Introduction to Chromatic Homotopy Theory

- Chromatic Homotopy theory is a part of stable htpy theory
- Every map $f: S^n \to X$, X pt space, induces $\Sigma f: \Sigma S^n \cong S^{n+1} \to \Sigma X$.

The stable htpy gps of X are

$$\pi_n^{st}(X) := \operatorname{colim}\left(\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots\right)$$

- Hard to get your hands on even in simple cases
- The stable htpy gps of spheres (or htpy gps of the sphere spectrum), $\pi_n(S)$, are the stable htpy gps of S^0:

$$\pi_n(S) := \pi_n^{st}(S^0) = \operatorname{colim}_m \pi_{n+m}(S^m)$$

Q. What do these gps look like?

- Let us go to the rational story:

Slogan: Rational htpy theory is easy

Corollary: Rational stable htpy theory is very easy (supposed to be, heuristic).

But can be turned into a thm:

Theorem (Serre): The Hurewicz map induces an isomorphism

$$\pi_n^{st}(X) \otimes \mathbb{Q} \xrightarrow{\cong} \widetilde{H}_n(X;\mathbb{Q})$$

Corollary: $\pi_n(S) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & n=0 \\ 0, & n>0 \end{cases}$

ie, the higher stable htpy gps of sphers are all torsion gps, and in fact Serre proved that $\pi_n(S)$ are finite for $n>0$.

Examples:

n	0	1	2	3	4	5	6	7	8	9
$\pi_n S$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2\times\mathbb{Z}/2$	$\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2$

Q. What's the pattern, what's going on here? Looking at this one would say that this is random.

- Let's start w/ providing examples of elem'ts in $\pi_n S$

"The only chance we can sucess in mathematics is when we can reduce a problem to liner algebra"

What can liner algebra tell us about $\pi_n(S)$?

- Note. $U(n)$ acts on $S^{2n} = \mathbb{C}^n \cup \infty$. (it acts on $\mathbb{C}^n$). We can think of this acton as a map

$$U(n) \longrightarrow Map_*(S^{2n}, S^{2n}) \cong \Omega^{2n}(S^{2n})$$

which induces

$$\pi_*(U(n)) \longrightarrow \pi_{*+2n}(S^{2n})$$

Taking colim $n \to \infty$, we get the complex J-homomorphism

$$J : \pi_*(U) \to \pi_*(S) . \qquad \text{a graded gp hom}$$

Here $U := \text{colim}\,(U(1) \hookrightarrow U(2) \hookrightarrow \dots)$ is the infinite unitary gp.

Q. ~~Why~~ is this useful? The rhs is mysterious, what we want to get a hand on, and the lhs is very simple, we understand pretty well:

Theorem (Bott periodicity): $\pi_m(U) \cong \begin{cases} \mathbb{Z}, & m \text{ odd} \\ 0, & m \text{ even} \end{cases}$

Since U is representing complex K-theory, this could be rephrased as

$$K^m(X) \cong K^{m+2}(X)$$

ie K is a periodic cohom theory.

Upshot. For each $n \geq 0$, the complex J-homomorphism determines a map

$$\mathbb{Z} \cong \pi_{2n-1}(U) \longrightarrow \pi_{2n-1}(S)$$

ie its image is a cyclic subgp of the finite gp $\pi_{2n-1}(S)$

Q (Adams). What does the image of the complex J-hom look like?

- To understand the answer, it is best to understand one prime at a time, ie the image is going to be a finite abelian grp, ie a sum $\bigoplus \mathbb{Z}/p$ for different p's. For simplicity let us take $p \neq 2$

- To begin w/ let us consider an upper bound for the image of the J-hom: for this recall a feature of complex K-theory: it is equipped w/ the Adams operations ie

$$\psi^g : K^0(X) \to K^0(X) \qquad g \geq 0 \text{ integer}$$

characterized by what it does to complex vb which are sums of line bundles:

$$\psi^g([L_1 \oplus \cdots \oplus L_k]) = [L_1^{\otimes g} \oplus \cdots \oplus L_k^{\otimes g}].$$

As $\pi_{2n-1}(U) \cong \pi_{2n}(KU) \cong \tilde{K}^0(S^{2n}) \cong \mathbb{Z}$, then

spectrum rep. K thy

ψ^g acts on $\pi_{2n}(KU)$ by multiplication by g^n.

