Categorification in topology and knot theory

Category number (Dan Freed) : loose measure of the amount of abstraction included in a mathematical idea, theorem, construction, ...

$$0 \underset{\text{forget maps}}{\overset{\text{add maps}}{\rightleftarrows}} 1 \underset{\text{forget functors}}{\overset{\text{add functors}}{\rightleftarrows}} 2 \rightleftarrows \cdots$$

0

Numbers

"Thm: Two numbers are the same"

1

Math. structures

- Sets
- Groups
- Vector spaces

"Thm: Two vs are the same"

(Requires a construction)

2

Categories

(= class of math str's)

"Thm: Two categories are equivalent"

(Requires a deeper construction)

Categorify : Add abstraction (structure) and gain information.

Examples:

(1) ($0 \longrightarrow 1$): Finite sets is a categorification of the natural numbers

$\mathbb{N}$	fSet
n	S_n = set w/ n elements
$\#S_n = n$	
$n + m$	$S_n \amalg S_m$
0	$\emptyset$
$n \cdot m$	$S_n \times S_m$
1	$\{*\}$
(m^n	Map(S_n, S_m))

Not unique, can also take vector spaces:

$\mathbb{N}$	fd vector spaces / $\mathbb{K}$
n	V_n = vs of dim n
$\dim V_n = n$	
$n + m$	$V_n \oplus V_m$
0	0
$n \cdot m$	$V_n \otimes V_m$
1	$\mathbb{R}$
$n - m$	?

Def: A finite-dimensional chain complex is a graded vector space

$$C = \bigoplus_{i \in \mathbb{Z}} C^i = \left(\cdots \to C^{-1} \xrightarrow{\partial} C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} C^2 \to \cdots \right)$$

st only finitely many C^i are nonzero and all C^i are fd, equipped with a degree -1 self map $\partial : C \to C$ st $\partial^2 = 0$.

The Euler characteristic of C is

$$\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \dim C^i .$$

$\mathbb{Z}$	fd Ch
$n > 0$	$C = C^0_n$
$n < 0$	$C = C^1_n$
$n = \chi(C)$	
$n + m$	$C \oplus D$ where $(C \oplus D)^i = C^i \oplus D^i$
0	$C = 0$
$n \cdot m$	$C \otimes D$, where $(C \otimes D)^i = \bigoplus_{p+q=i} C^p \otimes D^q$
1	$C = k^1$

(Actually $\mathbb{Z} \cong K_0(\text{fd } Ch_k / \text{chain htpy eq})$, K_0 Grothendieck ring)

② (1 → 2)

<u>Universal property of the free abelian group</u> : S set, A abelian group, then a map $\mathbb{Z}[S] = \bigoplus_S \mathbb{Z} \to A$ of groups is determined by a map of sets $S \to A$.

$$\begin{tikzcd} S \arrow[r, "\text{sets map}"] \arrow[d] & A \\ \mathbb{Z}[S] \arrow[ur, dashed, "\exists\, \text{gp hom}"'] & \end{tikzcd}$$

The categorification of this statement is the existence of an adjunction

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "\mathbb{Z}[-]"] \arrow[r, phantom, "\perp"] & \mathrm{Ab} \arrow[l, bend left, "U"] \end{tikzcd}$$

which means that there is a bijection

$$\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}[S], A) \cong \mathrm{Hom}_{\mathrm{Set}}(S, UA)$$

which is natural in S and A.

③ Categorification of the Euler characteristic of top spaces

Let K be a finite simplicial complex (more generally a Δ-complex or a CW-complex).

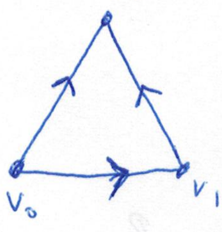
Let $c_i = \#$ i-simplices. Then the Euler characteristic is

$$\chi(K) = c_0 - c_1 + c_2 - c_3 + \dots$$

We never got to say why this is a topological invariant of $|K|$ (ie, if we pick another triangulation, the Euler char. will be the same).

Construction: let $C_n(K) := \mathbb{R}[\{n\text{-simplices}\}]$ and consider the chain complex $C(K) = \bigoplus_n C_n(K)$ with differential

$$\partial(\underbrace{[v_0, \dots, v_n]}_{n\text{-simplex}}) = \sum_{i=0}^{n} (-1)^i \underbrace{[v_0, \dots, v_{i-1}, v_{i+1}, \dots v_n]}_{(n-1)\text{-faces}}$$



$$\partial([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_1, v_2].$$

Exercise: $\partial^2 = 0$, so ∂ is indeed a differential.

- $C(K)$ is not a topological invariant of K, but its homology is:

$$H(C) := \frac{\operatorname{Ker} \partial}{\operatorname{Im} \partial}$$

(its degree 1 part, ie $H_1(C(K)) = \dfrac{\operatorname{Ker}(\partial : C_1(K) \to C_0(K))}{\operatorname{Im}(\partial : C_2(K) \to C_1(K))}$

is exactly $\pi_1(K)^{ab} \otimes_{\mathbb{Z}} R$).

