The Alexander polynomial: the algebraic way

Aarnout: If $G \cong \dfrac{F(x_1 \dots x_n)}{\langle r_1 \dots r_s \rangle}$, consider the maps

$$\frac{\partial}{\partial x_i} : \mathbb{Z}[F_n] \longrightarrow \mathbb{Z}[F_n] \quad , \quad \begin{tikzcd} \mathbb{Z}[F_n] \arrow[r, "\gamma"] \arrow[rr, bend right, "\rho"'] & \mathbb{Z}[G] \arrow[r, "a"] & \mathbb{Z}[G^{ab}] \end{tikzcd}$$

where $F_n = F(x_1 \dots x_n)$ free group in n letters $x_1, \dots, x_n$,

γ induced by the projection $F_n \longrightarrow F_n/_{\langle r_i \rangle} \cong G$

a induced by the abelianization $G \longrightarrow G/_{[G,G]} = G^{ab}$.

Then the Alexander matrix of $\langle x_i \mid r_j \rangle$ is $A = \left(\rho\left(\dfrac{\partial r_i}{\partial x_j} \right) \right)$.

Theorem: The elementary ideals $\underset{\subset \mathbb{Z}[G^{ab}]}{\mathcal{E}_k(A)}$ do not depend on the choice of presentation of G.

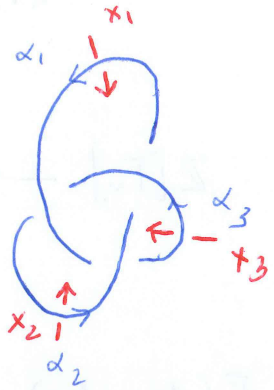
Aim: Create a new Knot invariant out of a presentation of the Knot group $\pi_K := \pi_1(S^3 - K)$ using the elementary ideals of the Alexander matrix of π_K.

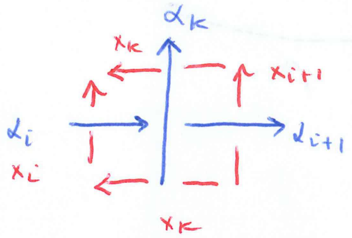
Since π_K is already a Knot invariant, what we create out of this will also be a Knot invariant.

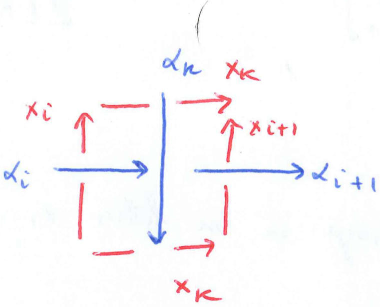
Proposition: Let K be a knot. Then $\pi_K^{ab} \cong \mathbb{Z}$.

Pf: By the Wirtinger algorithm, $\pi_K \cong \dfrac{F(x_1 \dots x_n)}{\langle r_1 \dots r_n \rangle}$,

where r_i is one of the following two relations:







r_i: $\quad x_K x_i = x_{i+1} x_K \qquad\qquad x_K x_{i+1} = x_i x_K$

In the abelianization, the relation r_i becomes $x_i = x_{i+1}$, so

$$\left(\frac{F(x_1 \dots x_n)}{\langle r_1 \dots r_n \rangle}\right)^{ab} \cong F(x_1) \cong \mathbb{Z} = \langle t \rangle$$

$$x_i \longleftrightarrow x_1 \longleftrightarrow 1 \longleftrightarrow t$$

$\square$

Exercise: Show that if L is a link of l components, then $\pi_1(S^3 - L)^{ab} \cong \mathbb{Z}^l$.

• The upshot is that our group ring at hand is $\mathbb{Z}[\langle t \rangle] = \mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials. Any element is of the form

$$p(t) = \sum_{i=n}^{m} a_i t^i, \qquad n \le m, \quad n, m \in \mathbb{Z}.$$
$$a_i \in \mathbb{Z}.$$

<u>Lemme</u>: The ring $\mathbb{Z}[t,t^{-1}]$ is a unique factorization domain, and its invertible elements are $\mathbb{Z}[t,t^{-1}]^{\times} = \{\pm t^{\pm n}\}$.

<u>Pf</u>: $\mathbb{Z}$ is a UFD (Euclides), therefore so is $\mathbb{Z}[t]$ (Gauss), and therefore

$\underset{\overset{\shortparallel}{\mathbb{Z}[t,t^{-1}]}}{\mathbb{Z}[t][t^{-1}]} = S^{-1}\mathbb{Z}[t]$, where $S = \{1, t, t^2, \dots\}$ (the localization of UFD is a UFD).

For the second part, we let $\varphi : \mathbb{Z}[t,t^{-1}] \longrightarrow \mathbb{Z}[t]$ be defined by

$$\varphi\left(p(t) = \sum_{i=n}^{m} a_i t^i\right) := t^{-n} \cdot p(t)$$

It is routine to check that φ is a ring homomorphism. Then, if $p(t) \in \mathbb{Z}[t,t^{-1}]^{\times}$ with inverse $q(t)$, we have

$$1 = \varphi(1) = \varphi(p(t) \cdot q(t)) = \varphi(p(t)) \cdot \varphi(q(t)).$$

Since the only units in $\mathbb{Z}[t]$ are ± 1, we get that $\varphi(p(t)) = \pm 1$ and therefore $p(t) = \pm t^{-n}$. $\square$

- In a unique factorization domain, we can define the gcd of a finite number of elements (and the proof is the same as for $\mathbb{Z}$).