<u>Theorem</u> (Adams conjecture, showed by Quillen): Applying ψ^g does not change the J-hom, ie

$$J \circ \psi^g = J.$$

- So, the image of the J-hom is a cyclic gp gen by some elmt, $[1]=J(1)$, $[1)$ gen of $\mathbb{Z}/n = \operatorname{Im} J$. But by the thm $[1] = [g^n]$, ie

$\mathbb{Z}/(g^n-1)$ is an upper bound for $\operatorname{Im} J$ (for bigger values $[g^n]\neq[1]$).

ie $[1]$ must be annihilated by g^n-1. And this is true for all g.

- If you are interested in the p-local component, you get the best possible information by choosing special values of g, namely take g to be a top generator for the group $\mathbb{Z}_p^\times$ of p-adic units.

- What about a lower bound? Let us write $\hat{K}$ for the p-completed K-theory. That is, $\hat{K}(X) = K(X)^\wedge_p$ where for an abelian gp A and p prime, the p-completion of A is

$$A^\wedge_p := \lim_n A/p^nA .$$

(eg if $A=\mathbb{Z}$ then $A^\wedge_p = \mathbb{Z}_p$ the p-adic numbers, and if A is finitely gen then $A^\wedge_p \cong A\otimes\mathbb{Z}_p$.

- So if we complete, now it turns out that if $p\nmid g$ then ψ^g defines an automorphism of $\hat{K}$,

$$\psi^g : \hat{K}(X) \to \hat{K}(U)$$

let us look at the fixed pts of this automorphism: write $\hat{K}^{\psi^g=1}$ to be the "fixed pts" of this automorphism. More precisely, in the world of spectra, $\hat{K}^{\psi^g=1}$ is the htpy fixed pts of ψ^g on $\hat{K}$. Ie, get a fiber sequence (in spectra)

$$\hat{K}^{\psi^g=1} \longrightarrow \hat{K} \xrightarrow{\psi^g - \mathrm{Id}} \hat{K}$$

- We can use this to understand the htpy of $\hat{K}^{\psi^g=1}$: we understand the htpy gps of $\hat{K}$ by Bott periodicity:

$$\hat{K}_{2n} \cong \mathbb{Z}_p \qquad \hat{K}_{2n+1} = 0$$

and $\psi^g - \mathrm{Id}$ acts on $\mathbb{Z}_p$ by multiplication by $g^n - 1$. So looking at the induced les we conclude

$$\pi_{2n}(\hat{K}^{\psi^g-1}) \cong 0 \quad , \quad \pi_{2n-1}(\hat{K}^{\psi^g-1}) = \mathbb{Z}_p / g^n - 1 .$$

<u>Theorem</u> (Adams) let p odd and g a top generator of $\mathbb{Z}_p^\times$. Then the map

$$\pi_{2n}(K) = \pi_{2n-1}(U) \xrightarrow{J} \pi_{2n-1}(S) \longrightarrow \pi_{2n-1}(\hat{K}^{\psi^g=1})$$

is surjective.

- This gives a lower bound which matches the upper bound!

<u>Upshot</u>. $(\mathrm{Im}\, J)_{(p)} = \begin{cases} 0, & (p-1) \nmid n \\ \mathbb{Z}/p^{k-1}, & n = (p-1)p^k m, \quad p \nmid m. \end{cases}$

- Putting together all the answers of "what happens at the prime p" we get

n	0	1	2	3	4	5	6	7	
$\pi_n S$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	...
$\mathrm{Im}\, J$	NA	$\mathbb{Z}/2$	0	$\mathbb{Z}/24$	0	0	0	$\mathbb{Z}/120$	...

