Homology and cohomology of algebras

- Right A-modules are essencially left A^{op}-modules: if M is a right A-mod with multiplication map $M \otimes A \xrightarrow{\mu'} M$, then it is a left A^{op}-mod with multiplication map $A \otimes M \xrightarrow{\tau} M \otimes A \xrightarrow{\mu'} M$.

- A-B bimodule: left A-module M which has a right B-module structure compatible in the sense that $a(mb) = (am)b$.

$$A\text{-}B \text{ bimodule} = A \otimes B^{op}\text{-left module} = A^{op} \otimes B \text{ right module.}$$
$$amb \longleftrightarrow (a \otimes b) \cdot m \longleftrightarrow m(a \otimes b)$$

- Tensor products: If M is a A-B-bimodule and N a B-C bimodule, the tensor product $M \otimes_B N$ is in a natural way a A-C-bimodule.

- If $f: A \to B$ is a k-alg hom, then ~~f~~ endows B with structures of A-B bimodule ~~and~~ (denoted by ${}_fB$) and B-A-bimodule (denoted by B_f).

- Let Alg_k be the cat of k-algebras. Now let $\widetilde{Alg_k}$ be the cat with same objects but arrows $\mathrm{Hom}_{\widetilde{Alg_k}}(A,B) := \{$ iso class of A-B-bimodules $\}$. Composition is given by the tensor product as stated before. Id is ${}_{Id}A = A_{Id}$.

  There is a natural functor $Alg_k \to \widetilde{Alg_k}$ being Id on obj and sending f to ${}_fB$.

- Morita equivalence: Two algebras A, B are Morita equivalent if they are isomorphic in the cat $\widetilde{Alg_k}$. Explicitly, if $\exists$ a A-B-bimodule P st there is a B-A-bimodule Q together with bimodule isomorphisms

$$P \otimes_B Q \xrightarrow{\simeq} A \quad , \quad Q \otimes_A P \xrightarrow{\simeq} B$$

which expresses the same arrow in $\widetilde{Alg}_k$.

<u>Fact</u>: P, Q are projective over A & B.

<u>Fact</u>: $A \cong_{Morita} B \Rightarrow Z(A) \cong Z(B)$.

So for commutative algebras, Morita = isomorphism.

<u>Example</u>: For $q \geq 2$, $M_{q \times q}(A) \simeq A$.

<u>TRACE MAPS</u>

<u>Def</u>: Let R be a ring and G an ab gp. A <u>trace</u> is a gp hom $\tau: R \to G$ st

$\tau(rs) = \tau(sr)$ $\forall s, r \in R$. It should be clear that

$$\{\text{traces } \tau: R \to G\} \cong \mathrm{Hom}(R/[R,R], G) \quad (\text{group iso})$$

"The universal trace" is the projection map $R \to R/[R,R] =: H_0(R)$

<u>Examples</u>: 1) $H_0(R) = R$ if R commutative

2) $H_0(M_p(R)) \cong H_0(R)$

3) Let $\tau_n: V^{\otimes n} \to V^{\otimes n}$ be $v_1 \otimes \cdots \otimes v_n \mapsto v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$. Then

$$H_0(T(V)) = k \oplus V \oplus \bigoplus_{n \geq 2} \mathrm{Coker}(\mathrm{Id} - \tau_n).$$

4) Let $A_1(k) = \dfrac{F(x,y)}{\langle [x,y]=1 \rangle}$ the <u>Weyl algebra</u>. Then $H_0(A_1(k)) = 0$.

(this is because $A_1(k) = [A_1(k), A_1(k)]$. Indeed

$$[x, x^m y^n] = n \cdot x^m y^{n-1}, \quad [x^m y^n, y] = m\, x^{m-1} y^n \quad)$$

ALGEBRAIC K-THEORY

Lemma. Let R be a ring. A left R-module M is projective finite-generated iff $\exists\, n \geq 1$ and $e \in M_n(R)$ idempotent ($e^2 = e$) st $M \cong e \cdot R^n = \operatorname{Im} e$.

