Recap of "Topologie en Meetkunde"

1. FUNDAMENTAL GROUP

- $\pi_1(X,p) := \dfrac{\{\text{loops at } p\}}{\simeq}$, $\sigma \simeq \sigma'$ is homotopy rel. $\{0,1\}$

Concatenation is compatible with the homotopy relation, so it is a group (<u>not</u> abelian in general)

<u>Properties</u> : 1) If X is path-connected, then the fundamental group does <u>not</u> depend on the base-point. Precisely, if $p,q \in X$ and γ is a path from p to q, then

$$\begin{aligned} \beta_\gamma : \pi_1(X,p) &\xrightarrow{\cong} \pi_1(X,q) \\ [\sigma] &\longmapsto [\gamma^{-1}\sigma\gamma] \end{aligned} \qquad \text{is gp isomorphism}$$

2) Homotopy of paths is preserved by cont maps (ie, if $X \xrightarrow{f} Y$, and $\sigma \simeq \sigma'$ in X, then $f\circ\sigma \simeq f\circ\sigma'$ in Y). This defines a gp hom. $f_* : \pi_1(X,p) \to \pi_1(Y, f(p))$.

3) Homotopic maps induce the same morphisms in fundamental groups up to an isomorphism : if $f,g : X \to Y$ are homotopic via $H : X \times I \to Y$, $H_0 = f$, $H_1 = g$, then the following diagram commutes:

$$\begin{tikzcd} & \pi_1(Y,y_0) \arrow[d, "\beta_\gamma", "\wr"'] \\ \pi_1(X,x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\ & \pi_1(Y,y_1) \end{tikzcd}$$

where $x_0 \in X$, $y_0 = f(x_0)$, $y_1 = g(x_0)$, and $\gamma(s) := H(x_0, s)$.

4) Homotopy equivalent spaces have isomorphic fundamental groups: if $f : X \to Y$ is a homotopy equivalence (ie, $\exists\, g: Y \to X$ st $g \circ f \simeq \mathrm{Id}_X$, $f \circ g \simeq \mathrm{Id}_Y$), then $f_* : \pi_1(X, x_0) \xrightarrow{\cong} \pi_1(Y, f(x_0))$ is an isomorphism.

<u>Consequence</u>: Every contractible space has trivial fundamental group:

$\pi_1(\mathbb{R}^n) \cong 0$, $\pi_1(\text{convex subset of } \mathbb{R}^n) \cong 0$, ...

* <u>Theorem</u>: $\pi_1(S^1, 1) \cong \mathbb{Z}$, and it is generated by $\sigma(t) = e^{2\pi i t}$: $[\sigma^n] \longleftrightarrow n$

Consequences:

<u>Theorem</u> (Brower fixed point): Every continuous map $\varphi : D^2 \to D^2$ has a fixed point

<u>Theorem</u> (Borsuk-Ulam): If $\varphi : S^2 \to \mathbb{R}^2$ is a continuous map, then there is $x \in S^2$ such that $\varphi(x) = \varphi(-x)$.

"At every instant of time, there is a point on Earth with the same pressure and temperature as its antipode".

<u>Theorem</u> (Hairy ball): Every vector field on S^2 vanishes at some point.

- Let S be a set. The <u>free group</u> generated by S is

$$F(S) := \left\{ \begin{array}{l} \text{finite sequences } s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \text{ of elements} \\ \text{of } S \text{ with exponents } \varepsilon_i = \pm 1, \text{ where there are} \\ \text{no elements of the form } s^{\varepsilon} s^{-\varepsilon} \end{array} \right\}$$

It is a group with neutral element the empty sequence and multiplication is concatenation

$$(s_1^{\varepsilon_1} \cdots s_m^{\varepsilon_m}) \cdot (\bar{s}_1^{\bar{\varepsilon}_1} \cdots \bar{s}_r^{\bar{\varepsilon}_r}) := s_1^{\varepsilon_1} \cdots \bar{s}_r^{\bar{\varepsilon}_r}$$

where if there is a piece $s^{\varepsilon} s^{-\varepsilon}$, one "simplifies".

- Let G_1, G_2 be groups. The <u>free product</u> of G_1 and G_2 is

$$G_1 * G_2 := \left\{ \begin{array}{l} \text{finite sequences } g_1 \cdots g_n \text{ of elemts} \\ \text{of } G_1 \amalg G_2 \text{, when there are not two} \\ \text{consequtive elements of the same group, and} \\ \text{neither of the elements is the neutral elemts of } G_1, G_2 \end{array} \right\}$$

This is a group with neutral element the empty sequence and product is concadenation

$$(g_1 \cdots g_n) \circ (\bar{g}_1 \cdots \bar{g}_r) := g_1 \cdots g_n \bar{g}_1 \cdots \bar{g}_r$$

where if two consequtive elemts belong to the same group, you consider the product in the group.

