KNOT THEORY

I : DEFINITION AND BASIC PROPERTIES

• Denote by $\mathcal{C}$ the category Top of top spaces and cont maps, Diff of smooth mnfds and smooth maps, or PL of triangulated spaces and piecewise linear maps.

<u>Definition</u>: A <u>knot</u> is an embedding $S^1 \hookrightarrow S^3$ in $\mathcal{C}$. A <u>link</u> of k components is an embedding $\coprod_k S^1 \hookrightarrow S^3$ in $\mathcal{C}$.

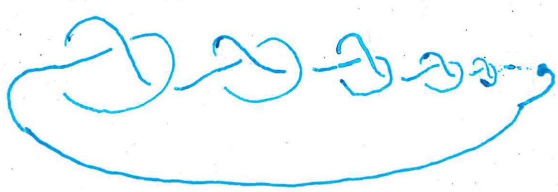
• The notion of equivalence of knots will be that we can deform one into the other. However this might lead all knots to be equivalent to the unknot in Top. We have to move the whole space.

<u>Definition</u>: Two knots K, K' are <u>equivalent</u> (or <u>ambient isotopic</u>) if there is an isotopy

$$H : S^3 \times I \to S^3$$

st $H_0 = \mathrm{Id}$ and $H_1(K) = K'$.

• We also want to avoid pathological knots, knotted infinitely many times, eg



<u>Definition</u>: A point $p \in K$ on a knot is <u>locally flat</u> if there is a nbhd $U \subset S^3$ of p st the pair $(U, U \cap K)$ is homeomorphic to $(D^3, \text{diameter})$.

A knot is <u>tame</u> if evy point is locally flat. Otherwise it is <u>wild</u>.

<u>Lemma</u>: Evy tame knot is equivalent to a PL knot (and a smooth knot).

• For this reason, from now on we will consider all knots to be PL.

- The following is a result of PL topology:

<u>Theorem</u>: Two knots are equivalent $\Leftrightarrow$ There is a (PL) orientation-preserving homeomorphism $h: S^3 \to S^3$ st $h(K) = K'$.

- We can perform connected sum of oriented knots as follows: let K_1, K_2 be oriented knots separated by some spheres. Choose locally flat neighbourhoods $U_1 \subset K_1$ and $U_2 \subset K_2$

<u>Definition</u>: The <u>connected sum</u> of two oriented knots $K_1 K_2$ is

$$K_1 \# K_2 := \frac{(K_1 - \overline{U}_1) \amalg (K_2 - \overline{U}_2)}{\partial U_1 = \partial U_2}$$

where the identification $\partial U_1 = \partial U_2$ is chosen so that the orientations match up.

<u>Proposition</u>: The connected sum of knots is well-defined on knot types, it is commutative, associative and the unknot acts as a unit. That is, the set of or. knots with $\#$ is a commutative monoid.

- There are more natural operations in the set of or. knots.

<u>Definitions</u>: let K be an oriented knot. The reverse knot $-K$ is the same knot with the opposite orientation. The <u>reflection</u> $\overline{K}$ is the image of K under a orientation-reversing homeomorphism (tipically $(x,y,z) \mapsto (x,y,-z)$)
Moreover, K is called

a) <u>invertible</u> if $-K = K$ (eg 3_1)

b) <u>amphichiral</u> if $\overline{K} = K$ or $-K$, otherwise it is <u>chiral</u> (eg 12_- & 3_1)

c) <u>fully symmetric</u> if $K = -K = \overline{K} = -\overline{K}$ (eg 4_1)

d) <u>totally anti-symmetric</u> if $K, -K, \overline{K}, -\overline{K}$ are all distinct. (eg 9_{32})

• What knots have an inverse for #? Surprisingly, only O = unknot!

<u>Proposition</u>: If $K \neq O$, then $K \# K' \neq O$ for any K'. (Pf by Mazur Swindle)

<u>Definition</u>: A (nontrivial) knot K is prime if $K = K_1 \# K_2$ implies $K_1 = O$ or $K_2 = O$.

• We will show that the notion of primeness is very closely related with the integers.

<u>Theorem</u> (Unique Prime Decomposition): Every nontrivial knot decomposes, in a unique way up to order, as connected sum of prime knots,

$$K = K_1 \# \dots \# K_n .$$

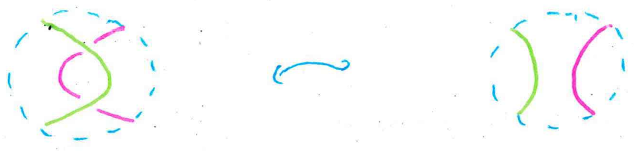
<u>Definition</u>: Let $K \subset \mathbb{R}^3 \subset S^3$ be a knot, and consider the projection $p: \mathbb{R}^3 \to \mathbb{R}^2$. Suppose that K is in "general position" wrt p, ie, the projection of two segments intersect in at most one point which for disjoint segments is not an endpoint, and that no point belongs to the projection of three segments.
The image of K in $\mathbb{R}^2$ together with "over and under" information is a <u>knot diagram</u>.

• Of course, an ambient isotopy gives certain isotopy of the knot diagram, and viceversa.

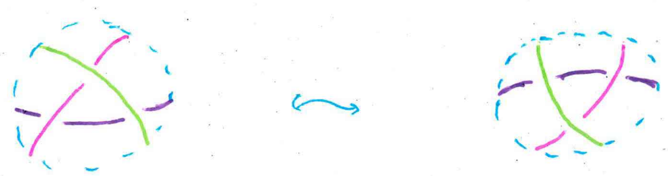
<u>Definition</u>: The <u>Reidemeister moves</u> are the following local moves of the knot diagram:

RI :

RII



RIII :



*Theorem (Reidemeister): Two planar diagrams correspond to isotopic links if and only if one can be obtained from the other by a finite sequence of Reidemeister moves and plane isotopies

DIAGRAM INVARIANTS

• The previous thm says that we can promote invariants of planar diagrams to knot invariants as long as they are invariant under the Reidemeister moves.

: For an oriented knot we consider the following signs for crossings

+1   −1

Definition: Let $L = K_1 \sqcup K_2$ a 2-component link. The linking number of K_1 and K_2 is

$$lk(K_1, K_2) := \frac{1}{2} \sum (\text{signed crossings})$$

and it preserves R. movements.

Definition: A link diagram D is 3-colorable if each segment of the diagram (from one underpass to the next) can be colored in 3 different colours st

i) All 3 colours are used

ii) At each crossing, either a single colour or all 3 appear.

• Colourability is preserved by R. Moves and it is a knot invariant. It detects that unknot $\neq$ trefoil, and trefoil $\neq$ figure-of-eight knot.

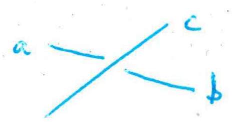
Lemma: Let $col_3(D)$ be the number of different tricolorings

1) $col_3(D)$ is a power of 3

2) $col_3(L_1 \# L_2) = \frac{1}{3} col_3(L_1) \cdot col_3(L_2)$.

• This idea can be promoted to n-coloring.

<u>Definition</u>: A link diagram D is <u>n-colourable</u> if every arc can be labelled by one of the numbers $0, 1, \dots, n-1$ such that at every crossing we have



$$\underline{a + b \equiv 2c \pmod{n}}$$

<u>Lemme</u>: let $col_n(D)$ be #n-colourings

1) n-colouring is a link invariant
2) $col_n(L_1 \# L_2) = \frac{1}{n} col_n(L_1) \cdot col_n(L_2)$.

• Recall that the <u>dihedral group</u> D_n is the group of isometries of the regular n-gon. It has a presentation

$$D_n \cong \langle g, \tau \mid g^n = \tau^2 = 1, \ \tau g \tau = g^{-1} \rangle$$

(g is the $\frac{2\pi}{n}$ rotation and τ a reflexion).

<u>Proposition</u>: There is a bijection $col_n(L) = Hom_{grp}(\pi_1(S^3 - L), D_n)$.

More precisely, if $\pi_1(S^3-L) = \langle x_1 \dots x_n \mid r_1 \dots r_n \rangle$ is a Wirtinger presentation, given a n-colourability the element x_i maps to τg^k, where k is the colour of the arc which correspond to x_i.

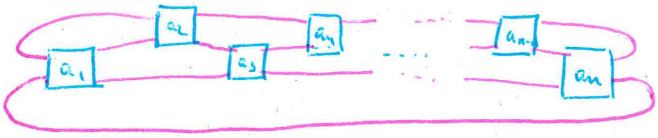
<u>Definition</u>: a) The <u>crossing number</u> of a knot is the minimal number of crossings needed for a diagram of the knot.

b) The <u>unknotting number</u> is the minimal number of self-crossing needed to turn a knot into the unknot.

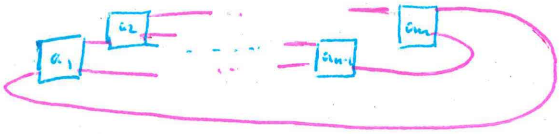
c) A knot is <u>alternating</u> if it has a diagram which alternates over's and under's.

d) The <u>bridge number</u> of a knot is the minimum number of local maxima (of the natural "height" function given by the z-coordinate) over all diagrams of a knot. A <u>rational knot</u> is a 2-bridge knot.

Rational knots have the following diagram:



n odd



n even

and each square represents $\boxed{a_i}$ =  $|a_i|$ twists, $a_i > 0$, and

$a_i < 0$ if the twists are in the opposite sense.

<u>Proposition</u>: Two rational knots are equivalent $\Longleftrightarrow$ they have the same fraction expansion

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

<u>Definition</u>: Let $K \subset sT = D^2 \times S^1$ be a knot inside a solid torus and let K' be a knot. A <u>satellite knot</u> of K' with <u>pattern</u> K is the image $h(K)$ by an embedding $h: sT \to S^3$ mapping sT to a regular nbhd of K'. Sometimes we say that K' is the <u>companion</u> of $h(K)$.

If K is a (p,q) torus knot (see later), then $h(K)$ is a <u>(p,q)-cable knot</u> if h maps a longitude of sT to a longitude of K'.

II : THE KNOT GROUP

Definition: Let $K \subset S^3$ be a knot (or more gen a link). The knot exterior is

$$X_K := \overline{S^3 - V} \qquad , \; V \text{ reg nbhd of } K \text{ (tubular nbhd)}$$

It is a compact 3-manifold with boundary, which is T (or $\coprod_i T$ for a link).

• Of course, if two knots K, K' are equivalents, then $X_K \cong X_{K'}$. An important result says that the converse is also true, X_K determines the isotopy type of K:

*Theorem (Gordon - Luecke, 1989): Two unoriented knots are equivalent if and only if there is a orientation preserving homeomorphism $X_K \xrightarrow{\cong} X_{K'}$.

Remark: This theorem is very false for links !!!

Theorem: 1) For an embedding $h: D^k \hookrightarrow S^n$, we have

$$\widetilde{H}_i(S^n - h(D^k)) \cong 0 \qquad \forall i$$

2) For an embedding $h: S^k \hookrightarrow S^n$ $(k < n)$ we have

$$\widetilde{H}_i(S^n - h(S^k)) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n-k-1 \\ 0, & \text{else.} \end{cases}$$

• This theorem (or Alexander duality) implies that $H_1(X_K) \cong \mathbb{Z}$ and 0 otherwise. But there is a straightforward argument: $H_1(X_K) \cong H_2(S^3, X_K) \overset{\text{exc.}}{\cong} H_2(sT, T) \cong \mathbb{Z}$. One even sees that it is generated by the class of a loop that runs the deleted tubular nbhd of X_K once.