- Let $\mathrm{Proj}(R)$ be the set of iso classes of f.g. projective modules over R. Recall that the forgetful $\mathrm{Ab} \to \mathrm{CMon}$ has a left adjoint K called the Grothendieck construction

$$\begin{tikzcd} \mathrm{CMon} \arrow[r, bend left, "K"] \arrow[r, phantom, "\perp"] & \mathrm{Ab} \arrow[l, bend left, "U"] \end{tikzcd}$$

Def: The 0-th algebraic K-group of R is $K_0(R) := K(\mathrm{Proj}(R))$, i.e.,

$$K_0(R) = \frac{\mathbb{Z}[\mathrm{Proj}(R)]}{\langle [M] + [N] = [M \oplus N] \rangle}$$

Proposition: If R is PID, then $K_0(R) \cong \mathbb{Z}$.

- Let us construct a morphism $\tau : K_0(R) \to H_0(R)$. Let $P \in \mathrm{Proj}(R)$, so $P \cong \operatorname{Im} e$ for some idempotent matrix e. Let $\tau(P) := \operatorname{tr} e \in H_0(R) = R/[R,R]$.

Theorem: $T(P)$ only depends on the iso class of P, and it is additive,

$$T(P \oplus Q) = T(P) + T(Q)$$

Definition: The Hattori-Stallings trace is the unique gp hom

$$\tau : K_0(R) \to HH_0(R)$$

st $\tau([P]) = T(P)$.

HOCHSCHILD (CO)HOMOLOGY

Def: Let A be an algebra and let M be a A-A-bimodule $= (A \otimes A^{op})$ left module.

The Hochschild (co)homology groups of A w/ coeff. in M are

$$H_n(A; M) := \mathrm{Tor}_n^{A \otimes A^{op}}(M, A) \quad , \quad H^n(A, M) := \mathrm{Ext}^n_{A \otimes A^{op}}(A, M)$$

By definition,

$$H_0(A, M) = M \otimes_{A \otimes A^{op}} A \quad , \quad H^0(A, M) = \mathrm{Hom}_{A \otimes A^{op}}(A, M)$$

- Denote $[A, M] := \mathrm{span}_k \{ am - ma : a \in A,\ m \in M \}$

$$M^A := \{ m \in M : am = ma \ \forall a \}$$

Lemma: $H_0(A, M) \cong M/[A, M] \quad , \quad H^0(A, M) \cong M^A.$

Pf (for homology): Note that $M \otimes_{A\otimes A^{op}} A = M \otimes_k A / \langle am \otimes b - m \otimes ba, \; m a \otimes b - m \otimes ab \rangle$.

The assignments

$$\begin{array}{ccc} M \otimes_{A\otimes A^{op}} A & \cong & M / [A, M] \\ m \otimes a & \longmapsto & [ma] \\ m \otimes 1 & \longleftarrow & m \end{array}$$

are inverses of each other. □

• How to compute $H_n(A, M)$ & $H^n(A, M)$? Need to exhibit projective resolutions:

Hochschild standard resolution: Consider the A-bimodule

$$C'_q(A) := A \otimes A^{\otimes q} \otimes A \qquad , \; q \geq 1$$

$$C'_0(A) = A \otimes A.$$

(the left / right action taking place on the leftmost / rightmost tensorand A. They are now free as A-bimodules. For $i = 0, \dots, q$ define

$d_i : C'_q(A) \to C'_{q-1}(A)$, $\quad a_0 \otimes \cdots \otimes a_{q+1} \longmapsto a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{q+1}$

$s : C'_q(A) \to C'_{q+1}(A)$, $\quad$ ——— $\longmapsto 1 \otimes a_0 \otimes \cdots \otimes a_{q+1}$

• Define $\partial' := \sum_{i=0}^{q} (-1)^i d_i$.

Lemma: $\partial'^2 = 0$ and $\partial' s + s \partial' = \mathrm{Id}$.

Corollary: The complex $(C'_q(A), \partial')$ is a free resolution of A by A-bimodules.