• To sum up, given a knot K we can consider a collection of ideals $\mathcal{E}_r(A) \subset \mathbb{Z}[t,t^{-1}]$, where A is the Alexander matrix of π_K.

<u>Definition</u>: let K be a knot. The <u>r-th Alexander polynomial</u> $\Delta_r(t)$ of K is the generator of the smallest principal ideal $(\Delta_r(t)) \subset \mathbb{Z}[t,t^{-1}]$ containing $\mathcal{E}_r(A)$, that is,

$$\Delta_r(t) := \gcd(\mathcal{E}_r(A)).$$

The generator is unique up to an invertible element of $\mathbb{Z}[t,t^{-1}]$, ie, up to multiplication by $\pm t^{\pm n}$.

The first Alexander polynomial $\Delta_1(t)$ is called <u>the Alexander polynomial of K</u> and is denoted by $\Delta_K(t)$. In other words,

$$\Delta_K(t) = \gcd\left((n-1)\times(n-1) \text{ minors of } A\right).$$

• We can make the computation a bit easier:

<u>Fact</u>: In the Wirtinger presentation $\pi_K \cong \langle x_1,\dots,x_n \mid r_1,\dots,r_n\rangle$ of a knot K, one of the relations r_i (any) is superfluous, that is, it is a consequence of the rest of relations.

- Consequently, π_K admits a presentation $\langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$, so a Alexander matrix of dimension $(n-1) \times n$.

<u>Corollary</u>: $\mathcal{E}_0(A) = 0$, so $\Delta_0(t) = 0$.

- At this point we would still have to compute n determinants to get $\Delta_K(t)$. The following proposition asserts that it is enough to take one!

<u>Proposition</u>: Let A be the Alexander matrix of a knot $A \in M_{n \times n}(\mathbb{Z}[t, t^{-1}])$ obtained from a presentation of π_K given by the Wirtinger algorithm. Then A is equivalent to any of its $(n-1) \times (n-1)$ submatrices. Consequently, if A' is one of them then

$$\Delta_K(t) = \det A'.$$

<u>Pf</u>: We already saw that we can drop one of the rows of A. Let us see now that we can drop any column. The key point is that, as in calculus, the free calculus satisfies the equality

$$f(x) - f(1) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cdot (x_j - 1)$$

(here $f(1)$ stands for the image of f under the morphism $\mathbb{Z}[F_n] \longrightarrow \mathbb{Z}$, $x_i \longmapsto 1$, i.e., $f(1)$ is obtained by substituting every x_i by 1). Applying this to $f = r_i$ we get

$$r_i - 1 = \sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j} (x_j - 1).$$

Since $\rho(r_i) = 1$ and $\rho(x_k) = \rho(x_j) = t \quad \forall k, j = 1, \dots, n$, applying ρ to the latter expression we get

$$0 = \rho(r_i) - 1 = \sum_{j=1}^{n} \rho\left(\frac{\partial r_i}{\partial x_j}\right) \cdot (\rho(x_j) - 1)$$

$$= (t-1) \cdot \sum_{j=1}^{n} a_{ij}.$$

Since $(t-1) \neq 0$ and $\mathbb{Z}[t, t^{-1}]$ is a integral domain, $\sum_{j=1}^{n} a_{ij} = 0$, ie, any column is linear combination of the rest. $\square$

<u>Remark</u>: Since we already know the expression of every r_i, it provides us a quick way to compute $\Delta_K(t)$: namely, for a negative crossing, r_i is $x_k x_i = x_{i+1} x_k$. The only non-zero derivatives are

$$\frac{\partial r_i}{\partial x_k} = 1 - x_i \quad , \quad \frac{\partial r_i}{\partial x_i} = x_k \quad , \quad \frac{\partial r_i}{\partial x_{i+1}} = -1$$

After abelianizing, this becomes

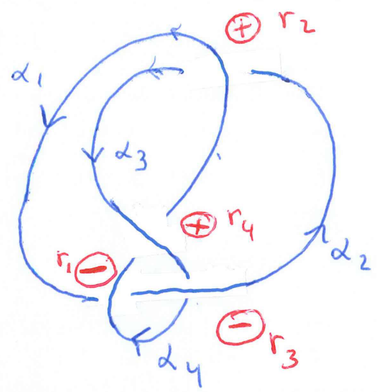
$$a_{ik} = 1 - t \quad , \quad a_{ii} = t \quad , \quad a_{i,i+1} = -1.$$

For a positive crossing, the coefficients are

$$a_{ik} = 1 - t \quad , \quad a_{ii} = -1 \quad , \quad a_{i,i+1} = t.$$

Later we only have to take the determinant of any $(n-1)\times(n-1)$ submatrix.

Example: Figure-of-eight knot 4_1



$$A = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} & \begin{pmatrix} t & -1 & 0 & 1-t \\ 1-t & -1 & t & 0 \\ 0 & 1-t & t & -1 \\ t & 0 & 1-t & -1 \end{pmatrix} \end{array}$$

$\Delta_K(t) = \det$ (any 3×3 submatrix)

$$= \begin{vmatrix} t & -1 & 0 \\ 1-t & -1 & t \\ 0 & 1-t & t \end{vmatrix} = -t^2 - t^2(1-t) + t(1-t)$$

$$= -t^2 - t^2 + t^3 + t - t^2 = t^3 - 3t^2 + t \doteq \underline{\underline{t^2 - 3t + 1}}.$$