Lemma: Let K, K' be knots. The linking number of K and K' is the integer $[K'] \in \mathbb{Z} \cong H_1(X_K)$.

- For a link, we have $H_1(X_L) \cong \bigoplus_{\# \text{comps}} \mathbb{Z}$.

Definition: Let K be a knot. The knot group is $\pi_K := \pi_1(X_K) = \pi_1(S^3 - K)$.

Example: If $K = $ unknot, then $X_K \cong s\mathbb{T} \simeq S^1$, thus $\pi_K \cong \mathbb{Z}$.

- We always have $\pi_K^{ab} = H_1(X_K) = \mathbb{Z}$, but when $\pi_K \cong \mathbb{Z}$? Only for the unknot!

Theorem (Unknotting): A (tame) knot is the unknot $\Longleftrightarrow$ $\pi_K \cong \mathbb{Z}$.

Definition: An embedding $h: D^k \hookrightarrow S^n$ is flat if h extends to an embedding $U \hookrightarrow S^n$, $D^k \subset U \subset \mathbb{R}^n$ open nbhd of D^k in $\mathbb{R}^n$. $h(D^k)$ is a flat disk.

Lemma (Alexander extension): Every homeomorphism $\varphi: \partial D^n \xrightarrow{\cong} \partial D^n$ extends to an homeomorph. $\tilde{\varphi}: D^n \xrightarrow{\cong} D^n$

Lemma (Unknotting): $K \subset S^3$ is the unknot $\Longleftrightarrow$ K bounds a flat disk in S^3.

TORUS KNOT

Definition: A torus knot is a knot that admits an embedding into $\mathbb{T} = S^1 \times S^1$. We let $T_{p,q}$ be a torus knot with $[T_{p,q}] = (p,q) \in \mathbb{Z} \oplus \mathbb{Z} = H_1(\mathbb{T})$.

- Explicit description: View $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ as the sphere in $\mathbb{C}^2$, and $S^1 \subset \mathbb{C}$ as the unit circle. Then

$$T_{p,q}(z) = \frac{1}{\sqrt{2}}(z^p, z^q)$$

($T_{p,q}$ winds the meridian p times and the longitude q times).

<u>Proposition</u>: The following equivalences hold:

1) $T_{p,\pm 1} \simeq T_{\pm 1,q} \simeq$ unknot

2) $T_{-p,-q} \simeq -T_{p,q}$

3) $T_{p,q} \simeq \overline{T}_{p,q} \simeq T_{-p,q}$

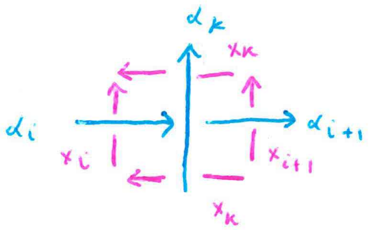
4) $T_{p,q} \simeq T_{q,p} \overset{3)}{\simeq} T_{-p,-q}$.

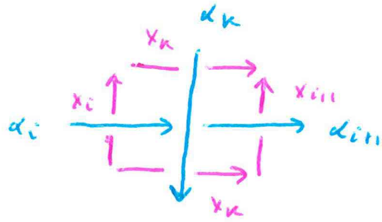
<u>Theorem</u> (Classification of torus knots):

$$T_{p,q} \simeq T_{(p',q')} \iff (p',q') \in \{(p,q),(q,p),(-p,-q),(-q,-p)\}.$$

In particular, $\pi_{T_{p,q}} \cong \langle x, y \mid x^p = y^q \rangle$.

* <u>Theorem</u> (Wirtinger): Let L be a link. Decompose its projection in a finite union of disjoint arcs $\alpha_1, \dots, \alpha_n$ ordered cyclically. Draw a small arrow x_i lying under α_i in a right handed fashion (each of them representing a loop in $S^3 - L$). At any crossing, let r_i be the relation depicted below, depending whether the crossing is positive or negative:





$r_i: \quad x_k x_i = x_{i+1} x_k$ $\qquad$ $r_i: \quad x_k x_{i+1} = x_i x_k$

Then

$$\boxed{\pi_1(S^3 - L) \cong \frac{F(x_1, \dots, x_n)}{\langle r_1, \dots, r_n \rangle}}$$

Examples: 1) $\pi_{3_1} = \langle a, b \mid aba = bab \rangle$. $3_1 \not\simeq$ unknot because there is a surjection

$\pi_{3_1} \to S_3$ but no surj $\mathbb{Z} \to S_3$ ($\mathbb{Z}$ can only surjects to an ab. grp!)

2) $\pi_{4_1} = \langle a, b \mid aba^{-1}bab^{-1} = bab^{-1}a \rangle$. $4_1 \not\simeq$ unknot because there is a surj

$\pi_{4_1} \to D_5$, $a \longmapsto \tau g^3$, $b \longmapsto \tau g$.

Remark: π_K is not a complete knot invariant!

Proposition: $\boxed{\pi_{K_1 \# K_2} \cong \pi_{K_1} *_{\mathbb{Z}} \pi_{K_2}}$

Corollary: Torus knots are prime.

THEOREMS OF PAPAKYRIAKOPOULOS

Theorem (Loop theorem): Let M be a 3-manifold with boundary such that the homomorphism $i_*: \pi_1(\partial M) \to \pi_1(M)$ induced by the inclusion is not injective, $\mathrm{Ker}\, i_* \neq 0$. Then there exists a PL embedding $h: (D^2, \partial D^2) \to (M, \partial M)$ st $h_{/\partial D^2}: S^1 \to \partial M$ is not nullhomotopic.

Lemma: Let M be a 3-mfld. If $f: (D^2, \partial D^2) \to (M, \partial M)$ is a map which is an embedding in some collar nbhd A of ∂D^2 with $f^{-1}(f(A)) = A$, then $f_{|A}$ extends to an embedding $g: D^2 \to M$.

Corollary: If $C \subseteq \partial M$ is a simple closed curve in the boundary of a 3-mfld and $[C] = 0 \in \pi_1(M)$, then C bounds a properly embedded disk in M.

*Theorem (Unknotting): $K \simeq$ unknot $\iff \pi_K \cong \mathbb{Z}$.

Theorem (Incompressibility of X_K): Let K be a knot, and $i_*: \pi_1(\partial X_K) \to \pi_1(X_K)$.

$$\boxed{\begin{array}{l} K \simeq \text{unknot} \iff \mathrm{Ker}\, i_* \neq 0 \\[1em] K \not\simeq \text{unknot} \iff \mathrm{Ker}\, i_* = 0 \end{array}}$$

Corollary: For a non-trivial knot K, π_K contains a subgp isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Corollary: if $K \neq$ unknot, then $K_1 \# K_2 \neq$ unknot for any K_2!

* Theorem (Sphere theorem): Let M be an orientable 3-manifold. If $\pi_2(M) \neq 0$, then there exists a PL-embedding $h : S^2 \hookrightarrow M$ which is not nullhomotopic.

* Theorem (Papa, 1957): Let K be a knot.

1) $\pi_n(X_K) \cong 0 \quad \forall n \geq 2$, ie, X_K is a $K(\pi_K, 1)$.

2) $\pi_1(X_K)$ is torsion-free.

- We already said that π_K is not a complete knot invariant, but it can be promoted to one!
- if K is a knot, in ∂X_K there is a simple closed curve m, representing a generator of $H_1(X_K)$, and a prefered longitude l s.t. $lk(l, K) = 0$

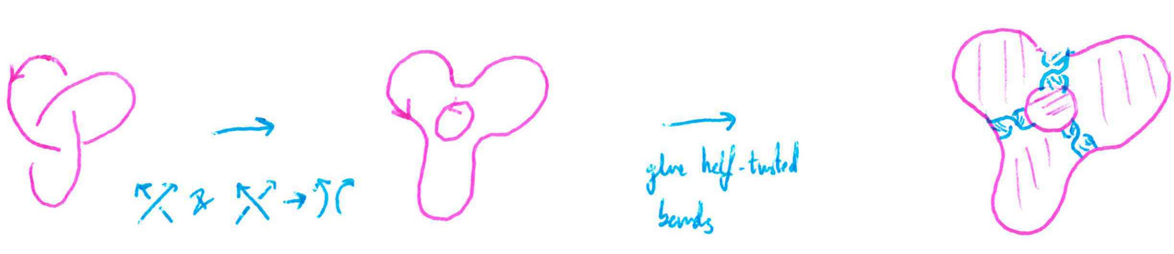
Definition: A peripheral system of a knot is a choice of homotopy class $[m], [l] \in \pi_K$ of ~~one~~ meridian and a longitude.

* Theorem (Waldhausen, 1966): Two knots K, K' are equivalent $\iff$ there exist peripheral systems $([m], [l])$ and $([m'], [l'])$ and ~~and~~ a isomorphism $\varphi : \pi_K \to \pi_{K'}$ st $\varphi([m]) = [m']$ and $\varphi([l]) = [l']$.

SEIFERT SURFACES

Definition: Let L be a oriented link. A Seifert surface is a connected compact oriented surface contained in S^3 which has L as its oriented boundary.

Theorem: Any oriented link in S^3 has a Seifert surface



• Recall the classification of surfaces with boundary:

<u>Theorem</u>: Any compact surface with boundary is homeomorphic to one of the following:

a) A sphere S^2 with $k>0$ holes
b) A connected sum of tori with $k>0$ holes
} orientable

c) A connected sum of proj. planes with $k>$ holes, } non orientable

where k is the number of boundary components of the surface.

<u>Definition</u>: Given a compact surface with boundary M, let M^* be M with k disks attached, which is a compact surface. The <u>genus</u> of M is the genus of M^*; and the <u>Euler characteristic</u> of M is $\chi(M) := \chi(M^*) - k$.

• In particular, if M is orientable, then $g(M) = 1 - \frac{1}{2} \cdot \chi(M^*)$.

<u>Definition</u>: The <u>genus</u> of a knot K is the minimum of the genus of all Seifert surfaces of K.

• Note that, if F is a Seifert surface constructed as above, then any of the disks is contractible and the half-twisted bands are htpic to unit intervals, so F is htpy eq to a graph with (s = number of Seifert circles) vertices and (n = number of crossings of the diagram) edges, so $\chi(F) = s-n$. So ...

$$g(F) = 1 - \frac{s-n+1}{2}$$

so $g(K) \leq 1 - \frac{s-n+1}{2}$.

<u>Corollary</u>: $g(K) = 0 \iff K \simeq$ unknot.

* <u>Theorem</u>: $\boxed{g(K_1 \# K_2) = g(K_1) + g(K_2)}$

<u>Corollary</u> (again!): $K_1 \not\simeq$ unknot $\Rightarrow K_1 \# K_2 \not\simeq$ unknot for any K_2!

<u>Corollary</u>: $K \# \overset{n}{\cdots} \# K \simeq K \# \overset{m}{\cdots} \# K \Rightarrow n = m$.

<u>Corollary</u>: A knot of genus 1 is prime.

Proposition: If $K = P \# Q = K_1 \# K_2$, P prime; then $K_i = P \# \text{sth}$ and $Q = \text{sth} \# K_{i\pm1}$.

