# DERIVED & TRIANGULATED CATEGORIES

## $\boxed{\text{I}}$. Abelian cat's and derived categories.

Def. $\mathcal{A}$ additive if

1) Ab-enriched
2) $\exists$ zero object $0$, ie $\quad \mathrm{Hom}(X,0) = 0 = \mathrm{Hom}(0,X)$
3) $\exists$ finite $\oplus$'s.

Eg. Ab, R-mod, mod-R, $VB(X)$, $Sh(X)$

vector bundles $/X$ — sheaves of ab. grp over $X$.

Def. $F: \mathcal{A} \to \mathcal{B}$ additive if $F_{X,Y}: \mathrm{Hom} \to \mathrm{Hom}$ a grp hom.

• Denote $h_X := \mathrm{Hom}(-,X) : \mathcal{A}^{op} \to Ab$.

A map $f: X \to Y$ induces a nat transf $h_f : h_X \Rightarrow h_Y$.

let $\mathrm{Ker}(h_f): \mathcal{A}^{op} \to Ab$
$$Z \mapsto \ker(h_X(Z) \to h_Y(Z)).$$

Def. $\mathrm{Ker} f$ is, if it exists, the object of $\mathcal{A}$ that represents $\ker h_f$, ie
$$\mathrm{Hom}(Z, \ker f) \cong \ker(h_f)(Z). \quad \left( = \mathrm{Ker}\left(\mathrm{Hom}(Z,X) \xrightarrow{f_*} \mathrm{Hom}(Z,Y)\right)\right)$$

If $h^X = \mathrm{Hom}(X, -) : \mathcal{A} \to Ab$,

$f : X \to Y$ induces $h^f : h^Y \to h^X \rightsquigarrow \ker(h^f) : \mathcal{A} \to Ab$.

Def Coker $f$ is the object in $\mathcal{A}$ that ~~corresp~~ represents $\ker(h^f)$,

$$\mathrm{Hom}(\mathrm{Coker}\, f, Z) \cong \ker(h^f)(Z).$$

Eg. In $VB(X)$ ker's and coker's might not exist!

If $f : X \to Y$, there are ~~two~~ canonical arrows

$$\ker f \to X \quad , \quad Y \to \mathrm{coker}\, f.$$

Def. $\mathrm{Coim}(f) := \mathrm{coker}(\ker f \hookrightarrow X)$

$\mathrm{Im}\, f := \ker(Y \to \mathrm{coker}\, f)$

Claim. If $\exists$ ker & coker of $f$, then $\exists\, \bar{f} : \mathrm{Coim}\, f \to \mathrm{Im}\, f$ st

$$\begin{tikzcd}
\ker f \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] & \mathrm{coker}\, f \\
 & \mathrm{Coim}\, f \arrow[r, "\bar{f}"] & \mathrm{Im}\, f \arrow[u] &
\end{tikzcd}$$

Def. An additive cat is abelian if $\forall f : X \to Y$, $\exists$ ~~ker~~, ker, coker and $\bar{f} : \mathrm{Coim} f \xrightarrow{\cong} \mathrm{Im} f$ is an iso.

Eg. The cat of filtered v.s. has ker's and coker's, but not abelian!

- In ab. cat's we have s.e.s's and we can apply the hom alg formalism to introduce derived functors.

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Write $Ch(\mathcal{A})$ = cochain complexes in $\mathcal{A}$

Exercise. $Ch(\mathcal{A})$ is an abelian cat.

Def. $H^i(A^\bullet) := \ker d^i / \mathrm{Im}\, d^{i-1}$, $A^\bullet \in Ch(\mathcal{A})$.

Def. $A^\bullet$ is exact at the i-th term if $H^i(A^\bullet) = 0$.

$A^\bullet$ exact if $H^i(A^\bullet) = 0$ $\forall i$.

- Ses to the right / left, a functor ~~the~~ that is left / right exact ...

Eg. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ exact

but $\mathrm{Hom}(-, \mathbb{Z})$ ( ) is left exact only

$- \otimes \mathbb{Z}/4$ ( ) — right ———.

**Theorem.** Let $F: \mathcal{A} \to \mathcal{B}$ be a left exact ~~/ ...~~ ~~$\mathcal{Z}$~~ functor, $\mathcal{A}$ w/ enough injectives.