- $H(C(K))$ can be viewed as another chain complex (with trivial differential),

and

$$\underline{\underline{\chi(H(C(K)))}} \overset{\text{Exercise}}{=} \chi(C(K)) = \underline{\underline{\chi(K)}}.$$

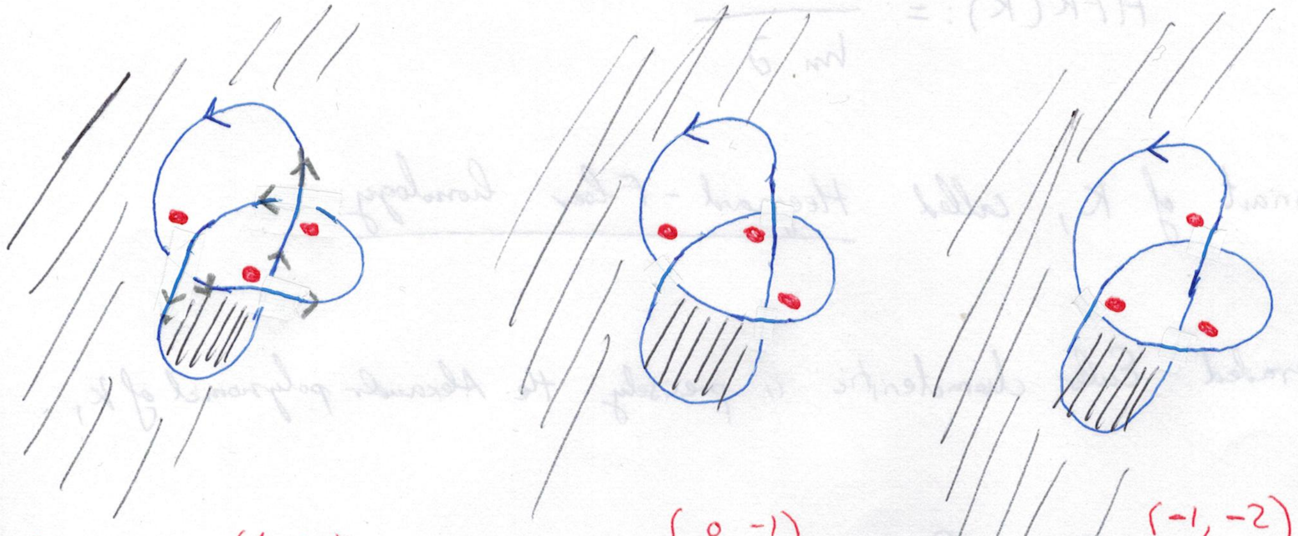
<u>Upshot</u>: The homology of K categorifies the Euler characteristic.

4) <u>Alexander polynomial</u>

Let K be a knot. Recall that $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ is an isotopy invariant of K. There is a categorification of Alexander in terms of a "graded Euler characteristic".

Let D be a knot diagram of K with n crossings. By an Euler char argument, D divides the plane in $n+2$ regions. Declare "forbidden" the external (unbounded) region and one adjacent to it, the rest are "allowed".

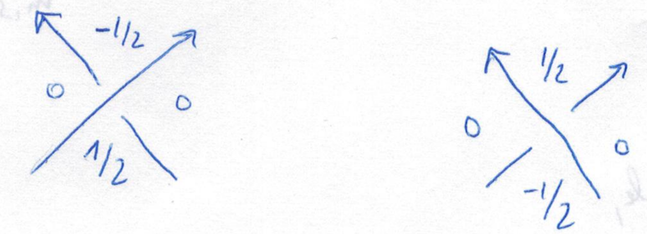
Definition: A Kauffman state for D is a choice of bijection between the crossings of D and the allowed regions.



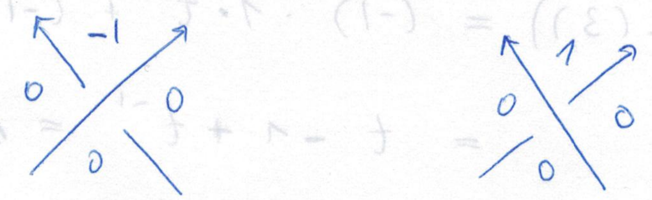
$(S, m) = (1, 0)$ $\qquad$ $(0, -1)$ $\qquad$ $(-1, -2)$

For each state, we want to assing two gradings according to the following rule:

Alexander grading $= S$



Maslov grading $= m$



Now let $CFK(D)$ be the free bigraded $\mathbb{Z}/2$-vector space generated by the Kauffman states, so in the example

$$CFK(D) = \mathbb{Z}/2_{(1,0)} \oplus \mathbb{Z}/2_{(0,-1)} \oplus \mathbb{Z}/2_{(-1,-2)}$$

Theorem: CFK(D) admits a differential of bidegree $(0,-1)$, and

$$\widehat{HFK}(K) := \frac{\ker \partial}{\operatorname{Im} \partial}$$

is an invariant of K, called Heegaard-Floer homology.

Its graded Euler characteristic is precisely the Alexander polynomial of K,

$$\Delta_K(t) = \chi_{gr}(\widehat{HFK}(K)) = \sum_{m,s} (-1)^m \dim \widehat{HFK}_{m,s}(K)\cdot t^s$$

$$= \sum_{m,s} (-1)^m \dim CFK_{m,s}(K)\cdot t^s$$

In the example,

$$\chi_{gr}(\widehat{HFK}(3_1)) = (-1)^0 \cdot 1 \cdot t + (-1)^{-1}\cdot 1 \cdot t^0 + (-1)^{-2} \cdot 1 \cdot t^{-1}$$

$$= t - 1 + t^{-1} = \Delta_{3_1}(t).$$

Upshot: Heegaard-Floer homology categorifies the Alexander polynomial.

Remark: HFK not only recovers Alexander but also strengthens its properties, eg $g(K) \geq \frac{1}{2}\,\text{breath}(\Delta_K(t))$ whereas $g(K) = \max \left\{ s : \widehat{HFK}_{m,s}(K) \neq 0 \text{ for some } m \right\}$.