• By def, $H_n(A,M) = \mathrm{Tor}_n^{A\otimes A^{op}}(M,A)$ are the homology grps of the complex

$$(M \otimes_{A\otimes A^{op}} C'_*(A), \ \mathrm{Id}\otimes\partial')$$

It can be made slightly simpler: there is an isomorphism

$$\varphi: M \otimes_{A\otimes A^{op}} C'_q(A) \xrightarrow{\cong} M \otimes A^{q} =: C_q(A,M)$$

$$m \otimes a_0 \otimes \cdots \otimes a_{q+1} \longmapsto a_{q+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_q$$

The differential $\mathrm{Id}\otimes\partial'$ under this iso becomes

$$\partial : C_q(A,M) \to C_{q-1}(A,M)$$

$$\partial(m\otimes a_1 \otimes \cdots \otimes a_q) := m a_1 \otimes \cdots \otimes a_q + \sum_{i=1}^{q-1} (-1)^i m \otimes a_1 \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q$$
$$+ (-1)^q a_q m \otimes a_1 \otimes \cdots \otimes a_{q-1}$$

Def: $(C_*(A,M), \partial)$ is called the <u>Hochschild standard chain complex</u>

• The story for $H^n(A,M)$ is similar: here $C^q(A,M) := \mathrm{Hom}_k(A^{\otimes q}, M)$ and

$$\delta(f)(a_0 \otimes \cdots \otimes a_q) = a_0 f(a_1 \otimes \cdots \otimes a_q) + \sum_{i=1}^{q} (-1)^i f(a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots a_q)$$
$$+ (-1)^{q+1} f(a_0 \otimes \cdots \otimes a_{q-1}) \cdot a_q.$$

(<u>Hochschild standard cochain complex</u>)

In practice, to compute $H_n(A,M)$ it is easier to find a free resolution of A and (?) tensor w/ M.

Example: let us compute $H_n(T(V),T(V))$.

Claim: A free resolution of $T(V)$ by free $T(V)$-bimodules is

$$\cdots 0 \to T(V)\otimes V\otimes T(V) \xrightarrow{\partial'} T(V)\otimes T(V) \to 0 \qquad (*)$$

where $\partial'(a\otimes v\otimes b) = av\otimes b - a\otimes vb$

Pf: The two groups are indeed free $T(V)$-bimodules. Consider the complex

$$0 \to T(V)\otimes V\otimes T(V) \xrightarrow{\partial'} T(V)\otimes T(V) \xrightarrow{\mu} T(V)$$

It suffices to check that this is acyclic, for $\ker\mu = \operatorname{Im}\partial$ and $\operatorname{Coker}\partial' = A\otimes A/\operatorname{Im}\partial = A\otimes A/\ker\mu \cong \operatorname{Im}\mu = A$.

To check that this is acyclic it is enough to show (by computation) that $\partial s + s\mu = Id_{A\otimes A}$ for some s (see Kassel pg 17).

So $(*)$ is a free resolution of A. If we tensor $(*)$ by $T(V)\otimes_{T(V)\otimes T(V)^{op}} -$ we get

$$0 \to T(V)\otimes V \xrightarrow{\partial} T(V), \text{ with } \partial(a\otimes v) = av - va,$$

Thus

$$H_K(T(V),T(V)) \cong \begin{cases} k\oplus V\oplus\left(\bigoplus_{m\geq 2}\dfrac{V^{\otimes m}}{(Id-\tau_m)V^{\otimes m}}\right), & K=0 \\ V\oplus\left(\bigoplus_{m\geq 2}(V^{\otimes m})^{\tau_m}\right), & K=1 \\ 0, & K\geq 2\end{cases}$$

Example: For $A = k[x]$, $k[x] \cong T(V)$ for V 1-dimensional. Thus

$$H_k(k[x], k[x]) \cong \begin{cases} k[x], & k=0 \\ x\cdot k[x], & k=1 \\ 0, & \text{else.} \end{cases}$$

Example: let $A_1(k) = \dfrac{F(x,y)}{\langle [x,y]=1 \rangle}$ the Weyl algebra, and let $V = \langle u, v \rangle$ 2-dim vs