Corollary: If P is prime, $P \# Q = P \# Q' \Rightarrow Q = Q'$.

***Theorem** (Unique prime decomposition): Any knot decomposes, in a unique way up to order, as connected sum of prime knots,

$$K = P_1 \# \cdots \# P_n .$$

Definition: Let K be a knot. The commutator subgroup is $\pi_K^{(1)} := [\pi_K, \pi_K] \subseteq \pi_K$.

- There is an obvious ses $1 \to \pi_K^{(1)} \to \pi_K \to \mathbb{Z} \cong \pi_K/\pi_K^{(1)} \cong H_1(X_K) \to 1$, which has a section $s: H_1(X_K) \to \pi_K$ sending the meridian m to its lift class, so

$$\pi_K \cong \pi_K^{(1)} \rtimes_\varphi \mathbb{Z} ,$$

where $\varphi: \mathbb{Z} \to \mathrm{Aut}(\pi_K^{(1)})$, $\varphi(m)(x) = m x m^{-1}$.

Theorem (Structure of $\pi_K^{(1)}$): Let K be a knot. For $\pi_K^{(1)}$ there are two possibilities:

1) If $\pi_K^{(1)}$ is fg, then $\pi_K^{(1)} \cong F(x_1, \dots, x_{2g})$, $g = g(K)$

2) If $\pi_K^{(1)}$ is not fg, then $\pi_K^{(1)} \cong \cdots G_{-1} *_{H_{-1}} G_0 *_{H_0} G_1 *_{H_1} G_2 *_{H_2} \cdots$

III : THE ALEXANDER POLYNOMIAL

Recall : M R-module, $\mathcal{E}_i(M)$ i-th Fitting ideal or elementary ideal generated by the minors of order $(m-i)$ of a presentation $F'_n \xrightarrow{f} F_m \to M \to 0$ of M. If R is UFD, then $\Delta_i(M) := \gcd(\mathcal{E}_i(M)) \in R$, i.e., the generator of the minimal principal ideal containing $\mathcal{E}_i(M)$.

$\operatorname{ord} M := \Delta_0(M)$ the order of M. Recall also that $\mathbb{Z}[F_n] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, F_n free ab gp of rank n.

Definition : Let X be a finite connected CW complex, let $F = \pi(X)^{ab} / \operatorname{Tors} \pi(X)^{ab}$ and let $\overline{X}$ be its universal free abelian cover, so $F \cong \operatorname{Aut}_X \overline{X}$ and $H_i(\overline{X}; \mathbb{Z})$ is a $\mathbb{Z}[F]$-module.

The i-th Alexander module of X is $A_i(X) := H_i(\overline{X}; \mathbb{Z})$ as $\mathbb{Z}[F]$-module, and the Alexander polynomial of X is

$$\Delta_X(t_1 \dots t_n) := \operatorname{ord} A_1(X) \in \mathbb{Z}[F_n] / \pm F_n = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] / \{t_1^{k_1} \cdots t_n^{k_n}\}$$

The image under the ring morphism $\mathrm{pr} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_1] = \mathbb{Z}[t, t^{-1}]$ sending $t_i \mapsto t$ is the reduced Alexander polynomial, $\Delta_X(t) = \Delta_X(t, \dots, t) = \mathrm{pr}\, \Delta_X(t_1 \dots t_n)$

Proposition : $\Delta_X(t) = \operatorname{ord} H_1(\check{X}; \mathbb{Z})$, where $\check{X} \to X$ is the covering corresponding with $\operatorname{Ker}(\pi_1(X) \to F \cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \to \mathbb{Z})$.

$$e_i \longmapsto 1$$

Definition : A covering space $X \to S$ with $\operatorname{Aut}_S X \cong \mathbb{Z}$ is called an infinite cyclic cover.

BACK TO KNOT THEORY

<u>Definition</u>: The (reduced) <u>Alexander polynomial</u> of a link L is the Alexander polynom of X_L. If $L = K$ a knot, it is just the (unred.) Alexander polynomial, $\Delta_L(t) := \Delta_{X_L}(t)$.

• (<u>Explicit construction of X_L^∞</u>): Denote by $X_L^\infty = \check{X}_L$ or $\overline{X}_K$ the infinite cyclic cover of the link/knot complement. It can be constructed as follows: let F be a Seifert surface for L, and let V be a tubular nbhd of L intersecting F in an annular nbhd of $L = \partial F$, and $\partial V \cap F$ is a longitude for K.

Now cut $X_K = \overline{S^3 - V}$ along the surface $X_K \cap F$, ie, choose a regular nbhd $\mathcal{U}$ of $X_K \cap F$ and let $X_F := \overline{X_K - \mathcal{U}}$. It is a 3-manifold with boundary $\partial X_F = F_+ \cup A \cup F_-$, where $F_\pm$ are copies of $F \cap X_K$ and $A \subseteq \partial X_K$ is an annulus.

There are obvious maps $r : F_+ \xrightarrow{\cong} F_-$ and $j_\pm : F_\pm \hookrightarrow X_K$. For any $i \in \mathbb{Z}$, ~~let~~ introduce a homeomorphic copy $X_F^{(i)}$ of X_F via an homeo $h_i : X_F \xrightarrow{\cong} X_F^{(i)}$. Then X_K^∞ is the infinite pushout

$$X_K^\infty := \cdots \cup_F X_F^{(i)} \cup_F X_F^{(i+1)} \cup_F \cdots$$

where at any step $F \xrightarrow{\cong} F_- \xhookrightarrow{j_-} X_K \xrightarrow{h_i} X_K^{(i)}$ and $F \xrightarrow{\cong} F_+ \xrightarrow{j_+} X_K \xrightarrow{h_{i+1}} X_K^{(i+1)}$ are the glueing maps.

The generator of the Galois group $\mathrm{Aut}_{X_K} X_K^\infty \cong \mathbb{Z}$ is $t : X_K^\infty \to X_K^\infty$ which takes the i-th copy of X_F to the $(i+1)$-th. More precisely, $t|_{X_F^{(i)}} = h_{i+1} \circ h_i^{-1}$.

The covering map $p : X_K^\infty \to X_K$ is $p|_{X_F^{(i)}} = \iota \circ h_i^{-1}$, $\iota : X_F \hookrightarrow X$

A very similar construction yields the n-fold cyclic cover $X_K^n \to X_K$ with Galois grp $\mathbb{Z}/n$.

• How to compute $\Delta_L(t)$? How to find a presentation?

<u>Proposition</u>: Let F be a connected, compact, orientable surface with boundary. Then there is a non-degenerate $\mathbb{Z}$-bilinear form

$$\langle \cdot , \cdot \rangle : H_1(S^3 - F) \times H_1(F) \longrightarrow \mathbb{Z}$$

inducing an iso $H_1(S^3 - F) \cong H_1(F)$. In particular, for oriented simple closed curves $\beta \subset S^3 - F$ and $\alpha \subset F$, we have $\langle \beta, \alpha \rangle = lk(\beta, \alpha)$.

• Recall the inclusion $j_- : F \longrightarrow X_F$ from above, which induces $(j_-)_* : H_1(F) \longrightarrow H_1(X_F) \cong H_1(S^3 - F)$.

<u>Definition</u>: Let F be a Seifert surface for an oriented link L. The <u>Seifert form</u> of L is the map

$$S : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$
$$(x, y) \longmapsto S(x,y) := \langle (j_-)_* x, y \rangle .$$

• If L has n components, then $H_1(F)$ is a free ab grp of rank $2g + n - 1$, where $g = genus(F)$, ie, $H_1(F) \cong \bigoplus_{2g+n-1} \mathbb{Z}$. So S is given by a matrix V, called the <u>Seifert matrix</u>.

• If a, b are simple closed curves (representing the generators of $H_1(F)$), then observe that

$$S([a],[b]) = lk(a^-, b) = lk(a, b^+)$$

where $j_-(a) = a^-$ and $j_+(b) = b^+$.

<u>Remark</u>: Let $\{a_1, \dots, a_{2g+n-1}\}$ be a basis for $H_1(F)$ and let $\{b_1, \dots, b_{2g+n-1}\}$ be its "dual basis" via the pairing (ie, basis of $H_1(S^3 - F)$). Then, in the $\{a_i\}$ basis, the Seifert matrix

$$V = \left(lk(a_i^-, a_j) \right) = \left(lk(a_i, a_j^+) \right)$$

has, as column vectors, the coordinates of a_j^+ in the basis $\{b_k\}$ and, as row vectors, the coordinates of $a_i^- \in H_1(S^3 - F)$

*Theorem: Let F be a Seifert surface of a link L and let V the Seifert matrix associated with respect to any basis of $H_1(F)$.

Then the matrix $\boxed{A := t \cdot V - V^T}$ presents the $\mathbb{Z}[t,t^{-1}]$-module $H_1(X_K^\infty)$, and it is called the Alexander matrix. In particular,

$$\boxed{\Delta_L(t) = \det A}.$$

Theorem: 1) If L is an oriented link, then $\Delta_L(t) = \Delta_L(t^{-1})$.

2) If K is an oriented knot, then $\Delta_K(1) = \pm 1$.
3) If L is a link with $\#L \geq 2$, then $\Delta_L(1) = 0$.

Corollary: For any knot K, Δ_K can be written as

$$\Delta_K(t) = a_0 + a_1(t+t^{-1}) + a_2(t^2+t^{-2}) + \cdots\cdots$$

where $a_i \in \mathbb{Z}$ and a_0 is odd.

Proposition: 1) $\Delta_L(t) = \Delta_{\bar{L}}(t) = \Delta_{-L}(t)$.

2) $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$.

3) $\Delta_{hK}(t) = \Delta_K(t) \cdot \Delta_{K'}(t^n)$, where hK is a satellite knot of K' with pattern K, and $[K] = n \in H_1(S\mathbb{T})$.

4) $\Delta_{L_1 \sqcup L_2}(t) = 0$.

• Call the breath of a Laurent pol. $\sum_{i=k}^{n} a_i t^i$ to $n-k$.

Corollary: $g(K) \geq \frac{1}{2} \cdot \text{breath } \Delta_K(t)$.

*Theorem: Let $t : X_K^\infty \to X_K^\infty$ be the gen of $\mathrm{Aut}_{X_K} X_K^\infty$. Then $\Delta_K(t)$ is, up to a unit, the characteristic polynomial of the linear map

$$t_* : H_1(X_K^\infty; \mathbb{Q}) \to H_1(X_K^\infty; \mathbb{Q}).$$

• Alexander can be also obtained via <u>Fox differential calculus</u>: if $\pi_K = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$ is a presentation of a knot group, let

$$A := \left(\rho\left(\frac{\partial r_i}{\partial x_j} \right) \right)$$

be the <u>Alexander matrix</u> of the presentation, where $\rho : \mathbb{Z}[F(x_1, \dots x_n)] \longrightarrow \mathbb{Z}[\pi_K] \xrightarrow{ab} \mathbb{Z}[t^{\pm 1}]$.

* <u>Theorem</u>: $\boxed{\Delta_K(t) = \gcd(\mathcal{E}_1(A))} = \Delta_1(A)$.

