 \subseteq \mathbb{C}^2 \longrightarrow S^2 \cong \mathbb{CP}^1, (z,w) \mapsto [z/w]$.
3. Real vector bundles of rank n are in one-to-one correspondence with principal $GL(n,\mathbb{R})$-bundles (this is essentially because vector bundles are determined by their transition functions, which are maps with values in $GL(n,\mathbb{R})$). For the complex case just replace $\mathbb{R}$ by $\mathbb{C}$.
4. If G is discrete and E is connected, then a principal G-bundle is the same thing as a regular covering map with group of deck transformations G.

Remark 1.2 For a principal G-bundle $E \longrightarrow X$, every fibre is homeomorphic (as a G-space) to G. This is a situation similar to vector bundles where every fibre is isomorphic (as a vector space) to $\mathbb{R}^n$.

Given principal G bundles $p_1 : E_1 \longrightarrow X$, $p_2 : E_2 \longrightarrow X$, a *bundle morphism* is a G-map $f : E_1 \longrightarrow E_2$ such that $p_1 = p_2 \circ f$. It is easily shown that this is enough for f to be an homeomorphism (check it!).

Construction 1.3 One of the operations that one cares the most when considering bundles is the pullback: if $f : B \longrightarrow X$ is a continuous map and $p : E \longrightarrow X$ is a principal G-bundle, we can consider the pullback diagram

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ B & \xrightarrow{f} & X \end{array}$$

Explicitly,

$$f^*E = B \times_X E = \{(b,e) \in B \times E : f(b) = p(e)\} \subseteq B \times E,$$

where the two new maps are the projections. Furthermore G acts by $(b,e)g := (b,eg)$. This turns f^*E into a principal G-bundle over B.

Denote by $\mathrm{Prin}_G(X)$ the set of isomorphism classes of principal G-bundles over X. The pullback defines a functor

$$\begin{aligned} F_G\colon \mathsf{Top}^{op} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \mathrm{Prin}_G(X) \end{aligned}$$

In the following it will be convenient to restrict ourselves to a reasonable subclass of base spaces that will allow us to skip some technicalities, namely to CW complexes[1] (this includes any compact smooth manifold and in general most of the spaces you care!)

Warning. For the rest of these notes all base spaces will be assumed to be (or at least to have the homotopy type of) CW complexes.

So consider instead

$$F_G\colon \mathsf{CW}^{op} \longrightarrow \mathsf{Set}.$$

Proposition 1.4 *Let $p : E \longrightarrow X$ be a principal G-bundle and let $f_0, f_1 : B \longrightarrow X$ be homotopic maps. Then the pullbacks f_0^*E and f_1^*E are isomorphic.*

Proof. Let $H : B \times I \longrightarrow X$ be the homotopy such that $H_0 = f_0$, $H_1 = f_1$. The homotopy lifting property for fibre bundles implies that $H^*E \cong f_0^*E \times I$, which in turn means that

$$f_1^*E = (H \circ i_1)^*E \cong i_1^*H^*E \cong i_1^*(f_0^*E \times I) = (f_0^*E \times I)_{|B\times\{1\}} \cong f_0^*E$$

where $i_1 : B \longrightarrow B \times I$ is the inclusion at 1. □

The upshot is that we get a functor

$$F_G\colon h\mathsf{CW}^{op} \longrightarrow \mathsf{Set}$$

from the (naive) homotopy category of CW complexes. A remarkable result due to Milnor is that this functor is representable[2]:

Theorem 1.5 (Milnor) *F_G is representable, that is, there exists a space BG such that*

$$[X, BG] \xrightarrow{\cong} \mathrm{Prin}_G(X)$$

for any CW complex X (and the isomorphism is natural on X).

In the above statement the brackets indicate the set of homotopy classes of maps $X \longrightarrow BG$. By the Yoneda lemma, this space BG is unique up to homotopy equivalence, and it is called the *classifying space* for G. The Yoneda lemma also says that such a natural isomorphism is induced by pulling back a principal bundle $EG \longrightarrow BG$ along the maps $X \longrightarrow BG$. One can show that EG is unique up to homotopy equivalence, and it is called the *universal bundle*.

