HEEGAARD - FLOER HOMOLOGY

(A) Y closed, oriented $\Rightarrow$ $CF(Y)$ chain complex, not an invariant ~~but~~ (depends on choice of description of Y), but the chain htpy type is an invariant of Y. $\rightsquigarrow$ $HF(Y) := H_*(CF(Y))$ (Ozsváth - Szabó)

(B) $K \subset S^3$ knot $\rightsquigarrow$ $CFK(K)$ chain complex. Same story: its chain htpy type is an invariant of K, to get an invariant $HFK(K) = H_*(CFK(K))$ (Ozsváth - Szabó; Rasmussen).

(C) Concrete flavour of before: $\widehat{HFK}(K)$ is a bigraded vs.

Its graded Euler char $= \Delta_K(t)$, so it "categorifies" Alexander.

Folklore: Δ_K gives bounds for $g(K)$: $\frac{1}{2}$ breath $(\Delta_K(t)) \leq g(K)$.

$\widehat{HFK}(K)$ detects genus (O-S), in particular detects the unknot.

Folklore: K fibered $\Rightarrow$ $\Delta_K(t)$ monic.

$\widehat{HFK}(K)$ detects fiberedness. (Niggini, Ni)

(D) There is a relation between (A) & (B): $CFK(K)$ determines

$HF(S^3_n(K))$

S^3 with n-surgery along K.

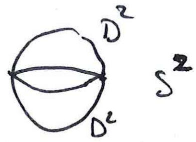
(A)

Q : How do we describe 3-manifolds?

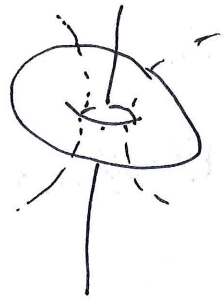
A : Heegaard diagrams.

Definition: A genus g handlebody is a nbhd of $\bigvee_g S^1$ in S^3. Alternatively, it is a 3-ball with g handles $D^2 \times D^1$ attached.

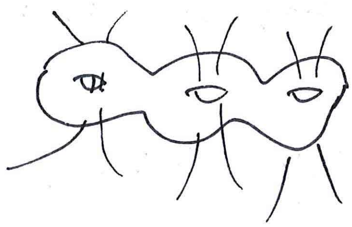
A Heegaard splitting of Y is a decomposition $Y = H_1 \cup_f H_2$ where H_1, H_2 are handlebodies (with the same genus) and $f : \partial H_1 \xrightarrow{\cong} \partial H_2$ is a orientation-reversing homeomorphism.

Examples: 1) $S^3 = D^3 \cup D^3$, Heegaard splitting of genus 0, just like 

2) $S^3 = \partial D^4 = \partial(D^2 \times D^2) = \partial D^2 \times D^2 \cup D^2 \times \partial D^2 = T^1 \cup T^1$.



3) S^3 has genus g H. splittings:



4) $L(p,q) = (S^1 \times D^2) \cup (S^1 \times D^2)$.

Theorem: Every closed, oriented 3-manifold admits a Heegaard splitting.

Pf: Pick a triangulation of a closed, oriented 3-mnfd Y. Let H_1 be a nbhd of 1-skeleton. The 1-skeleton is a finite graph so a nbhd of it is a handlebody.

Claim: $Y - H_1 = H_2$ is a handlebody. Why? H_2 is a nbhd of the dual 1-skeleton.

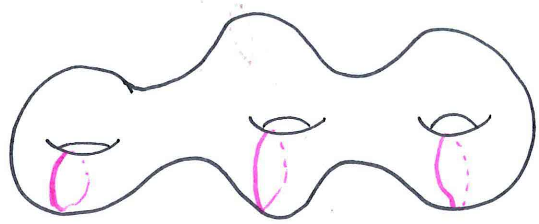
$\square$

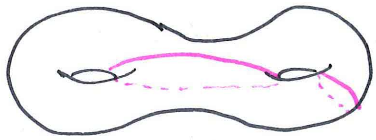
Definition: A set of attaching circles for a handlebody of genus g H is a set of simple closed curves $\{\gamma_1, \dots, \gamma_g\} \subset \Sigma := \partial H$ st

i) γ_i are pairwise disjoint

ii) $\Sigma - \gamma_1 - \dots - \gamma_g$ connected

iii) Each γ_i bounds a disk





* We can recover H from Σ and γ_i's by:

1) Thicken Σ to $\Sigma \times [0,1]$
2) Attach thickened disks along $\gamma_i \times \{0\}$. After this ~~two~~ one of the boundary components ("the interior") is homeomorphic to S^2
3) Fill in the resulting S^2 boundary with D^3.

Definition: A Heegaard diagram for $Y = H_1 \cup H_2$ is a ~~triple~~ quatruple $(\Sigma, \alpha, \beta, w)$ st

i) Σ closed, oriented surface of genus g

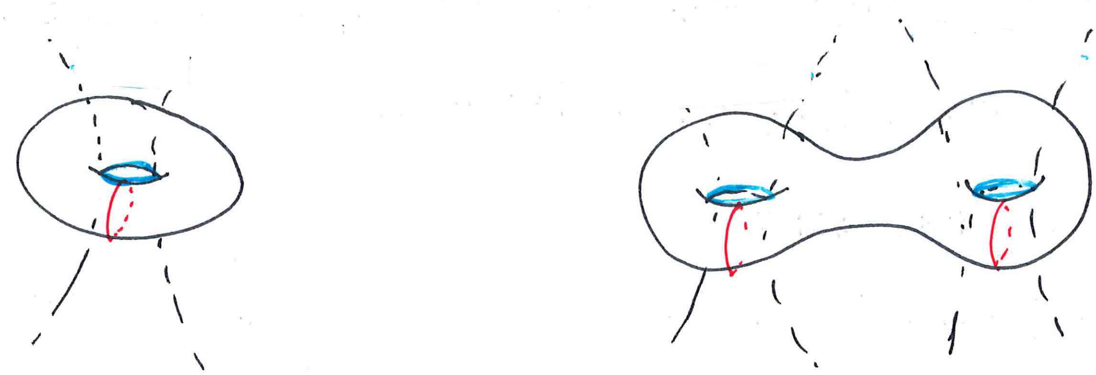
ii) $\alpha = \{\alpha_1, \dots \alpha_g\}$ set of attaching circles for H_1

iii) β ———————————————— H_2

iv) w basept in $\Sigma - \alpha - \beta$

Convention: α curves are red, β curves are blue.

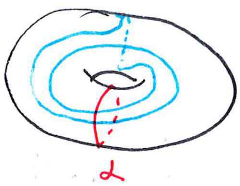
Examples:



- Y can be built from (Σ, α, β):

1) Thicken Σ to $\Sigma \times [0,1]$
2) Attach thickened disks along $\alpha_i \times \{0\}$ ("inside")
3) ——————— $\beta_i \times \{1\}$ ("outside")
4) The resulting boundary is $S^2 \sqcup S^2$

Example:



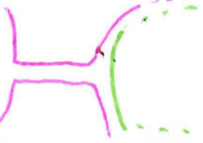
$\mathbb{RP}^3 \cong L(2,1)$.