Then $\exists!$ sequence of functors up to nat iso $R^i F : \mathcal{A} \to \mathcal{B}$ st

1) $R^0 F \cong F$

2) If $0 \to A \to B \to C \to 0$, $\exists$ a long ex. seq

$$0 \to FA \to FB \to FC \to (R^1F)A \to (R^1F)B \to (R^1F)(C) \to (R^2F)(A) \to \dots$$

$R^i F$ is called the $i$-th ~~te~~ derived functor of $F$.

Let $\mathcal{A}$ w/ enough projectives,

**Theorem.** If $G: \mathcal{A} \to \mathcal{B}$ is right exact, $\exists!$ $L^i G : \mathcal{A} \to \mathcal{B}$ st

1) $L^0 G \cong G$

2) $0 \to A \to B \to C \to 0$ $\exists$ l.e.s.

$$\dots \to (L^2G)(C) \to L^1G(A) \to L^1G(B) \to L^1G(C) \to GA \to GB \to GC \to 0$$

$L^i G$ is the $i$-th left derived functor of $G$.

<u>Motivation</u> (Grothendieck §, 1957)

$X$ top space $\rightsquigarrow$ $Sh(X)$ sheaves of ab gp on $X$,

has enough inj & proj.

$\Gamma : Sh(X) \to Ab$ the global section functor, it is <u>left exact</u>

$\rightsquigarrow$ get $R^i\Gamma : Sh(X) \to Ab$

Given an ab gp $A$, let $\underline{A} \in Sh(X)$ the constant functor,

$$\Gamma(U, \underline{A}) = A^{\oplus \# \pi_0 U}$$

<u>Thm</u>: $R^i\Gamma(\underline{A}) \cong H^i(X; A)$

$\boxed{\text{II}}$. How to compute derived functors?

Injective object $I$:

$$\begin{tikzcd} Y \arrow[dr, dashed] & \\ X \arrow[u, hook] \arrow[r] & I \end{tikzcd}$$

$\Longleftrightarrow$ $Hom(-, I)$ exact

Proj obj $P$.

$$\begin{tikzcd} & Y \arrow[d] \\ P \arrow[ur, dashed] \arrow[r] & Z \end{tikzcd}$$

$\Longleftrightarrow$ $Hom(P, -)$ exact

<u>Ex</u>. Inj (proj) in $\mathcal{A}^{op}$ = proj (inj) obj in $\mathcal{A}$

Def. $\mathcal{A}$ has enough inj if any obj $X \in \mathcal{A}$ can be embedded into a inj, ie $\exists$ a mono $X \hookrightarrow I$.

$\mathcal{A}$ has enough proj if any $X \in \mathcal{A}$ admits a surjection from a proj, ie $\exists$ epi $P \twoheadrightarrow X$.

Def. An injective / proj resolution of an object $X$ is an exact sequence

$$0 \to X \to I_0 \to I_1 \to I_2 \to \cdots$$

$$\cdots \to P_2 \to P_1 \to P_0 \to X \to 0.$$

The deleted resolution is

$$I^\bullet(X) = (I_0 \to I_1 \to \cdots)$$
$$P^\bullet(X) = (\cdots \to P_1 \to P_0).$$

Note. $H_n\left(\begin{matrix} I^\bullet(X) \\ P^\bullet(X)\end{matrix}\right) = \begin{cases} 0, & i \neq 0 \\ X, & i = 0 \end{cases}.$

- Homotopic maps of complexes induce the same map on homology.

Lemma. Let $g : X \to Y$ a morphism, and $P^\bullet(X)$, $P^\bullet(Y)$ resolutions. Then $g$ upgrades to a chain map $g^\bullet : P^\bullet(X) \to P^\bullet(Y)$ st $H_0^\bullet(g^\bullet) = g$.

$$\left(\begin{tikzcd} \to P_2^X \arrow[d] \arrow[r] & P_1^X \arrow[d] \arrow[r] & P_0^X \arrow[d] \arrow[r] & X \arrow[d, "g"] \arrow[r] & 0 \\ \to P_2^Y \arrow[r] & P_1^Y \arrow[r] & P_0^Y \arrow[r] & Y \arrow[r] & 0 \end{tikzcd}\right)$$

Def. $f : C_* \to D_*$ map of complexes quasi-iso if it induces isos on homology

Remark. View $X$ as a chain complex concentrated in deg 0. Then

For any proj resolution $P^\bullet(X)$ there is a quasi-iso $P^\bullet(X) \xrightarrow{\simeq} X$

Same for inj.