• <u>(Recipe for $\Delta_K(t)$)</u>: The previous can be even simplified if we have a presentation $\pi_K = \langle x_1 \dots x_n \mid r_1, \dots, r_m \rangle$ st $\rho(x_i) = \rho(x_j)$ $\forall i,j$, like the <u>Wirtinger presentation</u>.

① Label segments of the knot diagram cyclically $\alpha_1, \dots, \alpha_n$.

② At each crossing $\alpha_i \longrightarrow \uparrow^{\alpha_k} \rightarrow \alpha_{i+1}$ or $\alpha_i \longrightarrow \downarrow^{\alpha_k} \longrightarrow \alpha_{i+1}$, consider the relation

$$\begin{cases} -x_i + t\, x_{i+1} + (1-t)\, x_k, & \text{for positive crossing} \\ t\, x_i - x_{i+1} + (1-t)\, x_k, & \text{for negative crossing} \end{cases}$$

③ Construct the matrix A with one column for every generator and one row for every relation as above,

$$A = \begin{array}{c} \\ r_1 \\ \vdots \\ r_m \end{array} \begin{array}{c} \begin{array}{ccccccc} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_k & \cdots \end{array} \\ \left(\begin{array}{ccccccc} & & -1 & t & & 1-t & \\ & & & & & & \\ & & & & & & \end{array} \right) \end{array}$$

④ $\Delta_K(t) = \det$ of <u>any</u> $(n-1) \times (n-1)$ minor of A!

V : JONES POLYNOMIAL (1984)

• Useful for results about diagrams a knot can have.

* **Theorem**: There exists a unique map

$$V : \{\text{oriented links in } S^3\} \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

assigning to any link L a polynomial $V(L)$ called the <u>Jones polynomial</u>, such that

i) $V(\text{unknot}) = 1$

ii) If L_+, L_-, L_0 are the same links except in the nbhd of a point where they are like

L_+ L_- L_0

then their Jones polynomials satisfy the following <u>skein relation</u>

$$\boxed{t^{-1} V(L_+) - t\, V(L_-) + (t^{-1/2} - t^{1/2})\, V(L_0) = 0}$$

• Since any link can be changed to the unlink of c unknots by changing crossings in some diagram, it follows that the skein formula computes ~~all~~ $V(L)$ of all links.

<u>Proposition</u> (Properties):

1) $V(L \amalg O) = (-t^{-1/2} - t^{1/2})\, V(L)$

2) $V(\amalg_k O) = (-t^{-1/2} - t^{1/2})^{k-1}$

3) $V(K \# K') = V(K)\, V(K')$

4) $V(K \amalg K') = (-t^{-1/2} - t^{1/2})\, V(K)\, V(K')$.

• Let us see how to construct it in two different ways:

Definition: The **Kauffman bracket** is a function

$$\langle\ \rangle : \{\text{unoriented links diagrams}\} \longrightarrow \mathbb{Z}[A, A^{-1}]$$

characterized by

i) $\langle \bigcirc \rangle = 1$

ii) $\langle D \amalg \bigcirc \rangle = (-A^{-2} - A^{2}) \langle D \rangle$

iii) $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

• Of course, the bracket poly of a link with n crossings can be expressed as a linear sum of 2^n diagrams with no crossings.

Lemma: 1) The Kauffman bracket changes by a R I move in the following fashion

$$\langle \text{ᘐ}\!- \rangle = -A^{3} \langle - \rangle \quad , \quad \langle \gamma \rangle = -A^{-3} \langle - \rangle$$

2) The Kauffman bracket does not change by R II & R III moves

$$\langle \asymp \rangle = \langle \asymp \rangle \quad , \quad \langle \times\!\!\times \rangle = \langle \times\!\!\times \rangle$$

Remark: If $\overline{D}$ denotes the reflexion diagram (all unders are overs), then $\langle \overline{D} \rangle = \overline{\langle D \rangle}$, where the latter bar stands for the involution $A \mapsto A^{-1}$ in $\mathbb{Z}[A, A^{-1}]$.

Definition: Let D be an oriented diagram. Its **writhe** $w(D)$ is the signed sum of all its crossings.

• Observe that $w(D)$ does not change by RII or RIII, but $w(D)$ does change by $+1$ or -1 with a RI move (thus it is <u>not</u> a knot invariant).

<u>Proposition</u>: If D is a diagram of an oriented link L, then

$$(-A)^{3w(D)} \cdot \langle D \rangle$$

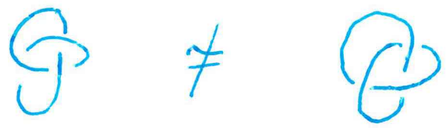
is an invariant of L. Moreover, the previous expression lies in $\mathbb{Z}[A^2, A^{-2}] \subset \mathbb{Z}[A, A^{-1}]$.

<u>Definition</u>: The <u>Jones polynomial</u> of an oriented link is the Laurent poly. in $t^{1/2}$ defined by evaluation in A^{-2}:

$$V(L) := (-A)^{3w(D)} \cdot \langle D \rangle \Big|_{t^{1/2} = A^{-2}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}].$$

<u>Remark</u>: It is unknown if there exist a non-trivial knot with $V(K) = 1$. However it is known that there are links with $V(L) = 1$. ($\#L > 1$)

• Of course, $V(\bar{L}) = \overline{V(L)}$, which shows that the right and left handed trefoils are not equivalent:



$-t^{-4} + t^{-3} + t^{-1}$ $\qquad\qquad$ $-t^{4} + t^{3} + t$.

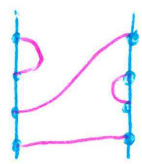
Alexander couldn't distinguish reflexions !!

<u>Remark</u>: $V(K)$ or V(link with odd # components) $\in \mathbb{Z}[t, t^{-1}] \subset \mathbb{Z}[t^{1/2}, t^{-1/2}]$.

<u>JONES AND BRAIDS</u>

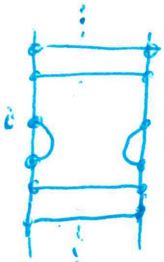
• Let us see how to connect both. This was the original way that Jones poly appeared.

• Consider the set D_n, $n \in \mathbb{N}$, of diagrams of two parallel lines with n vertices on each, such that vertices are connected by edges lying in between the two lines and no crosses are allowed.



<u>Definition</u>: let $n \in \mathbb{N}$, R comm ring and $\tau \in R$ a parameter. The <u>Temperley-Lieb algebra</u> is the free R-algebra on the basis D_n, with multiplication given by concatenation of diagrams, with the rule that if a circle forms, one removes it and multiply the result by τ. It is denoted by $TL(n,\tau)$.

• Clearly, it is associative with unit (~~and~~ the unit is ), and diagrams

$e_i =$  generate $TL(n,\tau)$, $\{e_1, \dots, e_{n-1}\}$.

• The TL-algebra admits a presentation

$$TL(n,\tau) = \frac{R[e_1, \dots, e_{n-1}]}{\left(e_i^2 = \tau e_i,\ e_i e_{i\pm 1} e_i = e_i,\ \underset{|i-j|\geq 2}{e_i e_j = e_j e_i}\right)}$$

• It is a fact that $\dim_{\mathbb{C}} TL(n,\tau) = \frac{1}{n+1}\binom{2n}{n}$, when $R = \mathbb{C}$.

<u>Definition</u>: let $n \in \mathbb{N}$, R a ring and $q \in R^\times$. The <u>Hecke algebra</u> is the free R-algebra generated by elements $g_1, \dots, g_{n-1}$ subject to the relations

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \qquad g_i g_j = g_j g_i \quad (|i-j|\geq 2), \qquad g_i^2 = (q-1) g_i + q.$$

• There are a composite of surjective maps

$$\mathbb{C}[B_n] \twoheadrightarrow H(n,q) \twoheadrightarrow TL(n, q^{1/2}), \qquad R = \mathbb{C}$$

which also exhibits the Hecke algebra and the TL algebra as a quotient of $\mathbb{C}[B_n]$.

• We can recover Jones (more precisely, Kauffman bracket) from TL: let $R = \mathbb{Z}[A, A^{-1}]$, and define

$$\rho : B_n \longrightarrow TL(n, -A^{-2} - A^{2})$$
$$\sigma_i \longmapsto A \cdot e_i + A^{-1} \cdot 1$$

<u>Proposition</u>: The previous map is a group homomorphism.

<u>Definition</u>: Let B be a R-algebra. A <u>trace</u> is an R-linear map $tr : B \to R$ satisfying $tr(b_1 b_2) = tr(b_2 b_1)$.

• Given a diagram $\beta \in D_n$, denote by $\hat{\beta}$ its "closure".

<u>Definition</u>: The <u>Markov trace</u> is the map $tr : TL(n, -A^{-2} - A^{2}) \longrightarrow \mathbb{Z}[A, A^{-1}]$ defined by

$$tr(\beta) := (-A^{-2} - A^{2})^{c-n},$$

where $c = \#$ components of $\hat{\beta}$.

<u>Proposition</u>: The Markov trace is indeed a trace, and it is uniquely characterized by the property that

$$X \in TL(n-1, -A^{-2} - A^{2}) \Rightarrow tr(X e_{n-1}) = \frac{1}{-A^{-2} - A^{2}} \cdot tr(X).$$

(here we view $TL(n-1, t) \subset TL(n, t)$).

$*$ <u>Theorem</u>: Let $\beta \in B_n$. Then $\langle \hat{\beta} \rangle = tr(\rho(\beta)) \cdot (-A^{-2} - A^{2})^{n-1}$, so

$$\boxed{V(\hat{\beta})(A^{-4}) = (-A)^{3w(\hat{\beta})} (-A^{-2} - A^{2})^{n-1} \cdot tr\, \rho(\beta).}$$

CONSEQUENCES OF JONES

- The Jones poly. allows to state results in terms of diagrams of the link, in particular with alternating diagrams.
- A link diagram is <u>connected</u> if it canot be covered by two disjoint balls.

***Theorem**: Let L be a link.

1) breath $V(L) \leq \#$(crossings of any connected diagram of L)

2) If D is a connected, alternating and reduced diagram of L, then <u>breath $V(L) = \#$(crossings of D)</u>

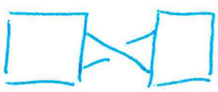
- This also says that for conn. alternating, reduced diagrams, #crossings is a link invariant!

<u>Corollary</u> (Tait): If L has a connected, reduced alternating diagram D, then it has no diagram of less crossings than D, ie

$$\boxed{\text{crossing number of } L = \#(\text{crossings of } D) = \text{breath } V(L)}$$

<u>Corollary</u>: If K_1, K_2 are alternating knots, then <u>$c(K_1 \# K_2) = c(K_1) + c(K_2)$</u>

- In the previous, a <u>reduced</u> link diagram is a diagram where every crossing touches four "different" regions



<u>NOT</u> reduced

GEOMETRY OF ALTERNATING LINKS

<u>Definition</u>: a) A link of more than 2 components is <u>split</u> if there is a 2-sphere in $S^3 - L$ separating S^3 into two balls, each containing a component of L.

b) A link diagram D is <u>split</u> if there is a simple closed curve in $S^2 - D$ separating S^2 into two disks, each containing a component of D.

- It is clear that L is split $\Longleftarrow$ it has a split diagram D. It is also clear that if D is split then L is split. But in general it won't be true that if L is split any diagram is split. But it is when D is alternating!