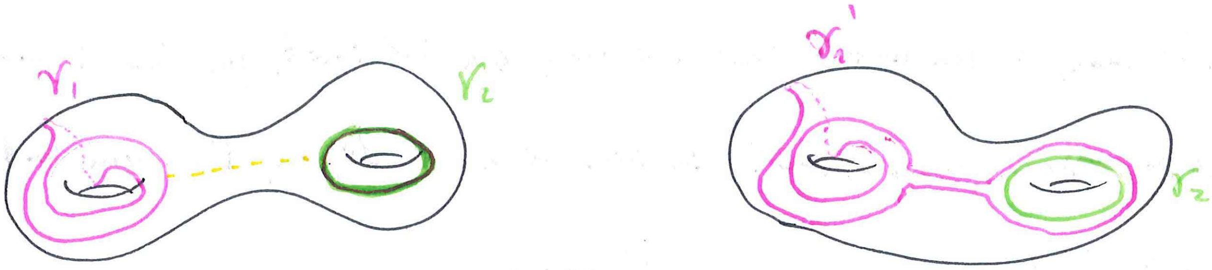
- There are different H. diagrams that represent the same 3-manifold. Is there any relationship between them?

Definition: Let $\{\gamma_1 \dots \gamma_g\}$ be a set of attaching circles for a a handlebody H. The Heegaard moves are:

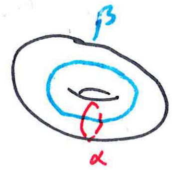
a) Isotopy through disjoint simple closed curves (away from basepoint)

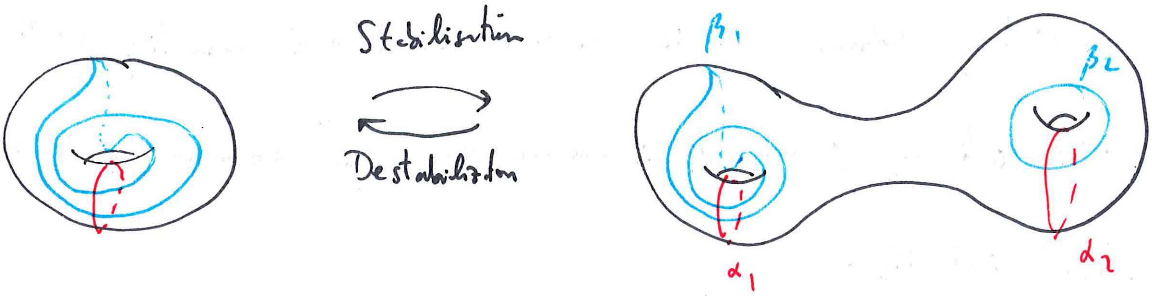
b) Handleslide: choose a path from a pt in γ_i to a pt in γ_j missing w.

Then turn  into  parallel copy of the green one



So we pass from $\{\gamma_1, \dots \gamma_g\}$ to $\{\gamma_1', \gamma_2, \dots \gamma_g\}$. Of course, one has to convince oneself that if γ_1 bounds a disk, so does γ_1'.

c) (De)Stabilisation : If (Σ, α, β) is a H. diagram, we take connected sum with 



(this is just $M \# S^3 \cong M$)

*Theorem (O-S) : There is a bijection

$$\left\{\begin{array}{c}\text{closed, connected}\\ \text{3-manifolds}\end{array}\right\} = \{\text{Heegaard diagrams}\} \Big/ \text{Heegaard moves}.$$

Remark : This is just a re-statement of the theorem which says that

$$\left\{\begin{array}{c}\text{closed, connected}\\ \text{3-manifolds}\end{array}\right\} = \left\{\begin{array}{c}\text{isotopy class}\\ \text{of links}\end{array}\right\} \Big/ \text{Kirby moves}.$$

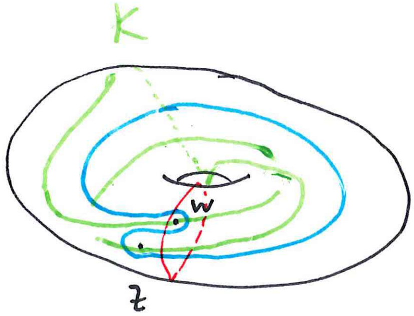
Remark: For knots, if you want to check that something is a knot invariant, you check R moves.

If you want to define a manifold invariant in terms of a H diagram, then you basically have to show that it does not depend on the H moves

Definition: Let $K \subset S^3$ be a knot. A doubly pointed Heegaard diagram for K is $(\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ where

i) $(\Sigma, \vec{\alpha}, \vec{\beta})$ is a H. diagram for $S^3 = H_1 \cup H_2$ (can be done more gen. for a 3-mfd M & $K \subset M$)

ii) $K = a \cup b$, where a is an arc in $\Sigma - \vec{\alpha}$ connecting w to z pushed slightly into H_1; and b is an arc in $\Sigma - \vec{\beta}$ connecting z to w pushed slightly into H_2.



K intersects Σ exactly in two pts: w, z

K = legendrian trefoil

• The notion of Heegaard moves for doubly pointed H. diagrams is mimicked in the obvious way.

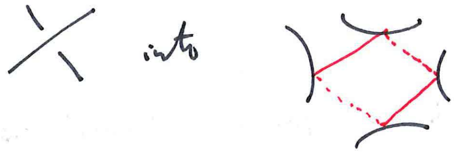
Theorem: There is a bijection

$$\left\{ \begin{array}{l} \text{isotopy classes} \\ \text{of knots in } S^3 \end{array} \right\} = \left\{ \begin{array}{l} \text{doubly pointed} \\ \text{Heegaard diagrams} \end{array} \right\} \Big/ \text{Heegaard moves}.$$

• There is a general recipe for the direct map from the previous bij, ie how to get a doubly pointed H diagram for a knot:

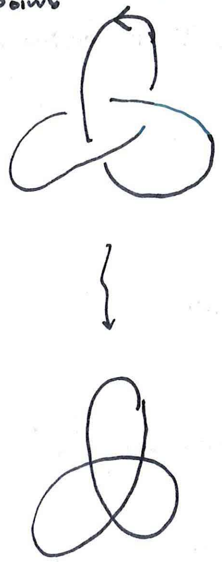
1) Consider a knot diagram, forget projections and consider the boundary of a 3-dim tub. nbhd of the resulting graph (∂ of a handlebody)

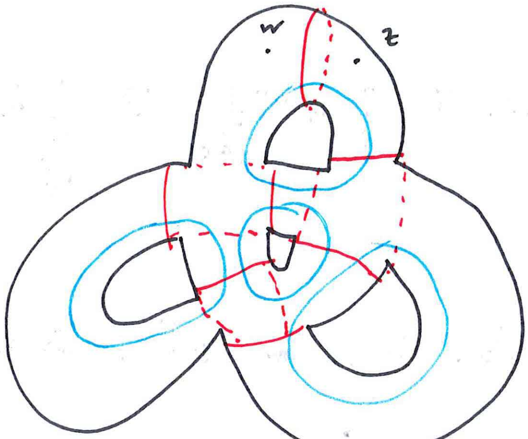
2) β-circles will come from the bounded regions of the knot 