Remark. If $\mathcal{A}$ has enough proj / inj) $\to$ proj (inj) resolutions exists.

Even for objects of $Ch(\mathcal{A})$! More precisely, for $C_*$ $\exists$ $P^\bullet(C_*)$ s.t.

$P^\bullet(C_*) \xrightarrow{\simeq} C_*$ quasi-iso and $P^\bullet(C_*)$ consists of proj objects.

How to compute derived functors?

let $\mathcal{A}$ be an ab cat w/ enough projectives, let $G : \mathcal{A} \to \mathcal{B}$ be right exact. For $X \in \mathcal{A}$ choose a proj resolution $P^\bullet(X)$ and define

$$(L^i G)(X) := H^{-i}(G(P^\bullet(X)))$$

If $F$ l.e. and $I^\bullet(X)$ inj res.

$$(R^i F)(X) = H^i(F(I^\bullet(X))).$$

# First def of derived category

Say $F: \mathcal{A} \to \mathcal{B}$ l.e. and $X \in \mathcal{A}$, so get $(R^iF)(X) \in \mathcal{B}$ $\forall i = 0,1,2,\dots$

Q. Can we pack the information about the $(R^iF)(X)$ $\forall i$ in one object for some category?

A. Yes ! That is the derived cat of $\mathcal{B}$, $D(\mathcal{B})$

In the end we'll obtain

$$RF : D(\mathcal{A}) \longrightarrow D(\mathcal{B})$$
$$X \longmapsto RF(X)$$

$\leftarrow$ contains all the info about $(R^iF)X$.

~~obj of $D(\mathcal{A})$ = cochain complexes~~

Def. $D(\mathcal{A}) := Ch(\mathcal{A})[\text{quasi-isos}]^{-1}$ the localisation.

- In this cat, a given complex is isomorphic to its projective resolution.

Goal. What structure does $D(\mathcal{A})$ have? Triangulated.

- There are enough proj in $R$-mod.

<u>Lemma</u>. $\mathcal{A}$ ab cat. Then $P$ proj $\Leftrightarrow$ for any surjection $X \overset{p}{\twoheadrightarrow} P$, there is an iso $X \cong P \oplus X'$.

Pf $(\Rightarrow)$
$$\begin{tikzcd} & P \arrow[dl, dashed, "i"'] \arrow[d, "\mathrm{id}"] \\ X \arrow[r, two heads, "p"] & P \end{tikzcd}$$

$\overset{\text{Ex.}}{\Longrightarrow}$ $X \cong P \oplus \ker p$

The proof is essentially the five lemma: if $X \twoheadrightarrow P$ surj then there is a ses
$$0 \to \ker p \hookrightarrow X \overset{p}{\twoheadrightarrow} P \to 0$$
and a splitting $\equiv$ section for $p$.

$(\Leftarrow)$ Consider the ~~diagr~~. lifting problem

$$\begin{tikzcd} & Y \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\exists ?"] \arrow[r, "\varphi"] & Z \end{tikzcd}$$

Let $X := \ker\left(Y \oplus P \xrightarrow{-\pi \oplus \varphi} Z\right)$.

Get a diagram
$$\begin{tikzcd} X \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & Y \arrow[d] \\ P \arrow[r, "\varphi"] & Z \\ p \arrow[r, mapsto] & \varphi(p) \end{tikzcd}$$

$y$

<u>Key observation</u>: $pr_2 : X \to P$ epi (comes from $(p, y)$).

By hypothesis $X \cong P \oplus X'$; from here easy.

Cor Proj $R$-mod $\equiv$ direct summands of free mod's.

Eg. $R = \mathbb{Z}/6$, then $\mathbb{Z}/2$ and $\mathbb{Z}/6$ are proj $\mathbb{Z}/6$-modules.