***<u>Theorem</u>**: If L is a split link and it has an alternating diagram D, then D is split!

(Menasco, 1984)

• There are more things that hold for alternating links:

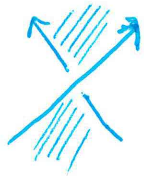
*Theorem (Murasugi '58, Crowell '59): Given an alternating oriented link, the Seifert algorithm applied to a reduced alternating diagram produces a Seifert surface of minimal genus.

(†)

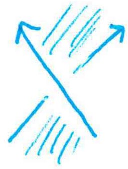
• Recall that for a knot K, $g(K) \geq \frac{1}{2}$ breath Δ_K.

Lemma: Every link diagram is 2-colourable, i.e, it admits a "checker-board" colour. (Jordan thm)

Definition: Let D be a link diagram with a checkerboard colour. The incident number $I(c)$ of a crossing c is defined by



$+1$



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*Theorem (Murasugi, '58): If D is an alternating diagram of an alternating knot K with constant incident number along its crossings, then $g(K) = \frac{1}{2}$ breath Δ_K.

Question (Fox): What is an alternating diagram? (i.e, top nature)

*Theorem (Howie, 2015): Let K be a non-trivial knot. Then K is alternating $\Longleftrightarrow$ there exist a pair of Seifert surfaces Σ, Σ' of K satisfying

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial\Sigma, \partial\Sigma') = 2.$$

Here $i(\partial\Sigma, \partial\Sigma') = | lk(K, \sigma) - lk(K, \sigma') |$, where $\sigma = \Sigma \cap \partial X_K$ and $\sigma' = \Sigma' \cap \partial X_K$.

(†) • The previous theorem has a converse! Call a connected, oriented, alternating diagram special if one of the spanning surfaces built of of "checkerboard colour" is orientable

*Theorem (Hirasawa-Sakuma '97, Banks 2011): A Seifert surface for a special alternating link L has minimum genus $\Longleftrightarrow$ it is obtained by applying the Seifert algorithm to a special alternating diagram of L.

VI: CONWAY NORMALIZATION AND SIGNATURES

• Issue concerning Δ_K: indeterminacy by $\pm t^k$. We would like to pick one and sum.

<u>Definition</u>: Let F be a Seifert surface for $L \subset S^3$ a link. Let $D^2 \times I \subset S^3$ be a solid cylinder such that $(D^2 \times I) \cap F = D^2 \times \{0,1\}$ and the solid cylinder is on the same side of F near $D^2 \times \{0,1\}$. Then the Seifert surface

$$F' := F - (D^2 \times \{0,1\}) \underset{\partial D^2 \times \{0,1\}}{\cup} (\partial D^2 \times I)$$

is said to be <u>obtained from F by surgery along the arc</u> $0 \times I$

* <u>Theorem</u>: Any two Seifert surfaces F, F' of an oriented link $L \subset S^3$ are related by a sequence of Seifert surfaces

$$F = \Sigma_1, \Sigma_2, \dots, \Sigma_r = F'$$

such that Σ_i is obtained from Σ_{i-1}, or Σ_{i-1} is obtained from Σ_i by surgery along an arc, or Σ_i and Σ_{i+1} are isotopic.

<u>Definition</u>: Two square matrices $A, B \in M_n(\mathbb{Z})$ are said <u>S-equivalent</u> if they are related by a sequence of changes of $\mathbb{Z}$-basis (elementary transformations) ($A \sim PAP^{-1}$, $\det P = \pm 1$) and transformation of the following kind:

$$A \sim \left(\begin{array}{c|c|c} A & x & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right) \in M_{n+2}(\mathbb{Z}) \quad , \quad A \sim \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline x^t & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right) \in M_{n+2}(\mathbb{Z})$$

* <u>Theorem</u>: Any two Seifert matrices for an oriented link L are S-equivalent.

- That is, anything defined out of Seifert matrices that preserves the S-equivalence will be a link invariant!

<u>Lemma</u>: Let $V \in M_m(\mathbb{Z})$. The polynomial $\det(t^{1/2}V - t^{-1/2}V^T) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ does <u>not</u> depend on the S-equivalence class of A.

- Observe that $\det(t^{1/2}V - t^{-1/2}V^T) = t^{-r/2} \cdot \det(tV - V^T)$, so this is a multiple of the Alexander polyn. (multiple by a unit of $\mathbb{Z}[t^{+1/2}, t^{-1/2}]$), where $r = \dim V$

<u>Definition</u>: Let L be a link. The <u>Conway-normalized Alexander polynomial</u> of L is

$$\widetilde{\Delta}_L(t) := \det(t^{1/2}V - t^{-1/2}V) \quad , \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

where V is <u>any</u> Seifert matrix of L.

- Here $t^{1/2}$ should be thought as a formal root sq of t, ie, $\mathbb{Z}[t, t^{-1}] \subset \mathbb{Z}[t^{1/2}, t^{-1/2}]$.
- Recall that $H_1(F) \cong \bigoplus_{2g+m-1} \mathbb{Z}$, where $m = \#L$. In particular, whenever L has an odd number of components (eg when it is a knot), then the Seifert matrix V has even dimension, and $\widetilde{\Delta}_L \in \mathbb{Z}[t, t^{-1}]$.
- The same arguments given before show that $\widetilde{\Delta}_K(t) = \widetilde{\Delta}_K(t^{-1})$ and $\Delta_K(1) = +1$.

*<u>Theorem</u>: The Conway-normalized Alexander polynomial is the only link invariant

$$\widetilde{\Delta} : \{\text{oriented links}\} \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

such that

1) $\widetilde{\Delta}_{\text{unknot}}(t) = 1$
2) Whenever L_+, L_-, L_0 are as usual, then the C.n-Alex. satisfies the following skein relation:

$$\boxed{\widetilde{\Delta}_{L_+}(t) - \widetilde{\Delta}_{L_-}(t) - (t^{-1/2} - t^{1/2})\widetilde{\Delta}_{L_0}(t) = 0}$$

- Iterated application of this relation allow to compute $\widetilde{\Delta}_L$ of any link, since $\widetilde{\Delta}_{L_1 \sqcup L_2} = 0$. (in part. for unknot $\sqcup$ unknot)

• The previous says that $\tilde{\Delta}_L(t)$ is a polynomial in $(t^{-1/2}-t^{1/2})$, so by making the substitution $z = t^{-1/2}-t^{1/2}$ we get

<u>Definition</u>: The <u>Conway polynomial</u> of an oriented link $\nabla_L(z) \in \mathbb{Z}[z]$ is the polynomial produced by $\nabla_L(t^{-1/2}-t^{1/2}) = \tilde{\Delta}_L(t)$, where $z = t^{-1/2}-t^{1/2}$.

• By the theorem, it is characterized by the facts that $\nabla_{\text{unknot}}(z) = 1$ and the skein relation

$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z)$.

<u>Proposition</u> (Properties): Let L be an oriented link of $n = \#L$ components and let $\nabla_L(z) = \sum_{i\geq 0} a_i z^i$. Then

1) $a_i = 0$ for $i < n-1$ and $i \equiv n \pmod 2$.

2) If $L = K$ a knot, then $a_0 = 1$

3) If $n = 2$, then $a_1 = \mathrm{lk}(K_1, K_2)$, where K_1, K_2 are the components of L.

• Of course, $\nabla_{L_1 \amalg L_2}(z) = 0$.

<u>Remark</u>: $\nabla_K(z) = 1$ for K = Kinoshita-Terasaka knot, so it is not a complete knot invariant.

<u>SIGNATURE OF A KNOT</u>

<u>Definition</u>: Let V be a Seifert matrix of an oriented link $L \subset S^3$. The <u>signature</u> of L is the integer $\sigma(L) := p - q$, where (p,q) is the signature of the symmetric matrix $V + V^T$.

<u>Proposition</u>: The signature does not change under S-equivalence, ie, it is a link invariant.

<u>Lemma</u>: $\sigma(\bar{L}) = -\sigma(L)$, ie, $L \simeq \bar{L} \Rightarrow \sigma(L) = 0$.

<u>Remark</u>: The signature can be described in a more topological way, at least when $\Delta_K(0) = \pm 1$ a unit of $\mathbb{Z}$, one can show that, as $\mathbb{Z}$-module = ab gr, $H_1(X_K^\infty; \mathbb{Z})$ is free of rank $\leq 2g(K)$. In particular, $H^1(X_K^\infty; \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(X_K^\infty; \mathbb{Z}), \mathbb{Z})$ is also free of the same rank.

Let $[F, \partial F] \in H_2(F, \partial F; \mathbb{Z})$ be the fundamental class of F which restricts to the orientation of K, F is a Seifert surface, and let $j_\infty : F \hookrightarrow X_K^\infty$ be the inclusion at the copy indexed by 0, $X_K^{(0)}$. Then there is a quadratic form

$$Q : H^1(X_K^\infty, \partial X_K^\infty; \mathbb{R}) \times H^1(X_K^\infty, \partial X_K^\infty; \mathbb{R}) \longrightarrow \mathbb{R}$$

called the Trotter quadratic form, which is non-degenerate and its signature (p, q) satisfies that $\sigma(K) = p - q$.

VIII: SLICE KNOTS

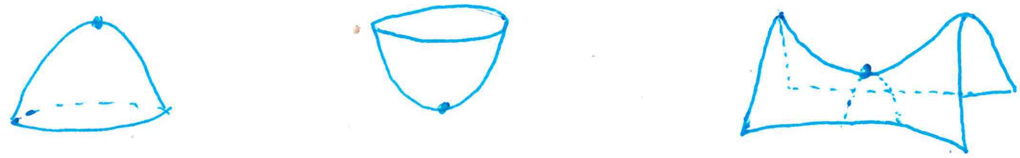
Definition: An embedding $h: D^2 \hookrightarrow D^4$ is *locally flat* if every point $x \in D^2$ has a nbhd U and a nbhd V of $h(x)$ st $(V, h(U)) \cong (\mathring{D}^4, \mathring{D}^2)$.

Definition: A knot $K \subset S^3 = \partial D^4$ is *slice* if there is a flat disk $D^2 \subset D^4$ st $K = \partial D^2 = D^2 \cap S^3$.

- Local flatness implies flatness, ie, $D^2 \subset D^4$ has a regular nbhd $N \cong D^2 \times I^2$.
- Every knot is the boundary of a 2-disk in D^4 (namely, the cone $CK \subset D^4$) but this disk is not loc. flat at the apex.

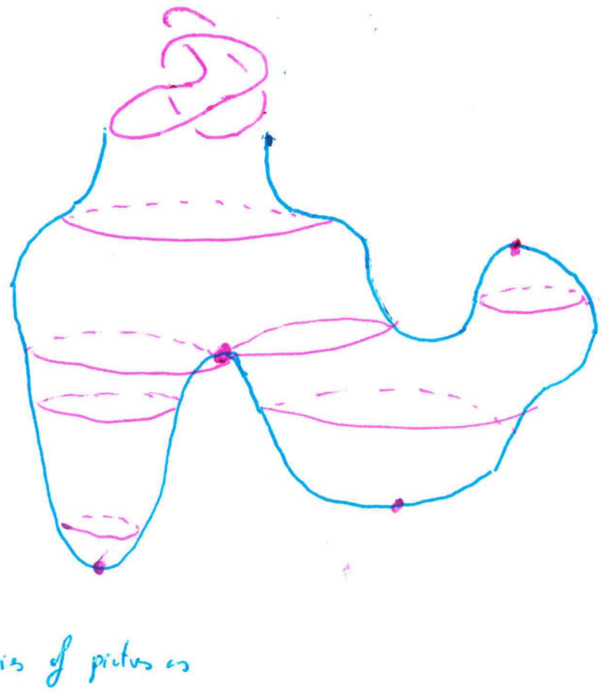
Lemma: For any knot K, $K \# -\overline{K}$ is slice.