3) α-circles come from turning crossings 

+ one extra meridian

4) points are set to the sides of the meridian, according to the orientation of K





Goal: From a H. diagram $H = (\Sigma, \alpha, \beta, w)$ for Y, build chain complexes

$\widehat{CF}(H)$ fin gen graded chain complex over $\mathbb{Z}/2$

$CF^-(H)$ fin gen free graded chain complex over $\mathbb{Z}/2[u]$, $\deg u = -2$

The chain htpy type of these complexes will preserve H moves, thus

$$\left.\begin{array}{l} \widehat{HF}(H) = H_*(\widehat{CF}(H)) \\ HF^-(H) = H_*(CF^-(H)) \end{array}\right\} \text{ are invariants of } Y.$$

• We will write HF° for $\circ = \wedge, -$.

Properties :

1) $HF^\circ(Y) = \bigoplus_{\substack{s \in spin^c(Y) \\ \updownarrow 1-1 \\ H^2(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})}} HF^\circ(Y,s)$, splitting coming from the level of chain complexes

$\widehat{HF}(Y,s)$ is a fin gen graded v.s over $\mathbb{Z}/2$

$HF^-(Y,s)$ is a fin gen graded module over $\mathbb{Z}/2[u]$, so free part $\oplus$ torsion part, ie

$$HF^-(Y,s) \cong \Big(\bigoplus_i \mathbb{Z}/2[u]_{(d_i)}\Big) \oplus \Big(\bigoplus_j \mathbb{Z}/2[u]\big/(u^{n_j})_{(e_j)}\Big)$$

d_i, e_j denote grading of $1 \in \mathbb{Z}/_2[u]$. Why u^{n_j}? It should be a polynomial, but such a polyn has to be homogeneous in degree because I am saying $HF^-(Y,s)$ is graded. So since $\deg u = -2$, every homogeneously graded polyn is a monomial, so only u^{n_j}.

Furthermore, if Y is a rational homology sphere S^3, then $HF^-(Y,s)$ has exactly one free summand :

$$HF^-(Y,s) \cong \underbrace{\mathbb{Z}/2[u]_{(d)}}_{\text{the } d\text{-invariant}} \oplus \underbrace{\Big(\bigoplus \mathbb{Z}/2[u]_{(e_j)}\big/(u^{n_j})\Big)}_{=:\ HF_{red}(Y,s)}$$

<u>Definition</u>: A $\mathbb{Q}HS^3$ (rat hom sph) Y is an <u>L-space</u> if $HF_{red}(Y,s)=0$ for all $s \in \mathrm{spin}^c(Y)$.

Alternatively, it is a 3-mfd which has the same HF-homology as a lens-space

2) Relation between $-$ & $\wedge$ flavours? On the chain level, there is a ses

$$0 \to CF^-(Y,s) \xrightarrow{\cdot u} CF^-(Y,s) \to \widehat{CF}(Y,s) \to 0$$

thus an exact triangle

$$\begin{tikzcd} HF^-(Y,s) \arrow[rr, "\cdot u"] & & HF^-(Y,s) \arrow[dl] \\ & \widehat{HF}(Y,s) \arrow[ul] & \end{tikzcd}$$

3) If Y is a $\mathbb{Q}HS^3$, then $\dim \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$

and $=$ $\Leftrightarrow$ Y is an L-space

4) <u>Künneth formula</u>:

chain htpy eq

$$CF^\circ(Y_1 \# Y_2, s_1 \# s_2) \simeq CF^\circ(Y_1, s_1) \otimes_{\mathbb{Z}/2} CF^\circ(Y_2, s_2).$$

5) <u>Cobordism map</u>: If W^4 is a smooth compact 4-mfd with

$$\partial W^4 = -Y_0 \amalg Y_1$$

equipped with a spin^c-structure s, then it induces a map

$$F^\circ_{W,s} : HF^\circ(Y_0, s|_{Y_0}) \longrightarrow HF^\circ(Y_1, s|_{Y_1})$$

6) <u>Surgery exact triangle</u> : let Z be a compact 3-mfld with $\partial Z \cong T^2$, let $\gamma_0, \gamma_1, \gamma_\infty$ be simple closed curves on ∂Z st

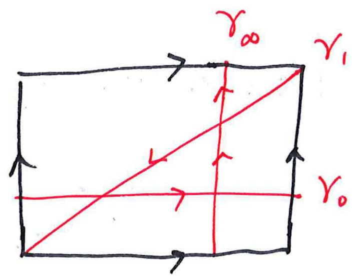
$$-1 = \#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma_\infty) = \#(\gamma_\infty \cap \gamma_0)$$

let $Y_i = Z \cup (S^1 \times D^2)$ st $\gamma_i = \{pt\} \times \partial D^2$ (ie perform Dehn filling along each of these curves)

then there is an exact triangle

$$\begin{array}{ccc} \widehat{HF}(Y_0) & \longrightarrow & \widehat{HF}(Y_1) \\ & \nwarrow \quad \swarrow & \\ & \widehat{HF}(Y_\infty) & \end{array}$$

<u>Remark</u> : Analogous exact triangle for HF^- but one needs to work over $\mathbb{Z}/2[[u]]$.



Example: For $Z = S^3 - N(K)$ for a knot K, the previous ~~see~~ triangle becomes

$$\begin{tikzcd} \widehat{HF}(S^3_n(K)) \arrow[rr] & & \widehat{HF}(S^3_{n+1}(K)) \arrow[dl] \\ & \widehat{HF}(S^3) \arrow[ul] & \end{tikzcd}$$

The Heegaard Floer chain complex

Let $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, w)$ a H. diagram of genus g, α's, β's intersecting transversally. Let

$$\mathrm{Sym}^g(\Sigma) := \overbrace{\Sigma \times \cdots \times \Sigma}^{g} / S_g \leftarrow \text{sym gp of } g \text{ elmts}$$

$$= \text{unordered } g\text{-tuples of pts in } \Sigma$$

Fact: This is a smooth manifold.