Eg: $\mathcal{A} = Ab$, $A$ ab gp, $A \otimes_{\mathbb{Z}} - : Ab \to Ab$ r.e.

What is $\mathrm{Tor}_i(A,-) := L^i(A \otimes_{\mathbb{Z}} -)$ ?

$\mathrm{Tor}_i(A, \mathbb{Z}/p)$ ?

First take a proj res. of $\mathbb{Z}/p$: $0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p$

The deleted res: $\{0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z}\}$.

$\downarrow A \otimes_{\mathbb{Z}} -$

$$0 \to A \xrightarrow{\cdot p} A$$

So we only have two non-trivial derived functors:

$(L^0(A \otimes_{\mathbb{Z}}))(\mathbb{Z}/p) = A/pA$

$L^1 = {}_pA$ p-torsion.

$L^i = 0 \quad \forall i > 1.$

## Derived cat

$F : \mathcal{A} \to \mathcal{B}$ r.e. $\rightsquigarrow$ want to def $LF : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$

$LF(C_\bullet) := F(\underbrace{P^\bullet(C_\bullet)})$

proj res of $C_\bullet$, ie a complex formed by proj obj which is q.iso to $C_\bullet$.

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Localization of $\mathcal{C}$ wrt a class of arrows $S$. $\rightsquigarrow \mathcal{C}[S^{-1}]$.

Goals. What do arrows in $\mathcal{D}(\mathcal{A})$ look like?

- If $S$ satisfies the "Ore conditions" then $\mathcal{C}[S^{-1}]$ can be described explicitly.

- $\mathcal{A} \hookrightarrow \mathcal{D}(\mathcal{A})$ embedding.

- What structure does $\mathcal{D}(\mathcal{A})$ have?

$\mathcal{D}(\mathcal{A})$ is almost never abelian, but it is triangulated.

Geometrical motivation

We already saw that the cat of vector bundles $VB(X)$ over an alg variety $X$ is not abelian, $\nexists$ Ker's

But we can do $VB(X) \hookrightarrow \mathcal{A}$ abelian

Given $E \in VB(X) \rightsquigarrow \mathcal{E}$ sheaf on $X$; $U \subset X$ then

$\mathcal{E}(U) := \Gamma(U, E)$ section of $E$ over $U$

(algebraic maps $U \to E$)

## $\boxed{\text{IV}}$. Describe $\mathcal{D}(\mathcal{A})$ more explicitly.

<u>Ore conditions.</u>

Let $S$ be some ~~set~~ collection of arrows. If $S$ satisfies the Ore conditions, then $\mathcal{C}[S^{-1}]$ can be described explicitly:

1) Id's belong to $S$

2) (Two-out-of-three prop). If two out of three arrs of $f, g, gf$ belong to $S$, so does the third.

3) Given a diagram

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ X \arrow[r, "g"] & Z \end{tikzcd} \quad \text{w/ } g \in S \quad \Rightarrow \quad \text{can be completed to} \quad \begin{tikzcd} W \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & Y \arrow[d, "f"] \\ X \arrow[r, "g"] & Z \end{tikzcd} \quad \text{w/ } \tilde{g} \in S.$$

4) Given a diag

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & Y \arrow[r, "g"] & Z \end{tikzcd} \quad \text{w/ } g \in S \quad \text{w/ } gf_1 = gf_2 \quad \Rightarrow \quad \exists\, \tilde{g} : W \to X \text{ st } \tilde{g} \in S$$

$$\begin{tikzcd} W \arrow[r, "\tilde{g}"] & X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & Y \end{tikzcd} \quad \text{st } f_1\tilde{g} = f_2\tilde{g}.$$

**Proposition.** If $S$ satisfies the Ore conditions then $\mathcal{C}[S^{-1}]$ can be described as

$$\mathrm{ob}(\mathcal{C}[S^{-1}]) = \mathrm{ob}(\mathcal{C})$$

$$\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X,Y) = \left\{ \text{eq classes of spans } \begin{tikzcd} & W \arrow[dl, "S \ni s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \right\}$$