- Things to read off:
  - The J-hom is non-trivial, but it is not really giving us everything. In particular it is not giving anything in even degrees.
  - If we zoomed out we would see that the J-hom is not really seeing that much from $\pi_n S$. Ie, in large dimensions $\mathrm{Im}\, J$ is only a small part of $\pi_* S$. Yet it is a part that we can <u>completely</u> understand.

- Asking the "right" question can reveal orderly (periodic) behaviour amidst apparent chaos.

<u>Ex</u>. At an odd prime p, $\mathrm{Im}\, J \subset \pi_{2n-1}(S)$ is nontrivial exactly when $(p-1) \mid n$.

- You can think of chromatic htpy theory as an attempt to generalise the previous story to get more complete information about the stable htpy grps of spheres.

- What was important in the prevs story? the spectrum K. Now, what happens if we replace it by some other coh. theory?

Essential features of K:

- Bott periodicity
- Adams operations

- Let E be a multiplicative cohoml theory, ie

$$E^* : \mathrm{Top}^{op} \longrightarrow \text{Graded (com) rings}$$
$$X \longmapsto E^*(X)$$

Say that E is <u>even</u> if $E^n(pt) = 0$ when n odd.

Say E <u>periodic</u> if $E^*(pt)$ contains an invertible elmt of deg 2.

<u>Example</u>. K is even periodic.

- I want to talk about things that you could do w/ an even periodic coh theory.

Recall. $\mathbb{CP}^\infty = BU(1)$, i.e. it is the space that classifies complex line bundles. There is a complex line bundle $\mathcal{O}(1) \to \mathbb{CP}^\infty$ st

$$[X, \mathbb{CP}^\infty] \xrightarrow{\cong} \left\{ \begin{array}{l} \text{Isomorph classes} \\ \text{of complex} \\ \text{line bundles on } X \end{array} \right\}$$

via pullback.

Now, $H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[t]$, $|t| = 2$, and t coh class well def up to sign.

If L is a line bundle on X classified by $f : X \to \mathbb{CP}^\infty$, then

$$c_1(L) := f^*(t) \in H^2(X; \mathbb{Z})$$

is the first Chern class.

. Now, what about our generalised coh theory? If E is an even periodic coh theory, then a more or less similar computation gives

$$E^*(\mathbb{CP}^\infty) \cong E^*(p)[[t]] \qquad , \quad |t| = 0$$

t depends on a choice. Now if L is a line bundle class by $f : X \to \mathbb{CP}^\infty$,

$$c_1^E(L) := f^*(t) \in E^0(X)$$

is the first Chern class in E-cohomology

• An important prop of Chern classes (in the usual setup) is that they are additive:

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

<u>Legend</u>. This all theory goes back to a mistake that Quillen made. When he was thinking about coh theories when you have a good notion of Chern class. Initially he assumed that this formula would be true in general but he quickly realised that that was not the case for Chern classes in generalised coh theories.

<u>Ex</u>: If $E = K$, $c_1^E(L) = [L] - 1$. This satisfies

$$c_1^E(L \otimes L') + 1 = \left(c_1(L) + 1\right)\left(c_1(L') + 1\right).$$

ie $\quad c_1(L \otimes L') = c_1(L) + c_1(L') + c_1(L)\, c_1(L')$.

• For a general even periodic coh theory, you don't expect that either of the formulas are going to be correct; but you can expect that there is always <u>some</u> formula, depending on E, ie

$$c_1(L \otimes L') = F(c_1(L), c_1(L'))$$

where F is a power series in two variables.

This follows from examining the universal case $X = \mathbb{CP}^\infty \times \mathbb{CP}^\infty$,

$$E^*(X) \cong E^*(pt)[[t_0, t_1]] .$$

For any of these even periodic coh theories there is some power series F w/ this property. In general this power series is complicated but it's not arbitrary: it will always satisfy.

- $F(0,t) = t$
- $F(t_0,t_1) = F(t_1, t_0)$ ⟵ conseq of the fact that $\otimes$ for line bundles is commutative (up to iso)
- $F(t_0, F(t_1,t_2)) = F(F(t_0,t_1), t_2)$

↖ conseq. that $\otimes$ associative up to iso.