- Consider the following subspaces of $\mathrm{Sym}^g(\Sigma)$:

$$\left.\begin{aligned} \mathbb{T}_\alpha &= \alpha_1 \times \cdots \times \alpha_g \\ \mathbb{T}_\beta &= \beta_1 \times \cdots \times \beta_g \end{aligned}\right\} \ \tfrac{1}{2}\text{-dim subspaces}$$

$$V_w = \{w\} \times \mathrm{Sym}^{g-1}(\Sigma) \qquad (\text{codim } 2)$$

$\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ 0-dim

• It will turn out that $\widehat{CF}(H)$ is generated over $\mathbb{Z}/2$ by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

<u>Definition</u>: let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. A <u>Whitney disk</u> from x to y is a continuous map

$$\phi : \mathbb{D} \longrightarrow Sym^g(\Sigma)$$

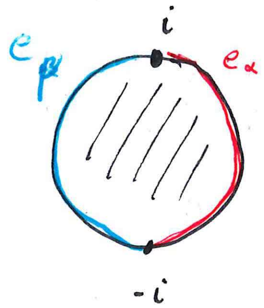
st

i) $\phi(-i) = x$

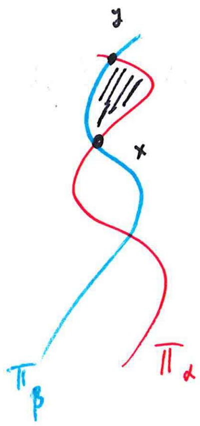
ii) $\phi(i) = y$

iii) $\phi(e_\alpha) \subset \mathbb{T}_\alpha$

iv) $\phi(e_\beta) \subset \mathbb{T}_\beta$



"<u>Cartoon picture</u>":



let $\pi_2(x, y)$ = set of hty classes of Whitney disks from x to y

• A choice of complex structure on Σ induces one on $Sym^g(\Sigma)$.

let $\mathcal{M}(\phi)$ = moduli space of holomorphic representatives of ϕ.

• Generically, $\mathcal{M}(\phi)$ will be a smooth manifold. Let

$\mu(\phi)$ = expected dimension of $\mathcal{M}(\phi)$

$n_w(\phi) = \#(\phi(\mathbb{D}) \cap V_w)$ alg intersection number

• There is a $\mathbb{R}$-action on $\mathcal{M}(\phi)$ (complex automorphisms that fix $\pm i$)

If $\mu(\phi) = 1$, then $\hat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$ is a compact 0-dim mnfd.

<u>Definition</u>: Let $\widehat{CF}(H) := \mathbb{Z}/2[\mathbb{T}_\alpha \cap \mathbb{T}_\beta]$ with differential

$$\partial : \widehat{CF}(H) \to \widehat{CF}(H),$$

$$\partial(x) := \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_w(\phi)=0}} (\# \hat{\mathcal{M}}(\phi)) \cdot y$$

"The differential counts holomorphic disks $x \to y$ in the complement of V_w"

* <u>Theorem</u> (Ozsváth - Szabó):

1) $\partial^2 = 0$

2) If H is a H diag for a 3-mfd Y, then $\widehat{HF}(H) := H_*(\widehat{CF}(H))$ is an invariant of Y.

Remarks: 1) There is a splitting on a chain level

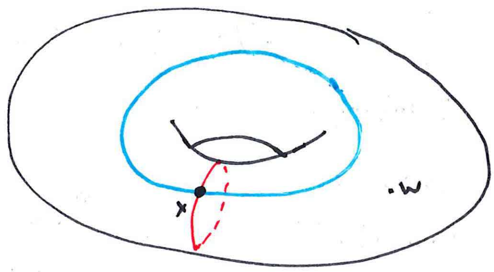
$$\hat{CF}(H) = \bigoplus_{s \in spin^c(Y)} \hat{CF}(H, s)$$

2) There is a relative $\mathbb{Z}$-grading: given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\phi \in \pi_2(x, y)$, then

$$gr(x) - gr(y) := \mu(\phi) - n_w(\phi).$$

Actually this relative $\mathbb{Z}$-graded can be lifted to an absolute $\mathbb{Q}$-grading from cobordism maps plus normalization of $HF^\circ(S^3)$

Example:

$H =$ 

$\Sigma = Sym^1(\Sigma)$

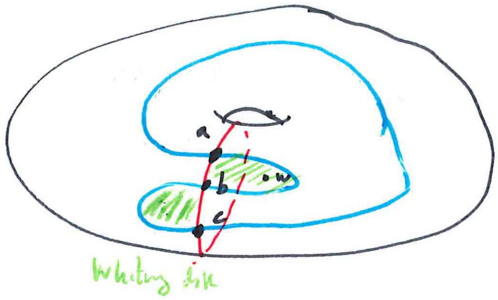
H Heegaard spl of S^3

$\mathbb{T}_\alpha = \alpha$, $\mathbb{T}_\beta = \beta$, so

$\hat{CF}(H) = \mathbb{Z}/2 \cdot x$ & $\partial = 0$ because the diff lower grading by 1, but there is an only gen!

So $\hat{HF}(S^3) \cong \mathbb{Z}/2_{(0)}$

Example :



Now $\widehat{CF}(H) = \mathbb{Z}/2 \cdot a \oplus \mathbb{Z}/2 \cdot b \oplus \mathbb{Z}/2 \cdot c \cong (\mathbb{Z}/2)^3$

$\partial c = b$

$\partial a = 0$ since the basepoint lies on the Whitney disk, $n_w(\phi) \neq 0$.

$\partial b = 0$.

$\Rightarrow H_*(\widehat{CF}(H)) \cong \mathbb{Z}/2_{(0)}$. (same)

· Similarly,

Definition : $CF^-(H) := \mathbb{Z}/2[u][\mathbb{T}_\alpha \cap \mathbb{T}_\beta]$ with

$$\partial : CF^-(H) \to CF^-(H)$$

$$\partial(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi) = 1}} \left(\# \hat{\mathcal{M}}(\phi) \cdot u^{n_w(\phi)}\right) \cdot y .$$

Theorem (OS): $\partial^2 = 0$, and the chain htpy type of this complex is independent of the choice of H diagram and choice of complex structure on Σ.

Example: The previous example, for $CF^-(H)$ is

$$CH^-(F) = (\mathbb{Z}/2[u])^3 \quad \text{and} \quad \begin{aligned} \partial a &= u \cdot b \\ \partial b &= 0 \\ \partial c &= b \end{aligned}$$

$$\Rightarrow H_*(CF^-(H)) \cong \mathbb{Z}/2[u].$$

Remark: $\widehat{CF}(H)$ can be recovered from $CF^-(H)$ by setting $u = 0$.

Knot Heegaard Floer homology

Starting point: doubly pointed Heegaard diagram $H = (\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ for $K \subset S^3$.

Goal: Build a chain complex $CFK(H)$ whose homology is an invariant of K.