Corollary 1.6 *Any principal bundle over a contractible space is trivial.*

A sensible question that I would like to address is: how do BG and EG look like? There are several answers to this depending on the level of sophistication that the audience is willing to hear. I will outline a pretty good one and not too sophisticated:

[1] Otherwise restrict to paracompact Haussdorff spaces on the base, or to numerable bundles.

[2] Alternatively one could show that F_G satisfies the hypothesis of the Brown representability theorem, which automatically yields the result.

Construction 1.7 (Milnor) Given a topological group G, the n-th fold join G^{*n} is the space of formal linear combinations $\sum_{i=1}^n t_i g_i$ where $0 \le t_i \le 1$ and $\sum_{i=1}^n t_i = 1$. This is topologised as

$$G^{*n} = \{((t_i, g_i)) \in \prod_{i=1}^n I \times G : 0 \le t_i \le 1, \sum_{i=1}^n t_i = 1\}/\sim$$

where the equivalence relation identifies $0g_i + \sum_{j\neq i} t_j g_j$ with $0g_i' + \sum_{j\neq i} t_j g_j$ for any $g_i, g_i' \in G$.

There is an obvious map $G^{*n} \longrightarrow G^{*(n+1)}$ given by $\sum_{i=1}^n t_i g_i \mapsto \sum_{i=1}^n t_i g_i + 0g_{i+1}$ for any $g_{i+1} \in G$. Then the universal bundle is defined by

$$EG := \mathrm{colim}_n\, G^{*n}.$$

This space comes with an obvious right action given by right multiplication on the g_i's. We let

$$BG := EG/G$$

or alternatively

$$BG := \mathrm{colim}_n\, G^{*n}/G.$$

It is not hard to check that the projection map $p : EG \longrightarrow BG$ is a principal G-bundle, trivialised by the cover $U_i := p(\{\sum t_j g_j : t_i \neq 0\})$. Furthermore EG is contractible by an argument similar to the one that shows that S^∞ is contractible. It turns out that this is everything you need for a principal G-bundle to be the universal one:

Proposition 1.8 *A principal G-bundle $E \longrightarrow X$ is universal if and only if E is contractible.*

Examples 1.9
1. If $G = \mathbb{Z}/2 = S^0$, then $(S^0)^{*n} \cong S^{n-1}$ (in general $S^n * S^m \cong S^{n+m+1}$). Then $EG = S^\infty$ and $BG = S^\infty/(\mathbb{Z}/2) = \mathbb{RP}^\infty$.
2. If $G = U(1) = S^1$, then $(S^1)^{*n} \cong S^{2n-1}$. Then $EG = S^\infty$ and $BG = \mathbb{CP}^\infty$.
3. If $G = SU(2) = S^3$, then $(S^3)^{*n} \cong S^{4n-1}$. Then $EG = S^\infty$ and $BG = \mathbb{HP}^\infty$.

The first of the previous examples is an interesting space in homotopy theory: $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2, 1)$. Recall that for a group G and $n \ge 1$, a path-connected space X is called an *Eilenberg-Maclane space* of type (G, n) or a $K(G, n)$ if

$$\pi_k(X) = [S^k, X]_* = \begin{cases} G, & k = n \\ 0, & k \neq n \end{cases}.$$

Any CW complex satisfying the above condition is unique up to homotopy equivalence.