- View an slicing disk as a movie, where the slicing disk intersects 3-spheres which move through it to produce either an ordinary knot or a knot with "singularities":

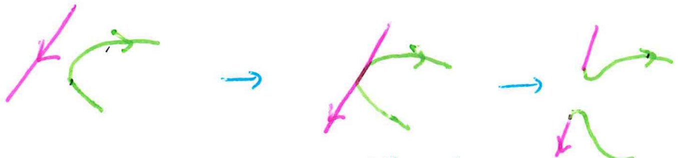


maximum minimum saddle point

These arise as



The saddle point picture has a series of pictures as



saddle point

Definition: A knot is *ribbon* if it bounds a "disk" only with "ribbon singularities", ie, there is a map $h: D^2 \to S^3$ with $\partial h(D^2) = K$ and the singularities of $h(D^2)$ are all double lines σ, $h^{-1}(\sigma) = \sigma_1 \amalg \sigma_2$, $\sigma_i \cong I$, st at least one of the $\sigma_i \subset \mathring{D}^2$.

Proposition. Ribbon knots are slice knots.

Conjecture (Fox): Slice knots are ribbon knots.

Theorem: Let $K \subset S^3$ be a slice knot and let F be a Seifert surface. Then the Seifert form associated to F, $S: H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$, vanishes in a half-rank direct summand of $H_1(F,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. That is, the Seifert matrix has the form

$$V = \left(\begin{array}{c|c} 0 & A \\ \hline B & C \end{array}\right).$$

Corollary: If $K \subset S^3$ is slice, then $\sigma(K) = 0$.

Corollary (Fox-Milnor, '66): If $K \subset S^3$ is slice, then the Alexander polynomial factors as $\Delta_K(t) = f(t) \cdot f(t^{-1})$, where $f(t) \in \mathbb{Z}[t]$.

Corollary: If $K \subset S^3$ is slice, then $\det K := |\Delta_K(-1)| = |a_0 - 2a_1 + 2a_2 - 2a_3 + \dots|$ is a square, where $\Delta_K(t) = a_0 + a_1(t^{-1}+t) + \dots$.

- Very few slice knots are known, eg 6_1, 8_8, ... It is ~~not~~ known whether the Conway knot is slice (has 0 Alexander and 0 signature).

KNOT CONCORDANCE GROUP

Definition: Let $(M,+)$ be a monoid.

IX: 3-MANIFOLDS BY SURGERY

Definition: Let M be a n-manifold. An elementary <u>r-surgery</u> consists of removing from M an embedded copy of $S^r \times D^{n-r}$ and replacing it with a copy of $D^{r+1} \times S^{n-r-1}$ by the obvious homeomorphism

$$\partial(S^r \times D^{n-r}) \xrightarrow{\cong} S^r \times S^{n-r-1} \xleftarrow{\cong} \partial(D^{r+1} \times S^{n-r-1}).$$

- In general, surgery means a sequence of r-surgeries, but for 3-manifolds, they can be performed "simultaneously", where the only possible thing will be 1-surgery.

Definition: A <u>framing</u> of a link $L \subset S^3$ is a regular nbhd by disjoint solid tori parametrized by $S^1 \times D^2$.

- In dimension 1, 2 & 3, PL, cont and smooth are the same:

Definition: Let K be a simplicial complex. A map $f: |K| \to \mathbb{R}^n$ is <u>piecewise linear</u> (PL) if it restricts to linear maps on each simplex.

An n-dim <u>PL manifold</u> is a polyhedron $|K|$ with the extra property that every point has a nbhd U with a PL homeomorphism $U \xrightarrow{\cong} \mathbb{R}^n$.

Theorem: For $n = 1, 2, 3$, there are equivalence of categories

$$\text{PL } n\text{-manifolds} \xrightarrow{\simeq} \text{top. } n\text{-manifolds} \xleftarrow{\simeq} \text{smooth } n\text{-manifolds}$$

where the arrows are the forgetful functors. In particular, all admits triangulations.

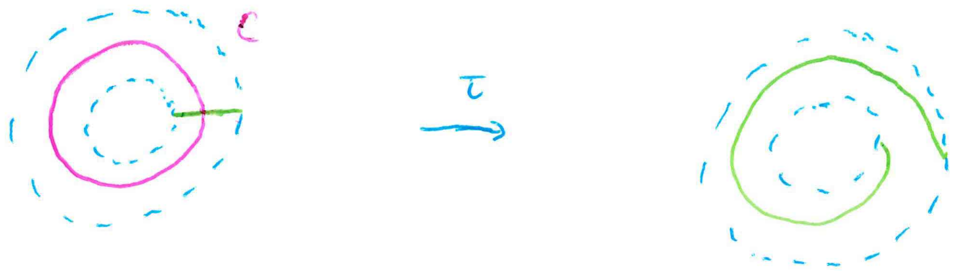
- The previous result means that for $n \leq 3$, any topological manifold has a unique PL structure and a unique smooth structure.
- For $n \geq 4$, both assertions are not true. However, it is true that every smooth manifold of any dimension has a unique PL structure, thus a triangulation.

• We will assume that 3-manifolds are PL.

• Recall that for a manifold M, $MCG(M) :=$ homeomorph $M \to M$ / isotopy. If M has boundary, then the homeos and the isotopies are rel. ∂M.

<u>Lemma</u>: Let M, N be 3-manifolds with homeomorphic boundaries. Then the gluing $M \cup_\varphi N$ of M and N along their boundary only depends on the isotopy class of the homeomorphism $\varphi : \partial M \xrightarrow{\cong} \partial N$.

<u>Definition</u>: Let Σ be a connected compact oriented surface, let $C \subset \Sigma$ be an embedded closed curve and U a tubular nbhd of C, $U \cong S^1 \times I$. A <u>Dehn twist about</u> C is (the isotopy class of) the homeomorphism $\tau : \Sigma \to \Sigma$ st $\tau_{|\Sigma - U} = \mathrm{Id}$ and $\tau_{|U}(e^{i\theta}, t) = (e^{i(\theta - 2\pi t)}, t)$.



• Denote by $\mathrm{Dehn}(\Sigma) \subseteq \mathrm{Aut}(M)$ the subgroup generated by the Dehn twists (which includes homeomorphisms isotopic to the identity).

• We set the following eq relation on simple closed curves in $\mathring{\Sigma}$:

$$\sigma \equiv \sigma' \iff \sigma' = h\sigma \qquad \text{for some } h \in \mathrm{Dehn}(\Sigma)$$

<u>Proposition</u>: Let σ, σ' be simple closed curves in $\mathring{\Sigma}$.

1) If σ, σ' intersect transversally at exactly one point, then $\sigma \equiv \sigma'$.

2) If σ, σ' do not separate Σ, then $\sigma \equiv \sigma'$.

<u>Corollary</u>: Let $(\sigma_1, \dots, \sigma_n)$, $(\sigma'_1, \dots, \sigma'_n)$ be two families of simple closed curves st neither $\cup \sigma_i$ nor $\cup \sigma'_i$ separate Σ. Then there is $h \in \mathrm{Dehn}(\Sigma)$ st $\sigma'_i = h\sigma_i$ $\forall i$ (and in particular $\sigma_i \equiv \sigma'_i$).

Definition: Let M be a n-manifold, and let $\varphi: \partial D^r \times D^{n-r} \longrightarrow \partial M$ be an embedding. To *attach a r-handle* to M is to perform the pushout $M \cup_\varphi (D^r \times D^{n-r})$.

- Note that $\partial(M \cup_\varphi (D^r \times D^{n-r}))$ is ∂M changed by a $(r-1)$-surgery.

Definition: A *handlebody* of genus g is the orientable 3-manifold obtained by a 3-disk with g 1-handles attached.

Definition: Let M be a closed, connected, orientable 3-manifold. A *Heegaard splitting* of M is a decomposition of M as $M = X \cup Y$, $X \cap Y = \partial X = \partial Y$, where X, Y are handlebodies.

Lemma. Let M be a 3-manifold with boundary and let $D^2_1, \dots, D^2_k$ be a collection of properly embedded 2-disks. If $\mathring{M} - \bigcup_i D^2_i \cong \coprod \mathring{D}^3$ a collection of open balls, then M is a handle body of genus $k-2$.

Proposition: Every closed, connected, orientable 3-manifold has a Heegaard splitting.

Theorem: Let M be a closed connected orientable 3-manifold. Then there are finite sets of solid tori $T_1, \dots, T_k \subset M$ and $T'_1, \dots, T'_k \subseteq S^3$ st $M - \bigcup T_i \cong S^3 - \bigcup T'_i$.

- The main result will state that we can get any 3-manifold "gluing back" the solid tori to S^3, but of course it will depend on the attaching maps. We think of these solid tori as a tubular neighbourhood of a link in S^3.

- Let $L \subset S^3$ be a link and $K \subset L$ one of the components, and let $T \subset S^3$ be a tub nbhd of K. Consider μ a *meridian* (simple closed curve bounding a disk in T) and a *longitude* λ (simple closed curve that intersects the meridian at exactly one point). View them as a basis of $H_1(\partial T; \mathbb{Z}) \cong \mathbb{Z}\mu \oplus \mathbb{Z}\lambda$.

- The point is that an attaching map $\partial(D^2 \times S^1) \xrightarrow{\varphi} \partial(S^3 - T) = \partial(S^1 \times D^2)$ (being homeomorphism) only depends, up to isotopy class (see before) on the homology class of the curve $\ell := \varphi(\partial D^2 \times *)$. So $[\ell] = (p, q)$ in the previous basis, and since ℓ is simple closed, $\gcd(p,q) = 1$. Note also that (p,q) & $(-p,-q)$ define the same curve (orientation does not play any role). This yields

Proposition: There is a bijection

$$\left\{\begin{array}{c}\text{isotopy class of}\\ \text{simple closed curves}\\ \text{on } \mathbb{T}\end{array}\right\} \;=\!=\; \mathbb{Q} \cup \{\infty\}$$

$$l \longmapsto p/q \, ,$$

and ~~...~~ corresponds to the meridian μ.

Definition: A framed link is a link L in which every component is labelled with a rational number $\mathbb{Q} \cup \infty$.

- These labels represent the isotopy class of curve to attach the solid torus $D^2 \times S^1$.

Definition: We say that M arises by rational surgery on a framed link L if M is the result of attaching solid tori to $S^3 \setminus N(L)$, $N(L) \cong \coprod T_i$, by simple closed curves corresponding to the framed link. If the labels are integers (i.e., $q = \pm 1$ in every component), we talk about integer surgery.

* Theorem: Every closed, connected, orientable 3-manifold can be obtained by performing integer surgery along a framed link $L \subset S^3$.

- One question one asks now is: how changing the link affects to the resulting manifold?

Remark: 1) If L is a integer framed link, then for every component K, in the frame $(p, 1)$, $p = lk(K, l)$.