A resolution of $A_1(k)$ by free $A_1(k)$-bimodules is given by

$$0 \to A_1(k) \otimes A_1(k)^{op} \otimes \Lambda^2 V \xrightarrow{\partial'} A_1(k) \otimes A_1(k)^{op} \otimes V \xrightarrow{\partial'} A_1(k) \otimes A_1(k)^{op}.$$

$$\partial'(1 \otimes 1 \otimes u \wedge v) = (1 \otimes y - y \otimes 1) \otimes v - (1 \otimes x - x \otimes 1) \otimes u$$

$$\partial'(1 \otimes 1 \otimes u) = 1 \otimes y - y \otimes 1$$

$$\partial'(1 \otimes 1 \otimes v) = 1 \otimes x - x \otimes 1$$

Tensoring w/ $A_1(k) \otimes_{A_1(k) \otimes A_1(k)^{op}} -$ yields

$$0 \to A_1(k) \otimes \Lambda^2 V \xrightarrow{\partial} A_1(k) \otimes V \xrightarrow{\partial} A_1(k)$$

$$\partial(a \otimes u \wedge v) = (ay - ya) \otimes v - (ax - xa) \otimes u$$

$$\partial(a \otimes u) = ay - ya$$

$$\partial(a \otimes v) = ax - xa$$

$$\rightarrow \quad H_k(A_1(k), A_1(k)) \cong \begin{cases} k, & k=2 \\ 0, & \text{else} \end{cases}$$

Related w/ the existence of a symplectic structure on the plane.

Theorem: Morita equivalent algebras have isomorphic Hochschild (co)homology groups.

More concretely, if (A, B, P, Q) is a Morita context, then the functor

$$\begin{aligned} \Phi : \text{bimod}_A &\longrightarrow \text{bimod}_B \\ M &\longmapsto Q \otimes_A M \otimes_A P \end{aligned}$$

(which are exact & send proj res. to proj) satisfies that

$$H_n(A, M) \cong H_n(B, \Phi(M)).$$

(and since $\Phi(A) \cong B$, the above claim follows).

HOCHSCHILD (CO)HOMOLOGY REVISITED

Given an algebra A, there are two canonical bimodules we can consider:

1) A with left / right multiplication

2) A^* w/ $(a_0 f a_1)(a) := f(a_1 a a_0)$, $f \in A^*$

Notation: $HH_*(A) := H_*(A, A)$ as in 1)

$HH^*(A) := H^*(A, A^*)$ — 2)

In particular, $HH_*(A)$ is the homology of the chain complex w/ $C_q(A) := A^{\otimes(q+1)}$, $q \geq 0$

and differential

$$\partial(a_0 \otimes \cdots \otimes a_q) = \sum_{i=0}^{q-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q + (-1)^q a_q a_0 \otimes \cdots \otimes a_{q-1}$$

It turns out that the dual chain complex $C_q(A)^* = \mathrm{Hom}(A^{\otimes q+1}, k)$ computes the cohomology $HH^*(A) = H^*(A, A^*)$. Since k is a field we also get

$$HH^*(A) \cong \mathrm{Hom}(HH_*(A), k).$$

From the formula for ∂^* in $C_q(A)^*$ it also follows that

$\partial^* = \delta : C^0(A) = A^* \to C^1(A) = A^* \otimes A^*$ is $\quad \delta(f)(a_0 \otimes a_1) = f(a_0 a_1 - a_1 a_0)$

so $HH^0(A) = \{\text{trace maps } A \to k\} = \mathrm{Hom}(HH_0(A), k)$.

CYCLIC (CO)HOMOLOGY

Let $t : C_q(A) \to C_q(A)$, $\quad t(a_0 \otimes \cdots \otimes a_q) = (-1)^q a_q \otimes a_0 \otimes \cdots \otimes a_{q-1}$

t defines an action $\mathbb{Z}/q \curvearrowright C_q(A)$ and similarly $\mathbb{Z}/q \curvearrowright C^q(A)$

- let $N := 1 + t + \cdots + t^q$, and let $\partial' : C'_q(A) = C_{q+1}(A) \to C_q(A) = C'_{q-1}(A)$.