• Simplest flavour:

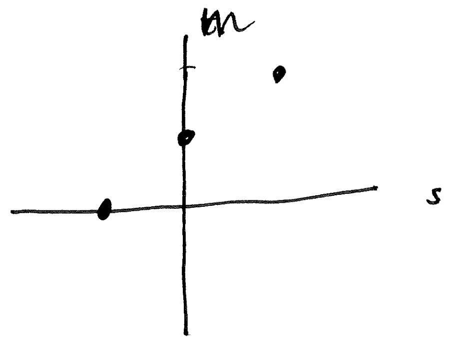
$$\widehat{HFK}(K) = \bigoplus_{m,s} HFK_m(K, s)$$

a bigraded vector space. It is the simplest version which categorifies Alexander:

*Theorem (Ozsváth–Szabó, Rasmussen): The graded Euler char of knot Floer homology is the Alexander polynomial:

$$\boxed{\Delta_K(t) = \sum_{m,s} (-1)^m \dim \widehat{HFK}_m(K, s) \cdot t^s.}$$

Example: $3_1 = -T_{2,3}$ left-handed trefoil



$\bullet = \mathbb{Z}/2$

$$\chi(\ ) = t^{-1} - 1 + t = \Delta_{3_1}(t)$$

$\widehat{HFK}$ not only categorifies ~~some~~ Alexander, but also strengthens some of its properties:

Properties:

1) let $\Delta_K(t) = a_0 + \sum_{s \geq 0} a_s (t^s + t^{-s})$ sym Alexander

Recall that $g(K) \geq \max\{s : a_s \neq 0\} = \frac{1}{2}$ breadth $\Delta_K(t)$

Theorem (O-S): $\widehat{HFK}$ detects the genus:

$$g(K) = \max\{s : \widehat{HFK}(K,s) \neq 0\}$$

(so eg the genus of the 3_1 is 1)

2) Recall: K fibered $\Rightarrow$ $a_{g(K)} = \pm 1$.

Theorem (Ghiggini, Ni): $\widehat{HFK}$ detects fiberedness:

$$K \text{ fibered} \iff \widehat{HFK}(K, g(K)) \cong \mathbb{Z}/2$$

• There are only two genus one, fibered knots : 3_1 & 4_1 . So

Corollary : $\widehat{HFK}$ detects 3_1 and 4_1 (and the unknot since it's the only knot of genus 0).

Construction of $\widehat{CFK}$:

Consider $\mathbb{Z}/2\,[u,v]$ as a bigraded ring where if $gr = (gr_u, gr_v)$ then $gr(u) = (-2, 0)$, $gr(v) = (0,-2)$. Let $A = \frac{1}{2}(gr_u - gr_v)$.

Definition : Let $\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ be a doubly pointed HD diagram for a knot K and let

$$CFK_{\mathbb{Z}/2[u,v]}(K) := (\mathbb{Z}/2[u,v])\,[\mathbb{T}_\alpha \cap \mathbb{T}_\beta].$$

with differential

$$\partial(x) := \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \; \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi) = 1}} \left(\# \hat{\mathcal{M}}(\phi) \cdot u^{n_w(\phi)} \cdot v^{n_z(\phi)} \right) \cdot y$$

Theorem (Ozsváth-Szabó, Rasmussen):

1) $\partial^2 = 0$

2) The chain htg type of $CFK(K)$ is a invariant of K.

Remark. Here we have a relative bigrading: given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(x,y)$

$$gr_u(x) - gr_u(y) = \mu(\phi) - 2n_w(\phi)$$

$$gr_v(x) - gr_v(y) = \mu(\phi) - 2n_z(\phi).$$

In particular, ∂ lowers gr_v, gr_u by 1, and preserves the Alexander grading

$$A := \frac{1}{2}(gr_u - gr_v).$$

Remark. Set $v=1$ and forget gr_v, ie, forget the z basepoint. So what we get is a (pointed) H. diagram for S^3. So if we take homology since $\mathbb{Z}/2[u,v]$ becomes $\mathbb{Z}/2[u]$, ~~it~~ we get $CF^-(S^3)$. Therefore

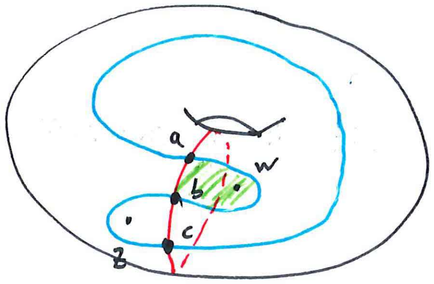
$$H_*(CFK_{\mathbb{Z}/2[u,v]}|_{v=1}(K)) \cong \mathbb{Z}/2[u]_{(0)}$$

This determines the absolute grading gr_u. Similarly

$$H_*(CFK_{\mathbb{Z}/2[u,v]}|_{u=1}(K)) \cong \mathbb{Z}/2[v]_{(0)}$$

determines the abs v-grading

Example :



$\partial a = u \cdot b$

$\partial b = 0$

$\partial c = v \cdot b$

To determine the absolute u-grading, set $v=1$. Then the last diff is $\partial c = b$, so

so $H_* = \ker \partial / \operatorname{Im} \partial = \dfrac{\langle b, a+uc \rangle}{\langle b \rangle} = \langle a+uc \rangle$

$\Rightarrow \quad gr_u (a+uc) = 0$. so $gr_u\, a = 0$ & $gr_u\, c = 2$

Also, since ∂ lowers gr_u deg by 1, so $gr_u\, vb = 1$, so $gr_u b = 1$.

	gr_u	gr_v	A
a	0	2	-1
b	1	1	0
c	2	0	1

Now we can compute the HF homology of -3_1 : (left handed)

$$H_*(CFK_{\mathbb{Z}/2[u,v]}(-3_1)) \cong \frac{\langle b, va+uc \rangle}{\langle ub, vb \rangle} \cong$$

$$\cong \underset{\substack{\uparrow \\ \text{gen by } va+uc}}{\mathbb{Z}/2[u,v]_{(0,0)}} \oplus \underset{\substack{\uparrow \\ \text{gen by } b}}{\mathbb{Z}/2_{(1,1)}}$$

Other flavours : HFK(K) is the most robust version, we can also have

$$\widehat{HFK}(K) = H_*\left(CFK_{\mathbb{Z}/2[u,v]/_{u=v=0}}(K)\right)$$ is a bigraded vs over $\mathbb{Z}/2$,

the m-grading comes from gr_u, s-grading comes from the Alexander grading.

Example : For the 3_1 left handed trefoil, if $u=0=v$, then the differentials become $\partial a = \partial b = \partial c = 0$. So

$$\widehat{HFK}(3_1) \cong \mathbb{Z}/2 \cdot a \oplus \mathbb{Z}/2 \cdot b \oplus \mathbb{Z}/2 \cdot c$$

This matches the previous example which recovers Alexander.

Another flavour : Set $v=0$. Then get $HFK^-(K) = H_*\left(CFK_{\mathbb{Z}[u,v]}|_{v=0}(K)\right)$

- There is a Künneth formula

$$CFK_{\mathbb{Z}/2[u,v]}(K_1 \# K_2) \simeq CFK_{\mathbb{Z}/2[u,v]}(K_1) \otimes_{\mathbb{Z}/2[u,v]} CFK_{\mathbb{Z}[u,v]}(K_2)$$

- The Floer complex behaves well w.r.t. mirror images of knots:

$$CFK_{\mathbb{Z}/2[u,v]}(\overline{K}) \simeq CFK_{\mathbb{Z}[u,v]}(K)^* \left(= \mathrm{Hom}(\quad , \mathbb{Z}/2[u,v])\right).$$

<u>Pf sketch</u>: If $H = (\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ H. diag for K, then

$H' = (-\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$ is a H. diagram for $\overline{K}$. There is a canonical identification

between gen's of $CFK(H)$ & $CFK(H')$: $\phi \in \pi_2(x, y)$ in $CFK(H)$

corresponds to $\phi' \in \pi_2(y, x)$ in $CFK(H')$, and going the other way is exactly

taking the dual. $\square$

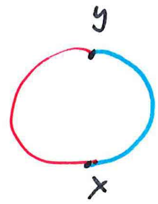
- <u>Important</u>: HFK does not detect ~~...~~ orientation reversal:

$$CFK_{\mathbb{Z}/2[u,v]}(K) \simeq CFK_{\mathbb{Z}/2[u,v]}(-K)$$

<u>Pf</u>: If $H = (\Sigma, \alpha, \beta, w, z)$ for K, then $H' = (-\Sigma, \beta, \alpha, w, z)$ is for $-K$.