2) The curve l can be thought as the other component of a (twisted) band ($\cong S^1 \times I$) along K, where one boundary component is K and the other is l. A framed link can be thought as a link with bands.

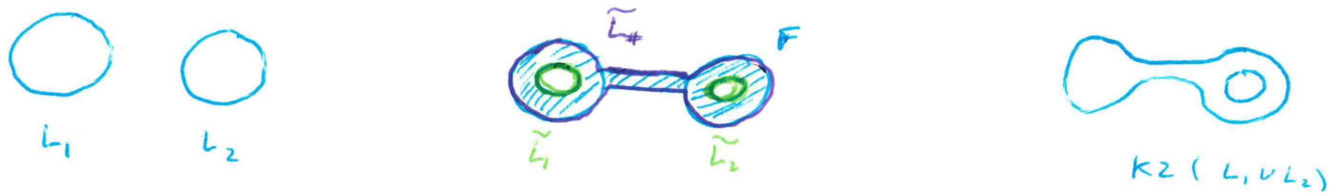
Definition: Let L be a link. The Kirby moves are the following transformations of links:

K1) Addition or removal of an unknotted, unlinked component of L with frame ± 1.

K2) "Addition of one component to another": Let $L = L_1 \cup \dots \cup L_m$ framed link, L_i knot, and let A_i the i-th band. Attach a $\overset{P=}{\text{band}} \cong$ rectangle to A_1, A_2. Then $F = A_1 \cup P \cup A_2 \cong$ disk with two holes. Then $\partial F = \tilde{L}_1 \cup \tilde{L}_2 \cup \tilde{L}_\#$, where $\tilde{L}_1, \tilde{L}_2$ are isotopic to L_1, L_2.

By definition, the link $\tilde{L}_\# \cup \tilde{L}_2 \cup L_3 \cup \dots \cup L_n$ is obtained from L through a transformation K2.

The new component $\tilde{L}_\#$ has a frame a band which is a collar nbhd of $\tilde{L}_\#$ on F, which is $n_1 + n_2 + lk(L_1, L_2)$



* <u>Theorem</u> (Kirby, 1978): Two manifolds M, M' obtained by performing integer surgery on framed links L, L' are homeomorphic $\Longleftrightarrow$ L, L' are related by a sequence of Kirby moves.

In other words,

$$\left\{ \begin{matrix} \text{closed, connected,} \\ \text{oriented 3-manifolds} \end{matrix} \right\} \;=\!=\; \left\{ \begin{matrix} \text{isotopy classes of} \\ \underbrace{\text{framed}}_{\text{integer}} \text{ links in } S^3 \end{matrix} \right\} \Big/ \text{Kirby moves.}$$

<u>Examples</u>:

1) Surgery on  $= S^1 \times S^2$

2) Surgery on  $= S^1 \times S^1 \times S^1$

3) Surgery on  = Poincaré homology sphere $\cong$ dodecahedron / opposite faces with $2\pi/10$ rotation.

4) Surgery on , $n = 0, 1, 2, \dots$ $= S^3$

(for details see Hu an Vo notes)

∘ The following harder theorem states that the only way to obtain $S^1 \times S^2$ from S^3 by 0-surgery on a knot K is the way described before, in other words, the unknot with frame 0 is not Kirby eq. to any non-trivial knot with frame 0.

<u>Theorem</u> (Gabai, 1987): If $S^1 \times S^2$ is obtained by performing 0-surgery along a knot K, then $K \simeq$ unknot.

- The proof uses "taut foliations".
- The famous result of Gordon-Luecke is also done via Dehn surgeries (here it's a corollary)

* <u>Theorem</u> (Gordon-Luecke, 1989): Non-trivial Dehn surgery on a non-trivial knot never yields S^3.

In other words, the unknot is the only knot so that S^3 can be obtained by non-trivial surgery on a knot.

* <u>Theorem</u> (Gordon-Luecke, 1989): Two knots are (isotopic) equivalent if and only if they have (orientation preserving) homeomorphic exteriors.

- In the previous theorem, "equivalent" means that there is a homeomorphism $X_K \xrightarrow{\simeq} X_{K'}$.

- Recall that in general an isomorphism $\pi_K \xrightarrow{\simeq} \pi_{K'}$ does <u>not</u> imply $K \simeq K'$ (it does if they have isomorphic peripheral systems, ie, under that isomorphism the meridian maps to the meridian and the longitude maps to the longitude). However, it does for prime knots

<u>Theorem</u> (Whitten, 1987): Let K, K' be <u>prime knots</u>. Then

$$\pi_K \cong \pi_{K'} \iff S^3 - K \cong S^3 - K'.$$

<u>Corollary</u>: If two prime knots have isomorphic fundamental group, then they are equivalent (<u>not</u> isotopic).

<u>BRANCHED COVERINGS</u>

- Let X be a top space and let $\Delta \subset X$. Say that Δ is <u>nowhere dense</u> if $\mathring{\bar{\Delta}} = \emptyset$.

<u>Definition</u>: A <u>branched covering map</u> is a cont map $p: E \to B$ such that there exists $\Delta \subset B$ nowhere dense with the property that

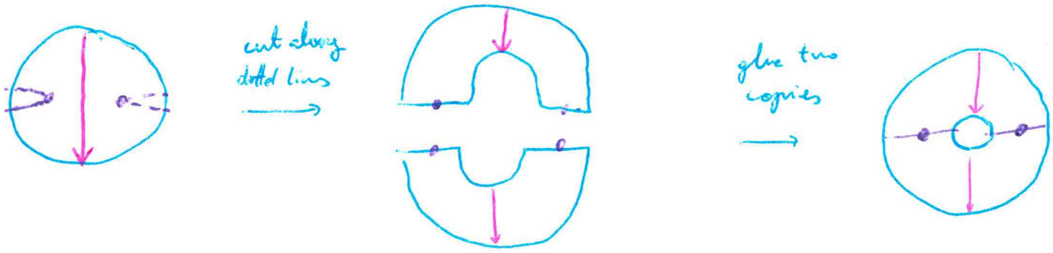
$$p|_{p^{-1}(B-\Delta)} : p^{-1}(B-\Delta) \to B - \Delta$$

is a covering map. We say that Δ is the <u>singular set</u>, and $B - \Delta$ the regular set.

We say that $x \in p^{-1}(\Delta)$ is a branch point with branch index k if p is k-to-1 near x.

Examples: 1) $E = ([-1,1] \times \{0,1\}) - \{(0,1)\} \xrightarrow{pr_1} [-1,1]$, it is a branched cover with $\Delta = \{0\}$.

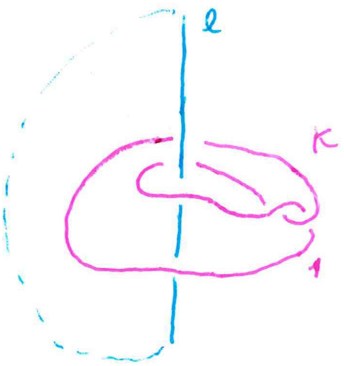
2) There is a branched covering map $D^2_- \to S^1 \times I$, 2-fold branched cover with 2 branched points:



3) $\mathbb{T} \# \mathbb{T} \to D^2$ − two holes, by projection.., is also a branched cover.

3) (Cyclic branched covering of S^3 along the unknot): Think of the unknot as a straight line in $\mathbb{R}^3 \cup \infty \cong S^3$. Define an action $\mathbb{Z}/n \curvearrowright S^3$ by rotation by $2\pi/n$ about the straight line. The quotient $S^3/\mathbb{Z}/n \cong S^3$, so the projection map $S^3 \to S^3$ is a n-fold branched cover with the unknot as singular set.

4) (Branched covering along 3_1): Perform 1-surgery along the knot K as depicted below, $K \simeq$ unknot, so the resulting mfld is S^3.



($K \cup \ell$ = Whitehead link)

The observation is that by deforming K into the standard unknot, and perform a Kirby 2 move, $K \cup \ell \equiv 3_1$,

X: QUANTUM INVARIANTS

Goal: Define 3-manifold invariants out of framed links.

- Since every 3-manifold arises by performing surgery on a framed link, we hope that we can associate to links some invariant that preserves Kirby moves.

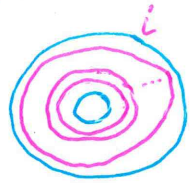
Definition: Let (Σ, P) be a surface with boundary Σ with a finite set $P \subseteq \partial\Sigma$ of points in the boundary. A link diagram in Σ is a collection of finitely many closed curves and arcs with endpoints in P, with the usual "over/under" information in crossings. If $P=\emptyset$, then the empty diagram is allowed.

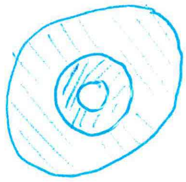
Definition: Let $\mathcal{L}(F)$ be the set of link diagrams in Σ, and let $A \in \mathbb{C}$. The linear skein of F is

$$\mathcal{S}(F) := \frac{\mathbb{C}[\mathcal{L}(F)]}{\langle D \sqcup \bigcirc = (-A^{-2}-A^{2})D \ , \ \times = A \cdot)(+ A^{-1} \cdot \asymp \rangle}$$

(where $\bigcirc$ stands for the boundary of an embedded disk in F).

Examples: 1) $\mathcal{S}(\mathbb{R}^2, \emptyset) \cong \mathbb{C}$, generated by the empty diagram. Of course $\mathcal{S}(S^2) \cong \mathcal{S}(\mathbb{R}^2)$.

2) $\mathcal{S}(S^1 \times I, \emptyset) \cong \mathbb{C}[\alpha]$, with base $\{1, \alpha, \alpha^2, \alpha^3, \dots\}$ where $\alpha^i =$ 

It actually has the usual structure of $\mathbb{C}$-algebra concatenating diagrams .

3) $\mathcal{S}(D^2, 2n \text{ pts}) \cong TL(n, A) = TL_n$, the Temperley-Lieb algebra. It was described earlier. Call $\{e_1, \dots e_n\}$ basis.

Proposition: Let $n \geq 1$ and let $A \in \mathbb{C}$ such that A^4 is not a k^{th}-root of unity for $k \leq n$. Then there exists an element $f^{(n)} \in TL_n$, called the Jones-Wenzl idempotent, satisfying the following properties:

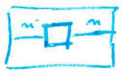
1) $f^{(n)} \cdot e_i = e_i \cdot f^{(n)} = 0$, $\quad i = 1, \dots, n-1$

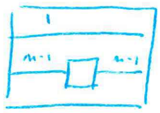
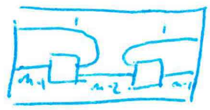
2) $f^{(n)} - 1$ belongs to the subalgebra generated by $e_1, \dots, e_{n-1}$ (without unit).

3) $f^{(n)}$ is idempotent, $f^{(n)} \cdot f^{(n)} = f^{(n)}$.