So we found an alg structure: a <u>formal gp law</u>.

<u>Def</u>. A <u>formal gp law</u> over R comm ring is a power series $F(s,t) \in R[[s,t]]$ satisfying the equalities above.

Eg: $F(s,t) = s+t$ (additive formal gp law)

Eg: $F(s,t) = s+t+st$ (multiplicative ———)

Def. Two formal gp laws F, F' are *isomorphic* if they differ by a change of coordinates, ie $\exists\, g \in R[\![u]\!]$ invertible st

$$F'(s,t) = g^{-1}\big(F(g(s), g(t))\big)$$

In this case we say that F, F' determine the same *formal group*

Moral: Every even periodic coh theory E determines a formal gp law over $R := E^0(pt)$ characterised by

$$c_1^E(L \otimes L') = F(c_1(L), c_1(L')).$$

This gives a construction

Even periodic coh theories E $\longrightarrow$ Comm rings R w/ a formal gp over R

topology

algebraic object, well-understood, they come up in alg number theory

It turns out that it is often possible to reverse this construction:

<u>Theorem</u> (Landweber). If F formal gp law over R comm ring, then under additional assumptions there is a unique even periodic coh theory E which gives rise to (R,F) under the preceding construction, ie so that $R \cong E^0(pt)$.

- I don't want to mention here what the assumptions are. But $R=\mathbb{Z}$ and $F(s,t) = s+t+st$ satisfies those assumptions, so K is the unique coh theory w/ such formal gp law. ~~the this~~
- A virtue of this is that it allows us to discover complex K-theory in a completely new way! It gives a "purely alg" approach.

Do we understand formal gps laws?

<u>Classification of formal gp laws</u>: let k be an alg closed field.

- If char $k = 0$, then any formal gp law is isomorphic to the additive fgl.
- If char $k = p > 0$, then any fgl is determined by an invariant called the <u>height</u>, $\in \mathbb{Z}_{>0} \cup \infty$.

<u>Ex</u>. F additive has height ∞

F multipl. has —— 1.

This was a story in pure algebra. What can we say about the corresponding coh th?

- Let k be a perfect field of char p and F/k fgl of height $0<n<\infty$.

<u>Theorem</u> (Morava). F comes from an even periodic coh theory $K(n)$.

The coh theories $K(n)$ are called <u>Morava K-theories</u>

<u>Ex</u>: $k = \underset{=\mathbb{Z}/p}{\mathbb{F}_p}$, $F(s,t) = s+t+st$, then $K(1) = K/p$. $\left(\begin{array}{l}\text{over } \mathbb{Z}\ F \text{ comes} \\ \text{from complex K-thy} \\ K\end{array}\right)$

<u>Theorem</u> (Lubin-Tate). If k, p, F and n are as before, F has a universal deformation $\widetilde{F}$ over the ring $R \cong \underbrace{W(k)[\![v_1,\dots,v_{n-1}]\!]}_{\uparrow \text{complete local ring w/ residue field } k}$

- The formal gp $\widetilde{F}$ arises from an even periodic coh th E_n, called <u>Morava E-theory</u>.

- This gives you a machinery to produce all kind of coh theories which are potentially interesting, but how can we exploit them? I want to take a detour to Bousfield localization:

Let E be a (co)homology theory. A map of spectra $f: X \to Y$ is an <u>E-equivalence</u> if it induces an isomorphism in E-(co)homology.

A spectrum Z is <u>E-local</u> if any E-equivalence $f: X \to Y$ induces a bijection

$$f_*: [Y, Z] \to [X, Z].$$

<u>Theorem</u> (Bousfield). For any spectrum X, there is an E-equivalence $f: X \to Y$ where Y is E-local.

- Standard abstract nonsense will tell you that such Y is unique, and denoted $L_E(X)$ the <u>E-localisation of X</u>.