<u>Lemma</u>: 1) $(\mathrm{Id} - t)N = N(\mathrm{Id} - t) = 0$ $\qquad$ 2) $\partial(\mathrm{Id} - t) = (\mathrm{Id} - t)\partial'$
3) $N\partial = \partial' N$.

- What 2) says is that $\partial(\mathrm{Im}(\mathrm{Id} - t)) \subset \mathrm{Im}(\mathrm{Id} - t)$, so ~~that~~ $\mathrm{Im}(\mathrm{Id} - t)$ forms a subcomplex.

<u>Definition</u>: The <u>cyclic complex</u> of A is $C_*^{cyc}(A) := C_*(A)/\mathrm{Im}(Id - t)$,

and the <u>cyclic homology</u> of A is $HC_*(A) := H_*(C_*^{cyc}(A))$.

• <u>Alternative description</u>: Consider the following ~~...~~ bicomplex:

$$C_{*,*}(A) = \begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ C_2 & \xleftarrow{Id-t} & C_2 & \xleftarrow{N} & C_2 & \xleftarrow{Id-t} & \\ \partial\downarrow & & \downarrow -\partial' & & \downarrow\partial & & \\ C_1 & \xleftarrow{Id-t} & C_1 & \xleftarrow{N} & C_2 & \xleftarrow{Id-t} & \\ \partial\downarrow & & \downarrow -\partial' & & \downarrow\partial & & \cdots \\ C_0 & \xleftarrow{Id-t} & C_0 & \xleftarrow{N} & C_0 & \xleftarrow{Id-t} & \end{array}$$

<u>Proposition</u>: Suppose $\mathbb{Q} \subset \mathbb{K}$. Then the totalisation $\underset{CC_*(A):=}{Tot}(C_{*,*})$ computes cyclic homology (cohom):

$$H_*(C_*^{cyc}(A)) \cong H_*(Tot(C_{*,*})). \quad (= HC_*(A))$$

(The coh theories are given by taking dual in the chain complex, or since K is a field, by taking dual in homology)

<u>CONNE'S LES</u>

• Recall the map $s: C_q(A) \to C_{q+1}(A)$. By the lemma at the bottom of ③, odd-numbered columns of $C_{*,*}$ are acyclic, so ~~...~~ in some sense they are useless and we want to get rid of them.

- Define the <u>Connes operator</u> $B := (\mathrm{Id} - t)\, s N : C_q(A) \to C_{q+1}(A)$

<u>Lemma</u>: 1) $B^2 = 0$, 2) $B\partial + \partial B = 0$.

- Consider now the bicomplex

$$B_{*,*}(A) := \begin{tikzcd} \vdots & \vdots & & \\ C_3 \arrow[d, "\partial"'] & C_2 \arrow[l, "B"'] \arrow[d, "\partial"] & C_1 \arrow[l, "B"'] \arrow[d, "\partial"] & C_0 \arrow[l, "B"'] \arrow[d] \\ C_2 \arrow[d, "\partial"'] & C_1 \arrow[l, "B"'] \arrow[d, "\partial"] & C_0 \arrow[l, "B"'] \arrow[d] & 0 \arrow[l] \arrow[d] \\ C_1 \arrow[d, "\partial"'] & C_0 \arrow[l, "B"'] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\ C_0 & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

and let $B_*(A) := \mathrm{Tot}\,(B_{*,*}(A))$, with differential D

<u>Proposition</u>: There is a chain hty eq $B_*(A) \to CC_*(A)$, inducing isos in homology

$$H_*(B_*(A), D) \cong H_*(C_*^{cyc}(A)) \cong H_*(CC_*(A))$$

(it uses the "perturbation lemma" of homol. algebra).

Definiton: A *mixed complex* is a chain complex C_* with two differentials ∂ of deg -1 and B of deg $+1$, compatible in the sense that $B\partial + \partial B = 0$. So C_* is both chain & cochain complex.

- The underlying chain complex (C_*, ∂) is called *Hochschild c.c.*, and its homology is denoted by $HH_*(C)$.