Theorem 1.10 *Let G be discrete (eg finite). Then a $K(G, 1)$ is a classifying space for G.*

Proof. Let X be a $K(G, 1)$ and let $\widetilde{X}$ be its universal cover. This means that $\widetilde{X}$ is simply connected and that has homotopy groups isomorphic to the ones of X for $k > 1$, which means that X is weakly contractible, hence contractible since it is a CW complex. We conclude by 1.8. □

Example 1.11 $B\mathbb{Z} = K(\mathbb{Z}, 1) = S^1$ and $E\mathbb{Z} = \mathbb{R}$ its universal cover. In particular for $G = \mathbb{Z}$ (or any discrete group)

$$\mathrm{Prin}_{\mathbb{Z}}(X) \cong [X, K(\mathbb{Z}, 1)] = H^1(X, \mathbb{Z}).$$

Another important property of the classifying space of a group is that it acts as a "delooping":

Theorem 1.12 *Let G be a topological group. Then ΩBG is weak homotopy equivalent to G. In particular*

$$\pi_k(BG) \cong \pi_{k-1}(G).$$

Proof. Consider the following diagram where the two rows are fibration sequences (the lower one is the path fibration):

$$\begin{array}{ccccc} G & \longrightarrow & EG & \longrightarrow & BG \\ \downarrow & & \downarrow & & \Vert \\ \Omega BG & \longrightarrow & PBG & \longrightarrow & BG \end{array}$$

By homotopying the maps if necessary I can make the diagram commutative. Applying the long exact sequence of a fibration together with the five-lemma we get the desired equivalence. □

A great advantage of Milnor's construction is that the passage from G to EG and BG is functorial: a group homomorphism $\varphi : H \longrightarrow G$ induces a continuous map

$$E\varphi : EH \longrightarrow EG, \qquad \sum t_i h_i \mapsto \sum t_i \varphi(h_i)$$

compatible with the group actions, so that it descents to a map $B\varphi : BH \longrightarrow BG$. This defines a functor

$$B : \mathsf{TopGps} \longrightarrow \mathsf{Top}.$$

If $\varphi : H \longrightarrow G$ is a group homomorphism and $E \longrightarrow X$ is a principal H-bundle, there is a well-known way to build a principal G-bundle via the balanced product

$$E \times_H G := E \times G/(eh, g) \sim (e, \varphi(h)g).$$

For the universal bundles, one can show that $B\varphi^* EG \cong EH \times_H G$, which shows

Proposition 1.13 *Let us denote by $f_E : X \longrightarrow BG$ the (homotopy class of) map corresponding to $E \longrightarrow X$. Then $E \times_H G \longrightarrow X$ is the G-bundle corresponding to the composite*

$$X \xrightarrow{f_E} BH \xrightarrow{B\varphi} BG.$$

A harder problem is the following: given $H \subseteq G$ a closed subgroup and $E \longrightarrow X$ a G bundle, does E come from a H-bundle? (one says that E has a *reduction* to H). That is, is $E \cong E' \times_H G$ for some H-bundle $E' \longrightarrow X$? In this language, the answer is very elegant and almost trivial:

Proposition 1.14 *A G-bundle $E \longrightarrow X$ has a reduction to H if and only if the classifying map f_E lifts to BH (up to homotopy),*

$$\begin{array}{ccc} & & BH \\ & \overset{\exists}{\nearrow} & \downarrow \\ X & \xrightarrow{f_E} & BG \end{array}$$

2 Geometric structures on vector bundles

Let us focus now on real vector bundles, aka principal $GL(n, \mathbb{R})$-bundles. To start with, the inclusion $O(n) \hookrightarrow GL(n, \mathbb{R})$ is a deformation retraction which in turn induces a (weak) homotopy equivalence $BO(n) \xrightarrow{\simeq} BGL(n, \mathbb{R})$. This means that real vector bundles of rank n are in bijection to the set $[X, BO(n)]$.

Example 2.1 For $n = 1$, $O(1) = \mathbb{Z}/2$ as topological groups so the set of real line bundles is in bijection to $[X, B\mathbb{Z}/2] \cong [X, K(\mathbb{Z}/2, 1)] \cong H^1(X; \mathbb{Z}/2)$. Given a line bundle E over X, the cohomology class corresponding under this bijection is known as the first *Stiefel-Whitney class* $w_1(E)$. There are higher Stiefel-Whitney classes that I will describe later.