There is a canonical identification between $CFK(H)$ and $CFK(H')$:

$\mathrm{Sym}^g(\Sigma)$ 



$\mathrm{Sym}^g(-\Sigma)$.

$\square$

<u>Note</u>: If $H = (\Sigma, \alpha, \beta, w, z)$ for K, then swapping w, z, $H' = (\Sigma, \alpha, \beta, z, w)$ is

a H. diagram for $-K$. In particular the previous result tells that $CFK_{\mathbb{Z}/2[u,v]}(K)$

is "symmetric" in u, v (up to chain htpy).

Q. How to compute HFK?

- If K has a genus 1 doubly pointed H. diagram then it can be computed from the definition, using Riemann mapping theorem to figure out the disks.

Q. How many knots with doubly pted H diagrams are there?

All torus knots, but there is a proper subset of knots that has a genus 1 H. diag.

- There is a variation of H diagram called grid diagram (Manolescu, O-S Thurston), a "multipointed" H. diagram, purely combinatorial, which produces a large chain complex ($n!$ gen's for a $n \times n$ grid). Eg: For 3_1, 5×5 grid $\Rightarrow$ 125 gen's....

- There is a fast algorithm for the $CFK_{\mathbb{Z}/2[u,v]}|_{u=0=v}$ complex, using "bordered algebras".

- If K is alternating, then $CFK_{\mathbb{Z}/2[u,v]}(K)$ is completely determined by $\Delta_K(t)$ and $\sigma(K)$: $\widehat{HFK}$ is supported on the diagonal $m = s + \frac{\sigma(K)}{2}$ wrt gr_v and Alex gr.

- Similarly, if K admits an L-space surgery (ie a $\mathbb{Q}HS^3$ with the same HF homology as a lens space), then $CFK_{\mathbb{Z}/2[u,v]}$ is determined by $\Delta_K(t)$. Eg: all torus knots.

HF and knot surgery

- The homogeneous polynomials in $\mathbb{Z}/2[u,v]$ is given by

$$A = \frac{1}{2}(gr_u - gr_v)$$

$gr_v \backslash gr_u$	$\cdots$	-4	-2	0
0	$\cdots$	u^2	u	1
-2		u^2v	uv	v
-4		u^2v^2	uv^2	v^2
$\vdots$				$\vdots$

$A=-1$ $\quad A=0 \quad A=1$

$A(u) = -1$

$A(v) = 1$

$A(uv) = 0$

Let $w := uw$.

- Recall that in $CFK_{\mathbb{Z}/2[u,v]}$, ∂ preserves the Alexander grading.
- As a chain complex of $\mathbb{Z}/2[w]$-modules, it splits as

$$CFK_{\mathbb{Z}[u,v]}(K) \cong \bigoplus_s \underbrace{CFK_{\mathbb{Z}/2[u,v]}(K,s)}_{\text{Alexander grading pieces}}$$

ie, ∂ respects the splitting

<u>Recall</u>: $H_1(S^3_n(K);\mathbb{Z}) \cong \mathbb{Z}/n \overset{1-1}{\longleftrightarrow} \mathrm{spin}^c(S^3_n(K))$

- This implies that

$$HF^-(S^3_n(K)) \cong \bigoplus_{[s]\in\mathbb{Z}/n} HF^-(S^3_n(K),[s])$$

• Relation between HF & HFK ?

* <u>Theorem</u> (Ozsváth - Szabó, Rasmussen) : let $K \subset S^3$, $N \geq 2g(K)-1$, and $|s| \leq \lfloor \frac{N}{2} \rfloor$. Then

$$\underset{w=u}{HF^-(S^3_N(K), [s])} \cong \underset{w=uv}{H_*\left(CFK_{\mathbb{Z}/2[u,v]}(K,s)\right)}$$

as relatively graded $\mathbb{Z}/2[w]$ - modules.

<u>Remark</u>. Can upgrade this to a isomorphism of absolutely graded modules.

Corollary. $$\widehat{HF}\,(S^3_N(K), [s]) \cong H_*\left(CFK_{\mathbb{Z}/2[u,v]}\big|_{uv=0}(K,s)\right).$$

<u>Example</u>: $Y = S^3_{+1}(3_1)$. (and $g(3_1)=1$)

Recall that $CFK_{\mathbb{Z}/2[u,v]}(3_1) \cong \langle a, b, c\rangle_{\mathbb{Z}/2[u,v]}$ with

	A	∂
a	-1	$u \cdot b$
b	0	0
c	1	$v \cdot b$

So the 0-part of the Alex grading is

$$CFK_{\mathbb{Z}/2[u,v]}(3_1, 0) \cong \langle va, b, u\cdot c\rangle_{\mathbb{Z}/2[w]} \quad (w=uv)$$

$\left\{ \begin{array}{l} v \text{ raises Alex grad by } 1 \\ u \text{ lowers Alex gr by } 1 \end{array} \right.$

Boundary map? $\begin{cases} \partial(va) = uvb = wb \\ \partial(b) = 0 \\ \partial(uc) = uvb = wb \end{cases}$

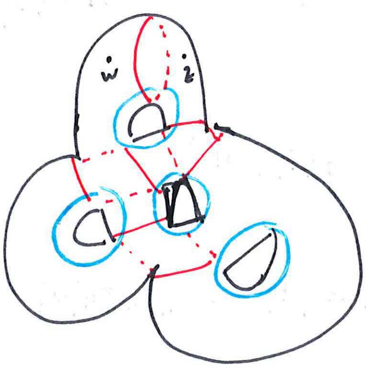
$\exists$ spin^c-struct on $Y = S^3_1(3_1)$ because $H_1(Y;\mathbb{Z}) \cong 0$.

So $HF^-(Y) \cong H_*\left(CFK^-_{\mathbb{Z}/2[u,v]}(3_1, 0)\right) \cong \langle va + uc, b\rangle / \langle wb\rangle$

$\cong \mathbb{Z}/2[w] \oplus \mathbb{Z}/2$

gen by $va+uc$ gen by b $\langle b\rangle/\langle wb\rangle$

<u>Idea of the proof</u>: Consider ~~the~~ H. diagram for a knot



How to build the 3-mfd from this?

