Equivalence between classical Kauffman bracket and Khovanov's modification:

Classical Kauffman bracket: Polynomial in $\mathbb{Z}[A, A^{-1}]$ determined by the rules

(1) $\langle \times \rangle = A \,\langle)(\rangle + A^{-1} \,\langle \asymp \rangle$

(2) $\langle L \coprod \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$

(3) $\langle \emptyset \rangle = 1$

The (unnormalised) Jones polynomial of a link L with diagram D is

$$J(L) := (-A)^{-3w(D)}\langle D \rangle.$$

Khovanov's modification: Polynomial in $\mathbb{Z}[q, q^{-1}]$ determined by the rules

(1′) $\langle \times \rangle' = \langle)(\rangle' - q \,\langle \asymp \rangle'$

(2′) $\langle L \coprod \bigcirc \rangle' = (q + q^{-1})\langle L \rangle'$

(3′) $\langle \emptyset \rangle' = 1$

Let $x(D)$ (resp. $y(D)$) be the number of negative (resp. positive) crossings of D. The (unnormalised) Jones polynomial of a link L with diagram D is

$$J(L) := (-1)^{x(D)} q^{y(D)-2x(D)} \langle D \rangle'.$$

Proposition 1. *Both notions are equivalent and yield the same Jones polynomial.*

Proof. The two notions are related by the following changes:

$$\langle D \rangle = A^{c(D)} \langle D \rangle' \qquad , \qquad A^2 = -q^{-1}$$

where $c(D)$ is the number of crossings of D. The equivalence follows by a computation: let $c = c(\times)$. Then $c()() = c(\asymp) = c - 1$ and (1) becomes

$$\begin{aligned} A^{c(\times)} \langle \times \rangle' &= A^{1+c()()} \langle)(\rangle' + A^{-1+c(\asymp)} \langle \asymp \rangle' \\ A^{c} \langle \times \rangle' &= A^{c} \langle)(\rangle' + A^{c-2} \langle \asymp \rangle' \\ \langle \times \rangle' &= \langle)(\rangle' - q \langle \asymp \rangle \end{aligned}$$

The Jones polynomial gets modified as follows:

$$\begin{aligned} J(L) &= (-A)^{-3w(D)} \langle D \rangle = (-A)^{-3(y(D)-x(D))} A^{y(D)+x(D)} \langle D \rangle' \\ &= (-1)^{-3(y(D)-x(D))} A^{-2y(D)+4x(D)} \langle D \rangle' = (-1)^{-3(y(D)-x(D))} (-q)^{y(D)-2x(D)} \langle D \rangle' \\ &= (-1)^{x(D)} q^{y(D)-2x(D)} \langle D \rangle'. \end{aligned}$$

□