~~with~~ (this is the analogue of a fraction, denoted $\underline{\underline{fs^{-1}}}$)

where

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \sim \begin{tikzcd} & W' \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

if $\exists\, \tilde{W}$ and $\tilde{W} \xrightarrow{s \in S} W$, $\tilde{W} \xrightarrow{s' \in S} W'$ st

$$\begin{tikzcd} & \tilde{W} \arrow[dl, "s"'] \arrow[dr, "s'"] & \\ W \arrow[d] \arrow[drr] & & W' \arrow[d] \arrow[dll] \\ X & & Y \end{tikzcd}$$

commutes

Composition of $\begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$ and $\begin{tikzcd} & W' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}$ is $(gt^{-1}) \circ (fs^{-1})$

$fs^{-1}$ $\qquad$ $gt^{-1}$ $\qquad$ $\shortparallel$

$(g\tilde{f})(s\tilde{t})^{-1}$

$$\begin{tikzcd} & & \tilde{W} \arrow[dl, "S \ni \tilde{t}"'] \arrow[dr, "\tilde{f}"] & & \\ & W \arrow[dl] \arrow[dr] & & W' \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

by (3)

Properties (exercise)

1) This is indeed an eq ~~condition~~ rel

2) Composition doesn't depend on eq classes

3) Comp is associative

4) $Id_X\ Id_X^{-1}$ is the identity

5) Composition does not depend on choice of $\tilde{t}$ and $\tilde{f}$.

~~Note~~

Under this ~~one~~ description,

$$j : \mathcal{C} \to \mathcal{C}[S^{-1}]$$

$$X \longmapsto X$$

$$(f : X \to Y) \longmapsto j(f) = \begin{tikzcd} & X \arrow[dl, "id"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \qquad \left( \text{ie } \text{"}\frac{f}{1}\text{"} \right)$$

$$g \circ f \longmapsto \begin{tikzcd} & X \arrow[dl, "id"'] \arrow[dr, "g \circ f"] & \\ X & & Z \end{tikzcd} = \begin{tikzcd} & & X \arrow[dl, "id"'] \arrow[dr, "f"] & & \\ & X \arrow[dl, "id"'] \arrow[dr, "f"] & & Y \arrow[dl, "id"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

$s \in S \longmapsto j(s)$ is an isomorphism, indeed ~~its inverse is~~

the inverse of $\begin{tikzcd} & X \arrow[dl, "id"'] \arrow[dr, "s"] & \\ X & & Y \end{tikzcd}$ is $\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "id"] & \\ Y & & X \end{tikzcd}$.

Easy to see that this satisfies the universal prop of localisation.

**Proposition.** If $\mathcal{C}$ additive and $S$ Ore $\Rightarrow$ $\mathcal{C}[S^{-1}]$ additive, and $j: \mathcal{C} \to \mathcal{C}[S^{-1}]$ is additive.

Pf: The trick : $\frac{f}{s} + \frac{g}{t} = \frac{\quad}{st}$

$$\begin{array}{cccc} & W & & W' \\ s \swarrow & \downarrow f & t \swarrow & \downarrow g \\ X & Y & X & Y \end{array}$$

## Explicit description of $\mathcal{D}(\mathcal{A})$

$\mathcal{C} = Ch(\mathcal{A})$

$S = QIS$

Issue. QIS does <u>not</u> satisfy the Ore condition in $Ch(\mathcal{A})$!

But if we pass to $Ch_h(\mathcal{A})$ = htpy class of chain complexes then it is!!

Exercise. $Ch(\mathcal{A}) \to Ch_h(\mathcal{A})$ is additive.

Theorem. 1) The class of q.i.s satisfies the Ore condition in $Ch_h(\mathcal{A})$

2) $\mathcal{D}(\mathcal{A}) \simeq Ch_h(\mathcal{A})[QIS^{-1}]$. More precisely,

$$\begin{tikzcd} & Ch(\mathcal{A}) \arrow[dl] \arrow[dr, "j"] & \\ Ch_h(\mathcal{A}) \arrow[rr, dashed, "j_h"'] & & \mathcal{D}(\mathcal{A}) = Ch(\mathcal{A})[QIS] \end{tikzcd}$$

$j_h$ the localisation functor.

Corollary. $\mathcal{D}(A)$ can be described explicitly as

ob $=$ cochain complexes.