You thicken your surface, you attach disks along the β-circles in $\Sigma \times 1$ and fill-in the boundary with a 3-ball, and along $\Sigma \times 0$ you attach disks along the α-circles and fill-in the remaining boundary with a 3-ball.

<u>Exercise</u>: If you attach disks along the α-circles except the meridian (in the interior) and you do not glue the 3-ball, then what you get is the knot complement.

One then uses a longitude to perform surgery. $\square$

• Now consider the $\mathbb{Z}/2[w]$-module

Alex grading

$$B_s = \left(CFK_{\mathbb{Z}/2[u,v]}(K) \otimes_{\mathbb{Z}/2[v]} \mathbb{Z}/2[v,v^{-1}] \, , \, s \right)$$

and consider the maps "multipl by" $v: B_s \to B_{s+1}$, $v^{-1}: B_{s+1} \to B_s$

Since one is the inverse of the other, they are iso, ie, $B_s \cong B_t$ $\forall s,t \in \mathbb{Z}$.

Even more, $B_s \simeq CF^-(S^3)$. The upshot is that

Corollary: $CFK_{\mathbb{Z}/2[u,v]}(K) \otimes_{\mathbb{Z}/2[v]} \mathbb{Z}/2[v,v^{-1}] \cong \bigoplus_{s\in\mathbb{Z}} B_s$.

$$\begin{tikzcd}
\vdots \\
B_{s+1} \arrow[loop right, "\cdot v"] \arrow[d, bend right, "\cdot v^{-1}"'] \\
B_s \arrow[loop right, "\cdot w"] \arrow[u, bend right, "\cdot v"'] \arrow[d, bend right, "\cdot v^{-1}"'] \\
B_{s-1} \arrow[loop above, "\cdot v"] \arrow[u, bend right, "\cdot v"'] \\
\vdots
\end{tikzcd}$$

Moreover, these complexes are sym $u \leftrightarrow v$, so

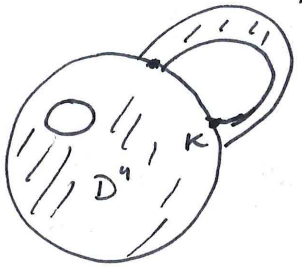
$$B_s \overset{\Phi_s}{\simeq} \left(CFK_{\mathbb{Z}/2[u,v]}(K) \otimes_{\mathbb{Z}/2[u]} \mathbb{Z}/2[u,u^{-1}] \, , \, s \right)$$

Theorem (O-S): The inclusion $i : CFK_{\mathbb{Z}/2[u,v]}(K,s) \to B_s$ induces the 2-handle cobordism map

$$i_* = F^-_{W,s} : HF^-(S^3_N(K), s) \to HF^-(S^3)$$

$(N \geq 2g-1)$

where $W = -(N\text{-trace}(K) - B^4) =$

$=$ 

N-framed 2-handle along K

(with reversed orientation)

so it is a cobordism from $S^3_N(K)$ to S^3.

• That is, an algebraic procedure (inverting v) has a geometric meaning! (the cobordism map).

Warning: For tech reasons, work over $\mathbb{Z}/2[u,v]$ instead of $\mathbb{Z}[u,v]$. Same holds for the B_s's.

• Consider the diagram

$$\begin{tikzcd}
CFK_{\mathbb{Z}/2[u,v]}(K) \arrow[r, "i_v"] \arrow[dr, "i_u"'] & CFK_{\mathbb{Z}/2[u,v]}(K) \otimes_{\mathbb{Z}/2[v]} \mathbb{Z}/2[v,v^{-1}] \\
& CFK_{\mathbb{Z}/2[u,v]}(K) \otimes_{\mathbb{Z}/2[v]} \mathbb{Z}/2[v,v^{-1}] \arrow[u, "\cong"', "\phi = \bigoplus_s \phi_s"]
\end{tikzcd}$$

and define the chain map

$$\Upsilon_n := i_v + v^n \circ \phi \circ i_u \quad : CFK_{\mathbb{Z}/2[[u,v]]}(K) \longrightarrow CFK_{\mathbb{Z}/2[[u,v]]}(K) \otimes_{\mathbb{Z}/2[v]} \mathbb{Z}/2[v,v^{-1}].$$

Recall from homological algebra that given a chain map $f: C_* \to D_*$, the <u>mapping cone</u> of f is $\mathrm{Cone}(f)_* := C_* \oplus D_*$ with

$$\partial(x,y) = (\partial^C x,\ f(x) + \partial^D y)$$

(over char 2 signs do no matter).

<u>Theorem</u> (O-S, Manolescu-Ozsváth): Let $n \in \mathbb{Z}$. Then

$$HF^-(S^3_n(K)) \otimes_{\mathbb{Z}/2[w]} \mathbb{Z}/2[[w]] \cong H_*(\mathrm{Cone}(\Upsilon_n)).$$

<u>Remarks</u>: 1) This can be upgraded to iso of abs graded $\mathbb{Z}/2[w]$-mods,

2) Similar formula for $\mathbb{Q}$-surgery.

<u>Corollary</u>: Let $n \in \mathbb{Z}$. Then

$$\widehat{HF}(S^3_n(K)) \cong H_*(\mathrm{Cone}(\hat{\Upsilon}_n))$$

where ^ are added to everything: $\hat{\Upsilon}_n = \hat{i}_v + v^n \circ \phi \circ \hat{i}_v$. Here hat means quotienting by uv:

$$\hat{i}_v : CFK_{\mathbb{Z}/2[u,v]/uv=0}(K) \longrightarrow CFK_{\mathbb{Z}/2[u,v]/uv=0}(K) \otimes_{\mathbb{Z}/2[v]} \mathbb{Z}/2[v,v^{-1}].$$

$\vdots \qquad\qquad \vdots$

Applications of the core constr.

Conjecture (Cosmetic surgery): Let K be a non-trivial knot in S^3. If

$$S^3_r(K) \underset{\substack{\text{orient}\\\text{pres.}}}{\cong} S^3_{r'}(K),$$

then $r = r'$. $(r, r' \in \mathbb{Q})$.

Theorem (Ni-Wu, 2010): Suppose K is a non-triv knot in S^3 with $S^3_r(K) \cong S^3_{r'}(K)$. Then $r = -r' = p/q$ where p, q coprime and $q^2 \equiv -1 \pmod p$.

Remark: Hanselman (2019) recently improved this to

$$r = -r' = \pm 2 \quad \text{or} \quad \pm \frac{1}{q}$$

using bordered Floer homology.

- Surgery obstructions:

The Thm by Lickorish-Wallace says that every 3-mfd can be obtained by surgery on a link.

Q: Can every 3-mfd be obtained by surgery on a knot?

A: No: $H_1(S^3_{p/q}(K); \mathbb{Z}) \cong \mathbb{Z}/p$. (cyclic)

Eg: $H_1(\mathbb{RP}^3 \# \mathbb{RP}^3; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$

• What if one forgets about this obstruction, i.e. looking at cyclic-hom. mnfds?