When $E = E_n$ is a Morava E theory, $L_{E_n}(X)$ is denoted $L_n(X)$.

- Now: fix a spectrum X and p prime. We can actually think of $L_n(X)$'s as approximations to X, which get better as n increases. These approximations are related to each other: they can be organised as an inverse system

$$\cdots \to L_3(X) \to L_2(X) \to L_1(X) \to L_0(X) := X_{\mathbb{Q}}$$

rationalisation of X

which is called the <u>chromatic tower</u> of X

Remark. $L_1(X)$ is related to the story about the J-hom:

$$\pi_n(L_1(S)) = \begin{cases} \mathbb{Z}_{(p)} & , n=0 \\ \operatorname{Im} J \subset \pi_n(S)_{(p)} & , n>0 \end{cases}$$

• Now, what can we say about the chromatic tower of X?

Theorem (Chromatic convergence thm, Hopkins-Ravenel). If X is a finite spectrum, then holim $L_n(X)$ recovers the p-localisation $X_{(p)}$.

Eg: If $X = S$ sphere spectrum, then this tells you that if you are interested in the p-local stable htpy gps of spheres, all the information is already contained in this ~~the~~ chromatic tower. All you need to do is to understand these localisations $L_n(S)$

• How to understand those? Fix a spectrum X, p prime and $n>0$ integers. There is a htpy pullback diagram

$$\begin{tikzcd} L_n(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_h"] & L_{K(n)}(X) \arrow[d] \\ L_{n-1}(X) \arrow[r] & L_{n-1}\left(L_{K(n)}(X)\right) \end{tikzcd}$$

This is non-trivial, but the takeaway is that if you want to understand L_n, it suffices to understand L_{n-1} and $L_{K(n)}$.

• Let us focus on $L_{K(n)}$. Understanding this comes down to understanding the appropriate generalisation of Adams operations.

Let F be a formal gp of height $n < \infty$ over $k = \overline{\mathbb{F}_p}$ (alg clos. of $\mathbb{F}_p$)
Let $\mathbb{G}_0$ be the (profinite) gp of automorphisms of F (keeping k fixed), and $\mathbb{G}$ the gp of automorphisms of (k, F) (pairs). These are related by the ses

$$0 \to \mathbb{G}_0 \to \mathbb{G} \to Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p) \to 0$$

<u>Theorem</u> (Hopkins - Miller). The groups $\mathbb{G}$ acts on the Morava E-theory spectrum E_n.

<u>Eg</u>: $n=1$, then $\mathbb{G}_0 \cong \mathbb{Z}_p^{\times}$, and the action of $\mathbb{G}_0$ ~~on~~ E_1 is (essentially) by Adams operations

• Why is this useful? Because we can look at the htpy fixed pts:

<u>Theorem</u> (Devinatz - Hopkins) Let E_n be the Morava E-theory associated to a formal gp of height n over $\overline{\mathbb{F}_p}$. Then $L_{K(n)}(S)$ can be recovered as the (continuous) htpy fixed pts for the action of $\mathbb{G}$ on E_n.

Eg. p odd and $n=1$, then $L_{K(1)}(S)$ is the spectrum $\hat{K}^{\psi^g=1}$.

Summary :

I have given you is an advertisement that sounds as a blueprint for understanding stable htpy theory. What do you need to do?

In principle

1) Start by writing down these Morava E-theories E_n. These are spectra satisfying something like Bott periodicity, and their htpy gps are completely known

2) Understand the symmetry gp G and pass to htpy fixed pts $E_n^{hG} = L_{K(n)}(S)$

3) The previous thing depends on n, but allowing n to vary, assemble these together for $n \leq m$ to the spectrum $L_m(S)$.

4) Pass to the limit $m \to \infty$, obtain the p-local sphere $S_{(p)}$ by the chromatic convergence thm. Taking htpy gps, you are done.

In practice, things get complicated very easily.

- For $n \geq 2$, G is non-commutative
- Its action G on E_n is very complicated (even at the level of htpy)

Nevertheless, the overarching framework is extremely useful for understanding stable htpy theory.