$\mathrm{Hom}_{\mathcal{D}(A)}(C^\bullet, D^\bullet) =$ equivalent classes of
$$\begin{tikzcd} & W^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr, "f"] & \\ C^\bullet & & D^\bullet \end{tikzcd}$$
, maps in $Ch_h$ !!

## Structure of $Ch_h(A)$

For a complex $(X^\bullet, d)$, put $(X^\bullet[1])^i = X^{i+1}$ w/ $(d_X[1])^i = -d_X^{i+1}$.

**Def.** If $f : X^\bullet \to Y^\bullet$ cochain map, the <u>cone of $f$</u> is the complex $\mathrm{Cone}(f)^\bullet$ given by

$$\mathrm{Cone}(f)^\bullet = Y^\bullet \oplus X^\bullet[1] \quad , \text{ and}$$

$$\begin{tikzcd} \arrow[r] & X^{i-1} \arrow[r] \arrow[d, "f"] & X^i \arrow[r] \arrow[d, "f"] & X^{i+1} \arrow[r] \arrow[d, "f"] & {} \\ \arrow[r] & Y^{i-1} \arrow[r] & Y^i \arrow[r] & Y^{i+1} \arrow[r] & {} \end{tikzcd}$$

$$d^i_{\mathrm{Cone}(f)} : Y^i \oplus X^{i+1} \to Y^{i+1} \oplus X^{i+2}$$

$$(y^i, x^{i+1}) \mapsto \left(d_Y y^i + f(x^{i+1}), -d_X(x^{i+1})\right)$$

there is a ses

$$0 \to Y^\bullet \to \mathrm{Cone}\, f \to X^\bullet[1] \to 0 .$$

This induces a les

$$H^{i-1}(X^\bullet[1]) \xrightarrow{\delta} H^i(Y^\bullet) \to H^i(\mathrm{Cone}\, f) \to H^i(X^\bullet[1]) \xrightarrow{\delta} H^{i+1}(Y^\bullet)$$

$$H^i(X^\bullet[1]) \cong H^{i+1}(X^\bullet) \xrightarrow{f_*} H^{i+1}(Y^\bullet)$$

this is precisely the map induced by $f$ !!

<u>Def</u>. A complex $X^\bullet$ is

- <u>acyclic</u> if $H(X^\bullet) = 0$
- <u>contractible</u> if $X^\bullet$ is htpy eq to the 0 complex.

<u>Ex</u>. $f : X \to Y$ chain htpy equivalence $\Leftrightarrow$ $\mathrm{Cone}(f)$ contractible.

<u>Remark</u>. Why doesn't QIS satisfy the Ore conditions in $Ch(\mathcal{A})$?

$$X^\bullet \to X^\bullet \to \mathrm{Cone}(\mathrm{Id})$$

<u>Remark</u>. $Ch_h(\mathcal{A})$ is <u>not</u> abelian: if $f$ is a monomorphism in $Ch_h(\mathcal{A})$, then $\exists\, g$ in $Ch_h(\mathcal{A})$ st $g \circ f$ htpy eq to Id.

But in $Ch_h$ we can define sequences that have properties analogue to ses's:

$$X^\bullet \to Y^\bullet \to \mathrm{Cone}(f) \to X^\bullet[1] \qquad (*)$$

<u>Notation</u>. $\mathrm{Hom}^i_{Ch_h}(X^\bullet, Y^\bullet) := \mathrm{Hom}_{Ch_h}(X^\bullet, Y^\bullet[i])$.

<u>Lemma</u>. For $C^\bullet \in Ch(\mathcal{A})$, the sequence $(*)$ induces the following l.e.s

$$\mathrm{Hom}^i_{Ch_h}(C^\bullet, X^\bullet) \to \mathrm{Hom}^i(C^\bullet, Y^\bullet) \to \mathrm{Hom}(C^\bullet, \mathrm{Cone} f) \to \mathrm{Hom}^{i+1}(C^\bullet, X^\bullet) \to \dots$$