<u>Theorem</u> (Auckly): There exist integral homology spheres that are not obtained by surgery on a knot in S^3 (hyperbolic, toroidal..)

• Using the cov formula one can also get this:

<u>Theorem</u> (Hom - Karakurt - Lidman): The $\mathbb{Z}HS^3$'s $\Sigma(2k, 4k-1, 4k+1)$, $k \geq 4$ are not obtained by surgery on a knot on S^3.

<u>Remark</u>: We have been using the existence of a splitting

$$\widehat{CF}(H) \cong \bigoplus_{s \in spin^c(Y)} \widehat{CF}(H, s)$$

for H H. diag of Y, that we never got to define. Actually, one could perfectly avoid to talk about spin^c-structures and just talk of homology classes as for 3-manifolds Y one has a bijection

$$spin^c(Y) \cong |H_1(Y;\mathbb{Z})|.$$

The previous splitting comes from partitioning $\mathbb{T}_\alpha \cap \mathbb{T}_\beta = \coprod_{s \in H_1(Y;\mathbb{Z})} (\mathbb{T}_\alpha \cap \mathbb{T}_\beta)_s$ as disj union of s-pieces, as follows: let $x = \{x_1, .. x_g\}$, $y = \{y_1 .. y_g\} \in \vec{\alpha} \cap \vec{\beta}$. Choose a collection of arcs $a \subset \vec{\alpha}$ st boundary $(a) = y_1 + \dots + y_g - x_1 \dots - x_g \in H_0(\Sigma;\mathbb{Z})$ and choose $b \subset \vec{\beta}$ st boundary (b) $= x_1 + .. + x_g - y_1 \dots - y_g \in H_0(\Sigma;\mathbb{Z})$. Then $a-b$ is a 1-cycle in Σ. Define $\varepsilon(x,y) := i_*[a-b] \in H_1(Y;\mathbb{Z})$, $i: \Sigma \hookrightarrow Y$.

There is an isomorphism $\displaystyle H_1(Y;\mathbb{Z}) \cong \frac{H_1(\mathrm{Sym}^g(\Sigma);\mathbb{Z})}{H_1(\mathbb{T}_\alpha;\mathbb{Z}) \oplus H_1(\mathbb{T}_\beta;\mathbb{Z})}$,

and under this iso, given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, can consider $\varepsilon(x,y)$. An obstruction to the existence of a Whitney disk from ~~x~~ to y is that $\varepsilon(x,y) \neq 0$. So we want $\varepsilon(x,y)=0$.

to have a ~~D~~ Whitney disk. For every $s \in H_1(Y;\mathbb{Z})$, let $x, z \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ st $\varepsilon(x,z)=s$.

Then let $(\mathbb{T}_\alpha \cap \mathbb{T}_\beta)_s := \{ x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta : \varepsilon(x,z) = s \}$. Given $x, y \in (\mathbb{T}_\alpha \cap \mathbb{T}_\beta)_s$, $\varepsilon(x,y)=0$ (because of the additivity of ε: $\varepsilon(x,y)+\varepsilon(y,z)=\varepsilon(x,z)$). In other words, there is an eq relation ~~on~~ on $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$: $x \sim y \Leftrightarrow \varepsilon(x,y)=0$, and ~~the~~ every class corresponds to a element in $H_1(Y;\mathbb{Z})$.

In the original work of O-S, they define ∂ directly using spin^c-structures.

<u>Appendix</u>: Spin^c-structures on 3-manifolds.

- Recall that $SO(3) \cong SU(2)/\pm 1 \cong U(2)/U(1)$, where $U(1)$ lies in $U(2)$ as the diagonal. The projection $U(2) \to SO(3)$ is a $U(1)$-principal bundle. (note $U(1) \cong S^1$ as a ~~top~~ space). If X is a CW-complex, then

$$\left\{ \begin{matrix} \text{principal} \\ U(1)\text{-bundles} \\ \text{over } X \end{matrix} \right\} = [X, BU(1)] = [X, K(\mathbb{Z},2)] = H^2(X)$$

Define $\mathrm{Spin}(3) := SU(2)$ and $\mathrm{Spin}^c(3) = U(2)$.

• Let M be a closed oriented 3-manifold. Recall that it admits a unique smooth structure. Endow M with a Riemannian metric and consider the associated principal $SO(3)$-bundle of oriented frames $\rho : Fr \to M$

<u>Definition</u>: A <u>spin^c-structure</u> on M is a lift of ρ to a principal $U(2)$-bundle.

More precisely, it is a (iso class of a) pair (F, α), where $F \to M$ is a principal $U(2)$-bundle and $\alpha : F/U(1) \xrightarrow{\cong} Fr$ is a iso of principal $SO(3)$-bundles.

$$\begin{tikzcd} & F \arrow[d] \\ Fr \arrow[ur, dashed] \arrow[r, "\rho"] & M \end{tikzcd}$$

<u>Fact</u>: Every orientable 3-manifold is parallelisable.

Thus the set $Spin^c(M) \neq \emptyset$.

• The group $H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$ acts on $spin^c(M)$ as follows: given a principal $U(1)$-bundle $E \to M$ (representing an elmt in $H^2(M;\mathbb{Z})$) and $F \to M$ a spin^c-structure, then $U(1)$ acts on $E \times F$ diagonally $\Rightarrow$ get $U(2)$-bundle $(E \times F)/U(1) \to M$. It is easy to see that this action of $H_1(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$ is free and transitive so $spin^c(M)$ is a homogeneous set over this group, ie, there is a bijection

$$spin^c(M) \cong |H_1(M;\mathbb{Z})| \cong |H^2(M;\mathbb{Z})|.$$

• Relation with (geometric) Euler structures

A vector field $u \in \mathfrak{X}(M)$ is called <u>non-singular</u> if it vanishes nowhere. Since $\dim M = 3$ odd, then $\chi(M) = 0$, so by the Poincaré–Hopf theorem, there are nonsingular vector fields.

Call two nonsing $u \sim v$ if for some $p \in M$, the restrictions of u, v to $M - \{p\}$ are homotopic in the class of non-sing vector fields on M.

The set of equivalent classes is denoted by $\mathrm{vect}(M)$ and are called <u>geometric Euler structures</u>

<u>Proposition</u>: There is a canonical $H_1(M;\mathbb{Z})$-equivariant bijection

$$\mathrm{vect}(M) \cong \mathrm{spin}^c(M).$$

<u>Remark</u>: $\mathrm{vect}(M)$ can be described in terms of a CW structure: consider $\hat{M}$ the maximal abelian cover of M. A family of open cells $\hat{e} = \{\hat{e}_i\}$ in $\hat{M}$ is <u>fundamental</u> if over each open cell in M lies exactly one $\hat{e}_i$ in $\hat{e}$. Let $\mathrm{Eul}(M)$ be the set of fundamental families modulo certain equivalence relation. These are called <u>combinatorial Euler structures</u>. Then one has

$$\mathrm{Eul}(M) \cong \mathrm{vect}(M).$$