- In a similar fashion as before, we can form a bicomplex $\mathcal{B}_{*,*}(C)$ and consider the complex $\mathcal{B}_*(C) = \mathrm{Tot}(\mathcal{B}_{**}(C))$. Its homology is called the *cyclic homology* of C, $HC_*(C)$

- It is clear that an algebra A gives rise to a mixed complex $(C_*(A), \partial, B)$. This construction actually defines a functor $\mathrm{Alg}_k \longrightarrow \mathrm{mixCh}_k$

- We can actually see $(C_*, \partial) \subset (\mathcal{B}_*(C), D)$ as a subcomplex, viewing (C_*, ∂) as the leftmost column of $\mathcal{B}_{**}(C)$. The quotient complex

$$\mathcal{B}_*(C)/C_* \cong s^2\mathcal{B}_*(C),$$ where $(s^2\mathcal{B}_*(C))_q = s^2\mathcal{B}_{q-2}(C)$ with the same differential D.

Morally, we get a ses of chain complexes

$$0 \to C_* \xrightarrow{I} \mathcal{B}_*(C) \xrightarrow{S} s^2\mathcal{B}(C) \to 0$$

which induces a les in homology, called the <u>Connes les</u>:

<u>Theorem</u> (Connes les):

$$HH_q(C) \xrightarrow{I} HC_q(C) \xrightarrow{S} HC_{q-2}(C) \xrightarrow{B}$$
$$\hookrightarrow H_{q-1}(C) \xrightarrow{I} HC_{q-1}(C) \xrightarrow{S} HC_{q-3}(C) \xrightarrow{B}$$
$$\cdots$$
$$\hookrightarrow HH_1(C) \xrightarrow{I} HC_1(C) \longrightarrow 0$$
$$\hookrightarrow HH_0(C) \xrightarrow{I} HC_0(C) \longrightarrow 0$$

<u>Corollary</u>: For any mixed complex C, we have $HC_0(C) \cong HH_0(C)$

Moreover, if $HH_i(C) = 0 \ \forall i > N$ for some N, then

$S : HC_{i+2}(C) \to HC_i(C)$ is an iso $\forall i \geq N$ and injective for $i = N-1$.

• By this reason S is called the <u>periodicity map</u>

<u>Examples</u>: 1) $HC_q(k) \cong \begin{cases} k, & q \text{ even} \\ 0, & q \text{ odd} \end{cases}$

2) $HC_q(A_1(k)) \cong \begin{cases} K, & q \text{ even} \geq 2 \\ 0, & \text{else} \end{cases}$

KÄHLER DIFFERENTIALS

- Let A be a commutative k-algebra. The A-module of *Kähler differentials* is the quotient of the free A-module generated by symbols $\{da : a \in A\}$, modulo the relation

$$d(aa') = a\,da' + a'\,da,$$

and it is denoted by Ω_A or $\Omega_{A/k}$.

- The condition says that $d : A \to \Omega_{A/k}$ is a derivation, but also follows that it is "the universal derivation": any other derivation $D : A \to M$ factors through d:

$$\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M).$$

- (Alternative description of $\Omega_{A/k}$): Let $I = \mathrm{Ker}(\mu : A \otimes A \to A)$. Then

$$\Omega_{A/k} \cong I/I^2, \quad \text{where} \quad da \leftrightarrow a \otimes 1 - 1 \otimes a.$$

- (de Rham complex). Let $\Omega^q_{A/k} := \Lambda^q \Omega_{A/k}$ the q-th exterior power of the A-module $\Omega_{A/k}$ ($\Omega^0_{A/k} = A$). The Kähler map $d : A \to \Omega^1_A$ extends to a cochain complex

$$0 \to A \xrightarrow{d} \Omega^1_A \to \Omega^2_A \to \Omega^3_A \to \cdots$$

called the *de Rham complex* of A, and d the *de Rham differential*.

$d : \Omega^q_{A/k} \to \Omega^{q+1}_{A/k}$ is defined by

$$d(a_0\, da_1 \wedge \cdots \wedge da_q) := da_0 \wedge da_1 \wedge \cdots \wedge da_q .$$

Of course, it satisfies $d^2=0$. Its cohomology $H^*_{dR}(A)$ is called the <u>de Rham cohomology</u> of A.