4) Consider the map $TL_n \cong \mathcal{S}(D^2, 2n\ pt) \longrightarrow \mathcal{S}(\mathbb{R}^2) \cong \mathbb{C}$ induced by assigning to every link diagram in $(D^2, 2n\ pt)$ its closure in $\mathbb{R}^2$, in the obvious way. Then the image Δ_n of $f^{(n)}$ under this map is

$$\Delta_n = \frac{(-1)^n \left(A^{2(n+1)} - A^{-2(n+1)}\right)}{A^2 - A^{-2}}.$$

Remarks: 1) The element $f^{(n)}$ is defined inductively as follows: if $f^{(n)} =$ , then

$$f^{(n)} = $$  $$- \frac{\Delta_{n-2}}{\Delta_{n-1}}$$ 

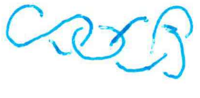
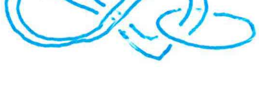
(extended linearly to all summands of $f^{(n)}$ in the general case). The assumption $A^4 \neq$ root of unity implies that $\Delta_k \neq 0$ so we can divide.

2) There is a map $TL_n \cong \mathcal{S}(D^2, 2n\ pts) \longrightarrow \mathcal{S}(S^1 \times I) \cong \mathbb{C}[\alpha]$ induced by an inclusion $D^2 \hookrightarrow S^1 \times I$ and do closure in the obvious way. The proof shows that the image of $f^{(n)}$ is the n-th Chebyshev polynomial $S_n(\alpha)$.

- Let $r \geq 3$. We define $\omega := \sum_{i=0}^{r-2} \Delta_i S_i(\alpha) \in \mathbb{C}[\alpha]$.
- Now let D be a link diagram of n components. There is a multilinear map

$$\langle -, \dots, - \rangle_D : \mathcal{S}(S^1 \times I) \times \overset{n}{\cdots} \times \mathcal{S}(S^1 \times I) \longrightarrow \mathcal{S}(\mathbb{R}^2)$$

induced by the following assignment of diagrams: since $\mathcal{L}(S^1 \times I) = \mathbb{N}_0 = \{\alpha^0, \alpha^1, \dots\}$, for any n-tuple $(\alpha^{i_1}, \dots, \alpha^{i_n})$, one considers the link diagram in $\mathbb{R}^2$ resulting from D after adding parallel copies of the components (as many as $i_j - 1$). (If α^0 appears, then that component is removed).

Example: $D =$ , then $\langle \alpha^2, \alpha^1, \alpha^0 \rangle_D =$ 

Proposition: let $r \geq 3$ and let $A \in \mathbb{C}$ st A^4 is a primitive r-th root of unity. If planar diagrams (of n-components) D, D' are related by Kirby moves of type II, then

$$\langle \omega, \dots, \omega \rangle_D = \langle \omega, \dots, \omega \rangle_{D'}$$

• Consider now the following framed Knot diagrams:

$U_+ =$  $+1$, $U_- =$  -1 , $U =$  0

Lemma: let $r \geq 3$ and A a primitive $4r$-th root of unity. Then $\langle \omega \rangle_U, \langle \omega \rangle_{U_+}, \langle \omega \rangle_{U_-} \neq 0$. Concretely,

$$\langle \omega \rangle_{U_+} \langle \omega \rangle_{U_-} = \langle \omega \rangle_U = \frac{-2r}{(A^2 - A^{-2})^2}$$

Definition: let $L = L_1 \cup \dots \cup L_n$ be a framed link. Its linking matrix $lk(L)$ is the matrix with entries the frames of the components in the diagonal and $lk(L_i, L_j)$ else.

• Observe that it defines a symmetric metric. Let (p, q) be its signature.

• if L changes by a K II move, $lk(L)$ changes by congruence, so the signature does not change.

* Theorem: let M be obtained by surgery on a framed link L, represented by a planar diagram D. Let (p, q) be the signature of $lk(L)$, let $r \geq 3$ and let $A \in \mathbb{C}$ a primitive $4r$-th root of unity. Then the quantity

$$\langle \omega, \dots, \omega \rangle_D \; \langle \omega \rangle_{U_+}^{-p} \; \langle \omega \rangle_{U_-}^{-q} \in \mathbb{C}$$

is a well-defined invariant of M, called the $SU_q(2)$-invariant (up to a factor).

METHODS TO COMPUTE QUANTUM INVARIANTS

<u>Lemma</u>: Let A be a $4r$-th root of unity.

1) $$\langle \omega \rangle_{U_+} = \frac{\sum_{n=1}^{4r} A^{n^2}}{2A^{(3+r^2)}(A^2-A^{-2})}$$

2) $$\langle \omega \rangle_{U_-} = \frac{-\sum_{n=1}^{4r} A^{-n^2}}{2A^{-(3+r^2)}(A^2-A^{-2})}$$

3) If $A = e^{\frac{i\pi}{2r}}$, then $\sum A^{n^2} = 2\sqrt{2r}\, e^{i\pi/4}$.

We now renormalize the $SU_q(2)$-invariant as follows: we let $\mu \in \mathbb{C}$ so that $\mu^{-2} = \langle \omega \rangle_U = \frac{-2r}{(A^2-A^{-2})^2}$. This implies that $\langle \mu\omega \rangle_{U_+} = \langle \mu\omega \rangle_{U_-}^{-1}$. This is, $\mu = \frac{A^2-A^{-2}}{\sqrt{-2r}}$.

<u>Definition</u>: Let $r \geq 3$ and let A be a primitive $4r$-th root of unity, and let D be a framed link diagram with signature $lk(D) = (p,q)$. The <u>quantum $SU_q(2)$-invariant</u> $I_A(M)$ of the manifold M obtained by surgery on D is

$$I_A(M) := \mu \cdot \langle \mu\omega, \dots, \mu\omega \rangle_D \cdot \langle \mu\omega \rangle_{U_-}^{p-q}.$$

<u>Examples</u>: 1) $I_A(S^3) = \mu = A^2 - A^{-2}/\sqrt{-2r}$

2) $I_A(S^1 \times S^2) = 1$.

3) $I_A(S^1 \times \Sigma_g) = (-2r)^{g-1} \sum_{i=0}^{r-2} \left(A^{2(i+1)} - A^{-2(i+1)}\right)^{2-2g} \in \mathbb{Z}$.

4) $I_A(S^1 \times \Sigma_2) = \frac{r^3-r}{6}$.

<u>Theorem</u>: The Jones polynomial of the (p,q)-torus knot is

$$V_{T_{p,q}}(t) = t^{\frac{(p-1)(q-1)}{2}} \cdot (1-t^2)^{-1} \cdot (1 - t^{p+1} - t^{q+1} + t^{p+q}).$$

XI : HOMFLY AND KAUFFMAN POLYNOMIALS

Theorem : There exists a unique assignment

$$P : \{\text{oriented links in } S^3\} \longrightarrow \mathbb{Z}[\ell^{\pm 1}, m^{\pm 1}]$$

called the HOMFLY polynomial, which satisfies

1) $P(\text{unknot}) = 1$
2) If L_+, L_-, L_0 are as usual, then

$$\boxed{\ell P(L_+) + \ell^{-1} P(L_-) + m P(L_0) = 0}$$

Theorem : There exists a unique assignment

$$\Lambda : \{\text{unoriented link diagrams}\} \longrightarrow \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

satisfying

1) $\Lambda(\text{unknot diagram with } 0 \text{ crossings}) = 1$.
2) Λ preserves RII & RIII.
3) $\Lambda(\ ) = a \cdot \Lambda(\ )$
4) If $D_+, D_-, D_0, D_\infty =$ are as usual, then $\Lambda(D_+) + \Lambda(D_-) = z\left(\Lambda(D_0) + \Lambda(D_\infty)\right)$.

Definition : The Kauffman polynomial is the assignment

$$F : \{\text{oriented links in } S^3\} \longrightarrow \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$$

defined by $F(L) := a^{-w(D)} \cdot \Lambda(D)$.

Proposition (Properties of P), let L, L_1, L_2 be links.

1) $P(L)(\ell, m) = P(L)(-\ell, -m)$

2) $P(\overline{L}) = \overline{P(L)}$, where the involution is $\overline{\ell} = \ell^{-1}$ & $\overline{m} = m$.

3) $P(L_1 \# L_2) = P(L_1) \cdot P(L_2)$ (independently of the selected components).

4) $P(L_1 \sqcup L_2) = -(\ell + \ell^{-1}) m^{-1} P(L_1) P(L_2)$

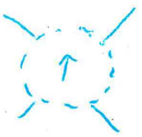
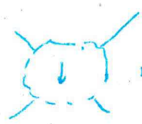
Proposition (Properties of F):

1) $F(L)(a,z) = F(L)(-a,-z)$

2) $F(\overline{L}) = \overline{F(L)}$, where the involution is $\overline{a} = a^{-1}$, $\overline{z} = z$.

3) $F(L_1 \# L_2) = F(L_1) F(L_2)$, (independently of the selected components)

4) $F(L_1 \sqcup L_2) = ((a + a^{-1}) z^{-1} - 1) F(L_1) F(L_2)$.

Proposition: P and F are unchanged by mutation, ie, the links  and  resulting by rotating π have the same P & F polynomials.

Remark: 1) There are infinitely many knots with the same HOMFLY polynomial.

2) Although the constructions of F & P are similar, one is not contained in the other, they are independent knot invariants.

* **Theorem**: The HOMFLY polynomial encodes both the Alexander polynomial and the Jones polynomial

$$\Delta_L(t) = P(L)\left(i, i(t^{1/2} - t^{-1/2})\right)$$
$$V_L(t) = P(L)\left(it^{-1}, i(t^{-1/2} - t^{1/2})\right)$$

where $i^2 = -1$.

Proposition: The Kauffman polynomial encodes the Jones polynomial:

$$V_L(t) = F(L)\left(-t^{3/4},\ (t^{-1/4}+t^{1/4})\right).$$

Definition: Let L be a link. The <u>braid index</u> of L $b(L)$ is the least number of strings needed to express L as the closure of a braid.

<u>Theorem</u> (Morton–Franks–Williams): Let L be a link and let E and e be the largest and smallest powers of l in the HOMFLY polynomial. Then

$$\boxed{b(L) \geq \frac{1}{2}(E-e)+1}$$

Notation: 1) If K is a knot, $P(K)$ can be expressed as $P(K) = \sum_{i\geq 0} p_i(l^2)m^i$, where $p_i(l^2) \in \mathbb{Z}[l^{\pm 2}]$, it is 0 when i is odd or big enough. To tabulate HOMFLY polynomials, we will use

$$(1\ 2\ \mathbf{2})\ (-2\ \mathbf{-2}\ -1)\ (\mathbf{1})$$

for $(l^{-4}+2l^{-2}+2)+(-2l^{-2}-2-l^2)m^2+m^4$. The numbers in the i-th bracket $(i\geq 0)$ give the coefficients in $p_{2i}(l^2)$, and the bold face number is the coefficient of l^0.

2) For a knot K, $F(K)$ can be expressed as $F(K) = \sum_{i\geq 0} q_i(a)z^i$ where $q_i \in \mathbb{Z}[a^{\pm 1}]$, $q_i(a)$ only contains odd powers of a when i is odd and only even if i is even. To tabulate Kauffman polyn, we use

$$(\mathbf{0}\ -2\ -1)\ (*\ 0\ 1\ 1)\ (\mathbf{0}\ 1\ 1)$$

for $(-2a^2-a^4)+(a^3+a^5)z+(a^2+a^4)z^2$. The numbers in the i-th bracket $(i\geq 0)$ give the coefficient of $q_i(a)$. If i even, the coefficient listed are the even powers of a, the bold face number being the coeff of a^0. If i odd, the coefficients are of the odd powers of a and $*$ separates negative & positive powers.