Def. Given $X^\bullet, Y^\bullet \in Ch(\mathcal{A})$, let $\underline{Hom}(X^\bullet, Y^\bullet) \in Ch(\mathcal{A})$ be

$$\underline{Hom}(X^\bullet, Y^\bullet)^i := \prod_{n \in \mathbb{Z}} Hom(X^n, Y^{i+n}),$$

w/ $d^i : \underline{Hom}(X^\bullet, Y^\bullet)^i \longrightarrow \underline{Hom}(X^\bullet, Y^\bullet)^{i+1}$

$$(f_n) \longmapsto \left(d_Y f^n - (-1)^i f^{n+1} d_X\right)_n$$

$$H^i(\underline{Hom}(X^\bullet, Y^\bullet)) = Hom_{Ch_h}(X^\bullet, Y[i]).$$

Applying $\underline{Hom}(C^\bullet, -)$ to $(*)$ get a ses in $Ch$:

$$0 \to \underline{Hom}(C^\bullet, Y^\bullet) \to \underline{Hom}(C^\bullet, \mathrm{Cone} f) \to \underline{Hom}(C^\bullet, X^\bullet[1]) \to 0$$

## $\boxed{V}$ Triangulated cat's

Last time: • $\mathrm{Cone}(f)$.

• $A \xrightarrow{f} B \to \mathrm{Cone} f \to A[1]$, and ~~th~~ for any $C$ cochain this sequence induces long exact sequences in $Ch_h(\mathcal{A})$.

Goal for today. Introduce triang cat's & prove $Ch_h(\mathcal{A})$ & $D(\mathcal{A})$ are triangulated.

**Def.** A <u>triangulated category</u> is an additive cat $\mathcal{T}$ together w/

1) an auto-equivalence $[1] : \mathcal{T} \longrightarrow \mathcal{T}$

2) a class of sequence ("~~triangle~~ <u>distinguished triangles</u>")

$$A \to B \to C \to A[1]$$

satisfying

(i) the class of distinguished triangles is closed under iso

(ii) $A \xrightarrow{\mathrm{Id}} A \longrightarrow 0 \longrightarrow A[1]$ is a distinguished triangle

(iii) Any map $f : A \to B$ is part of a distinguished triangle $A \xrightarrow{f} B \to C \to A[1]$

(iv) $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{u} A[1]$ distinguished $\Longleftrightarrow$ $B \xrightarrow{g} C \xrightarrow{u} A[1] \xrightarrow{-f[1]} B[1]$ disting.

(v) If the rows of this solid diag are dist. triangl. and the ~~solid~~ left hand side square comm., then the dashed arrow exists and the entire diag commutes

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, dashed, "c"] & A[1] \arrow[d, "a[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

(vi) <u>Octahedral axiom</u>: Given two disting. triangles in solid, then it can be extended to the following diagram where the two columns are also exact.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & Y \arrow[r] & Z \arrow[r] & X[1] \arrow[d, equal] \\
X \arrow[r] & Y' \arrow[r] & Z \arrow[r] & X[1]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, equal] \\
X \arrow[r, dashed] & Y' \arrow[r, dashed] \arrow[d] & Z \arrow[r, dashed] \arrow[d, dashed] & X[1] \arrow[d, dashed, "u[1]"] \\
& W \arrow[r, equal] \arrow[d] & W \arrow[r, dashed] \arrow[d, dashed] & Y[1] \\
& Y[1] \arrow[r, "v[1]"] & Z[1] &
\end{tikzcd}$$

<u>Lemma</u>. If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a dist. triang, then ~~#~~ any composite of two arrows is zero: $gf = hg = f[1]\,h = 0$.

Moreover the infinite sequence

$$\dots \to Z[-1] \to X \to Y \to Z \to X[1] \to Y[1] \to \dots$$

has the prop that any ~~co~~ 4-term sequence is an exact triangle.

(11)

<u>Lemma</u>. If $X \to Y \to Z \to X[1]$ dist. triang and $W$ ..., then ~~the~~ we have a l.e.s. of ab gps

$$
\cdots \to \mathrm{Hom}(W, Z[i-1]) \to \mathrm{Hom}(W, X[i]) \to \mathrm{Hom}(W, Y[i]) \to \mathrm{Hom}(W, Z[i]) \to \mathrm{Hom}(W, X[i+1]) \to \cdots
$$

<u>Corollary</u>. In axiom (V), if $a, b$ are isos, so is $c$.