• <u>(Ω_A, HH & HC)</u>: let Ω_A denote the mixed complex with one diff 0 and the second d. It follows directly that

$$HH_q(\Omega_A) \cong \Omega^q_{A/k}$$

$$HC_q(\Omega_A) \cong \Omega^q_{A/k} / \operatorname{Im} d \oplus H^{q-2}_{dR}(A) \oplus H^{q-4}_{dR}(A) \oplus \cdots$$

• <u>(HC & H_{dR})</u>: Assume char $k \neq 0$ (and A a <u>commutative</u> alg). Define

$$\mu : C_q(A) \longrightarrow \Omega^q_{A/k}$$
$$a_0 \otimes \cdots \otimes a_q \longmapsto a_0\, da_1 \wedge \cdots \wedge da_q$$

It is straightforward to check that

$$\mu \partial = 0, \qquad \mu B = d\mu.$$

This implies that μ defines a mixed complexes map

$$(C_*(A), \partial, B) \longrightarrow (\Omega^*_{A/k}, 0, d)$$

these maps in HH & HC :

$$HH_q(A) \longrightarrow \Omega^q_{A/k}$$

$$HC_q(A) \longrightarrow \Omega^q_{A/k}/\mathrm{Im}\, d \oplus H^{q-2}_{dR}(A) \oplus H^{q-4}_{dR}(A) \oplus \cdots$$

Theorem : For $q=0,1$, the above maps are isos,

$$HH_0(A) \cong A \cong \Omega^0_{A/k} \quad , \quad HH_1(A) \cong \Omega^1_{A/k} \quad , \quad HC_1(A) \cong \Omega^1_{A/k}/dA$$

Moreover, if A is a smooth algebra (e.g. $k[x,y]/(x^2+y^2-1)$) then the above maps are isos for all $q \geq 0$.

BRYLINSKI'S MIXED COMPLEX

Let A be a filtered algebra, $0 \subset A^0 \subset A^1 \subset \cdots$, $A = \bigcup_{q\geq 0} A^q$. The associated graded algebra $gr^\bullet A = \bigoplus_q A^q/A^{q-1}$, if commutative, is a Poisson algebra, meaning that it is endowed with a Poisson bracket

$$\{-,-\} : gr^\bullet A \times gr^\bullet A \longrightarrow gr^\bullet A$$

defined as follows: given $[a] \in gr^p(A)$ and $[b] \in gr^q(A)$, the commutator $[a,b] = ab - ba$ belongs to A^{p+q-1} (because should be 0 in $gr^{p+q}(A)$). Then $\{[a],[b]\} := \big[[a,b]\big] \in gr^{p+q-1}(A)$.

Lemma (Brylinski). Let A be a filtered algebra over k, char $k = 0$, such that $S = gr^\bullet A$ is commutative & smooth. Then there exists a mixed complex $(\Omega^*_{S/k}, \delta, d)$ where d is the de Rham differential and δ is of deg -1 given by

$$\delta(f_0\, df_1 \wedge \cdots \wedge df_q) = \sum_{i=1}^{q+1} (-1)^{i+1} \{f_0, f_i\}\, df_1 \wedge \cdots \overset{i}{\downarrow} \cdots \wedge df_q$$
$$+ \sum_{1 \le i < j \le q} (-1)^{i+j} f_0\, d\{f_i, f_j\} \wedge df_1 \wedge \cdots \overset{i}{\downarrow} \cdots \overset{j}{\downarrow} \cdots \wedge df_q$$

Theorem (Kassel): Let A, S be as before and suppose $S \cong k[x_1, \ldots, x_n]$. Then

$$HH_q(A) \cong HH_q(\Omega^*_{S/k}) \ldots$$
$$HC_q(A) \cong HC_q(\Omega^*_{S/k}).$$

Theorem: Let (C, ∂, B) be a mixed complex. There is a homological spectral sequence with E_2-page

$$E^2_{pq} = \begin{cases} H_{dR}^{q-p}(C), & q \geq p > 0 \\ HH_q(C)/\operatorname{Im}\partial, & q \geq p = 0 \\ 0 & \text{else} \end{cases}$$

converging to the cyclic homology of (C, ∂, B).