For the complex case this works almost in the same way except that now complex line bundles are in bijection to $H^2(X; \mathbb{Z})$ since BS^1 is a $K(2, \mathbb{Z})$. In this case the corresponding integral cohomology class is known as the first *Chern class* $c_1(E)$.

A model for $BO(n)$ (at least up to homotopy equivalence) is given by the infinite Grassmannian, the set of n-dimensional linear subspaces in $\mathbb{R}^\infty = \operatorname{colim}_n \mathbb{R}^n$. This is topologised as a quotient of the infinite Stiefel manifold. This space admits a CW structure which allows to compute its cohomology ring:

Fact. $H^\bullet(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \ldots, w_n]$, $|w_i| = i$.

For a vector bundle $E \longrightarrow X$ with classifying map $f_E : X \longrightarrow BO(n)$, the *i-th Stiefel-Whitney class* is $w_i(E) := f_E^*(w_i)$.

Now, similarly to the way we view vector bundles as $O(n)$-bundles, we can view oriented vector bundles as $SO(n)$-bundles. From 1.14 we directly obtain

Proposition 2.2 *Let $E \longrightarrow X$ be a real vector bundle with classifying map $f_E : X \longrightarrow BO(n)$. Then E is orientable if and only if f_E lifts (up to homotopy) to $BSO(n)$,*

$$\begin{array}{ccc} & & BSO(n) \\ & \overset{\exists}{\nearrow} & \downarrow \\ X & \xrightarrow{f_E} & BO(n) \end{array}$$

There is a beautiful story about the so-called Whitehead tower of $BO(n)$ for which the map $BSO(n) \longrightarrow BO(n)$ is just the first step. The space $BSO(n)$ can be realised as the set of oriented n-dimensional linear subspaces of $\mathbb{R}^\infty$ and $BSO(n) \longrightarrow BO(n)$ as the map that forgets the orientation on the subspaces. This map is a two-sheeted covering map, and in particular must be the universal cover of $BO(n)$ since $\pi_1(BSO(n)) = \pi_0(SO(n)) = 0$. So covering theory tells us that the lifting in 2.2 exists if and only if $(f_E)_*\pi_1(X) \subseteq p_*\pi_1(BSO(n)) \cong 0$, which happens precisely if $(f_E)_* : \pi_1(X) \longrightarrow \pi_1(BO(n))$ is the trivial map. It turns out that this happens precisely when the first Stiefel-Whitney class vanishes:

Proposition 2.3 *A real vector bundle of rank n is orientable if and only if $w_1(E) = 0$.*

Proof. Consider the following chain of isomorphisms:

$$H^1(X;\mathbb{Z}/2) \cong \mathrm{Hom}(H_1(X;\mathbb{Z}/2),\mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1(X),\mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1(X),\pi_1(BO(n))).$$

Here we have used the universal coefficients theorem for the first iso, the universal property of the abelianisation for the second one and $\pi_1(BO(n)) \cong \pi_0(O(n)) = \mathbb{Z}/2$ for the first one. Because of this sequence of isomorphisms is natural on X we obtain that $w_1(E)$ corresponds precisely to $(f_E)_*$. □

As stated above this is just the beginning of a bigger tower, called the *Whitehead tower* or tower of connected covers of $BO(n)$, which looks like

$$\begin{array}{c} \vdots \\ \downarrow \\ B5Brane(n) \\ \downarrow \\ BString(n) \\ \downarrow \\ BSpin(n) \\ \downarrow \\ BSO(n) \\ \downarrow \\ BO(n) \end{array}$$

One can argue in a homotopy-theoretical way to obtain obstruction classes for the existence of geometric structures on vector bundles, as we did before for the orientability. For instance, one can show that an orientable vector bundle $E \longrightarrow X$ has a spin structure if and only if $w_2(E) = 0$. For string structures, the criterion is harder, involving fractional Pontrjagin classes and Chern characters, see [SSS09].

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