<u>Pf</u>. By Yoneda lemma, enough to exhibit (natural) isos $\mathrm{Hom}(U, C) \xrightarrow{\cong} \mathrm{Hom}(U, C')$. But by the previous lemma this boils down to the five lemma. $\square$

<u>Def</u>. Given a triang cat $\mathcal{T}$, the cone of a map $f: A \to B$ is the object $C$ (unique up to iso by the previous corollary) st $f$ is completed to a triangle

$$A \xrightarrow{f} B \to C \to A[1]$$

<u>Warning</u>. Cone does <u>not</u> extend to a functor, this is saying that the arrow from (V) is not unique.

<u>Def</u>. A <u>dg category</u> is a cat enriched in $Ch^{\geq 0}(Ab)$.

Define $\mathcal{C}_{dg}(\mathcal{A})$ as the dg-cat w/

ob = ~~ch~~ cochain complexes

$$\mathrm{Hom}_{\mathcal{C}_{dg}(\mathcal{A})}(A^\bullet, B^\bullet) = \underline{\mathrm{Hom}}(A^\bullet, B^\bullet) \in \mathrm{Ch}(\mathrm{Ab})$$

$$\mathrm{Hom}^i = \prod_{n\in\mathbb{Z}} \mathrm{Hom}(A^n, B^{n+i})$$

For a dg cat $\mathcal{C}$, let $H^i(\mathcal{C}) = \begin{cases} \text{obj} = \text{same as } \mathcal{C} \\ \text{arr} = H^i(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \end{cases}$.

<u>Exercise</u>. $\mathrm{Ch}_h(\mathcal{A}) \simeq H^0(\mathcal{C}_{dg}(\mathcal{A}))$.

* <u>Theorem</u>. $\mathrm{Ch}_h(\mathcal{A})$ is triangulated, w/

- $[1]$ the shift functor
- triangles ~~...~~ ~~...~~ are of the form

$$A \xrightarrow{f} B \to \mathrm{Cone}\, f \to A[1]$$

$$\left(\mathrm{Cone}\, f := B \oplus A[1] \text{ w/ } d(b^i, a^{i+1}) = \left(f(a^{i+1}) + d b^i, -d a^{i+1}\right)\right)$$

* <u>Theorem</u>. $\mathcal{D}(\mathcal{A})$ is triangulated.

. Recall $\mathcal{D}(\mathcal{A}) \simeq Ch_h(\mathcal{A})[\text{Q IS}]^{-1}$.

• Suppose $\mathcal{T}$ is a triang cat and $\mathcal{S}$ a collection of morphisms that satisfies the Ore conditions. Then $\mathcal{T}[\mathcal{S}^{-1}]$ is additive.

Def. If $\mathcal{T}, \mathcal{S}$ as above, we say that $\mathcal{S}$ is compatible w/ the triangulated str if

1) $s \in \mathcal{S} \iff s[1] \in \mathcal{S}$.

2) In (iv), if $a, b \in \mathcal{S}$, then one can find $c \in \mathcal{S}$.

Proposition. If $\mathcal{S}$ Ore + compatibility w/ triang, then $\mathcal{T}[\mathcal{S}^{-1}]$ is also triangulated.

Pf. The shift functor is on obj the same as in $\mathcal{T}$. ( $\text{ob}\,\mathcal{T} = \text{ob}\,\mathcal{T}[\mathcal{S}^{-1}]$ ),

on arrows it's simply

$$\begin{tikzcd}[column sep=small] & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \longmapsto \begin{tikzcd}[column sep=small] & W \arrow[dl, "f{[1]}"'] \arrow[dr, "g{[1]}"] & \\ X & & Y \end{tikzcd}$$

Distinguished triangles are the smallest class closed under isos that contains the image under the loc. functor $j : \mathcal{T} \to \mathcal{T}[\mathcal{S}^{-1}]$ of the disting. triangles of $\mathcal{T}$.

Lemma. A ses $0 \to L \to M \to C \to 0$ in ~~K(...)~~ $Ch^{\geq 0}(\mathcal{A})$

induces a distinguished triangle

$$\begin{tikzcd} L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \mathrm{Cone}\, f \arrow[d, "\cong"] \\ L \arrow[r] & M \arrow[r] & C \end{tikzcd}$$

$\leftarrow$ iso in $D(\mathcal{A})$