- Let G_0, G_1, G_2 be groups and consider morphisms $\delta_1 : G_0 \to G_1$ and $\delta_2 : G_0 \to G_2$.
Consider the free product $G_1 * G_2$ and the composites

$$\begin{tikzcd} & G_1 \arrow[dr, hook] & \\ G_0 \arrow[ur, "\delta_1"] \arrow[dr, "\delta_2"'] \arrow[rr, bend left=60, "\varphi_1"] \arrow[rr, bend right=60, "\varphi_2"'] & & G_1 * G_2 \\ & G_2 \arrow[ur, hook] & \end{tikzcd}$$

, φ_i the composites

The <u>amalgamated product</u> of G_1 and G_2 over G_0 (wrt the morphisms δ_1, δ_2) is

$$G_1 *_{G_0} G_2 := \frac{G_1 * G_2}{\langle \varphi_1(g_0) = \varphi_2(g_0) \rangle_{g_0 \in G_0}}$$

where the quotient stands for the smallest normal subgroup of $G_1 * G_2$ which contains the subset $\{ \varphi_1(g_0)\varphi_2(g_0)^{-1} : g_0 \in G_0 \}$.

* <u>Theorem</u> (van Kampen) : Let X be a topological space and let $\mathcal{U}_1, \mathcal{U}_2 \subseteq X$ path-connected open subsets of X st $X = \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_0 := \mathcal{U}_1 \cap \mathcal{U}_2$ is non-empty and path-connected. Then

$$\pi_1(X, p) \cong \pi_1(\mathcal{U}_1, p) *_{\pi_1(\mathcal{U}_0, p)} \pi_1(\mathcal{U}_2, p),$$

where $p \in \mathcal{U}_0$ and the morphisms are induced by inclusions,

$$\begin{tikzcd}[column sep=small] & \mathcal{U}_1 \\ \mathcal{U}_0 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & \\ & \mathcal{U}_2 \end{tikzcd} \implies \begin{tikzcd}[column sep=small] & \pi_1(\mathcal{U}_1, p) \arrow[dr, hook] & \\ \pi_1(\mathcal{U}_0, p) \arrow[ur, "i_{1,*}"] \arrow[dr, "i_{2,*}"'] & & \pi_1(\mathcal{U}_1, p) * \pi_1(\mathcal{U}_2, p) \\ & \pi_1(\mathcal{U}_2, p) \arrow[ur, hook] & \end{tikzcd}$$

<u>Consequence</u> : <u>Fundamental group of connected cell-complexes</u>

Let X be a con. cell complex, $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X$, X_n the <u>n-skeleton</u>. This means that X_0 is a discrete space and X_n arises from X_{n-1} by attaching n-cells.

1) $\pi_1(X, p) \cong \pi_1(X_2, p)$

2) $\pi_1(X_1, p) \cong F(\sigma_1 \ldots \sigma_r)$, where $r = 1 - \chi(X_1)$ is the maximum number of 1-cells that we can remove from X_1 so that it is still connected, iow, the number of 1-cells we have to remove to find a maximal tree. The σ_i's are the generators of $\pi_1(\bigvee_{i=1}^{r} S^1)$ that we get after collapsing the maximal tree.

3) $\pi_1(X_2, p) \cong \dfrac{\pi_1(X_1, p)}{\langle \gamma_1, \dots, \gamma_k \rangle}$ where k is the number of 2-cells and γ_i's are loops running through the boundary of the 2-cells only once.

<u>Examples</u>: $\pi_1(S^n) \cong 0$, $(n \geq 2)$, $\pi_1(\mathbb{T}) \cong \mathbb{Z} \oplus \mathbb{Z}$,

$$\pi_1(\mathbb{K}) \cong \frac{F(a,b)}{\langle abab^{-1} \rangle} \overset{\text{notation}}{=} \langle ab \mid abab^{-1} \rangle ,$$

$\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$, $(n \geq 2)$; $\pi_1(\mathbb{RP}^1 \cong S^1) \cong \mathbb{Z}$.

2. CLASSIFICATION OF SURFACES

- <u>Surface</u> = connected compact topological manifold of dimension 2.
- A <u>triangulation</u> of a surface S is a homeomorphism $S \xrightarrow{\cong} K$, where K is a simplicial complex, ie, it is a space made out of glueing Δ^2's by their edges with the following rules

  (a) Every edge in K has two distint endpoints

  (b) For every two vertices there is <u>at most</u> one edge between them.

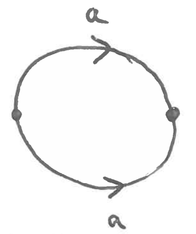
  (c) Every triangle has three different edges

  (d) For every 3 vertices there is <u>at most</u> one triangle spanned by them.

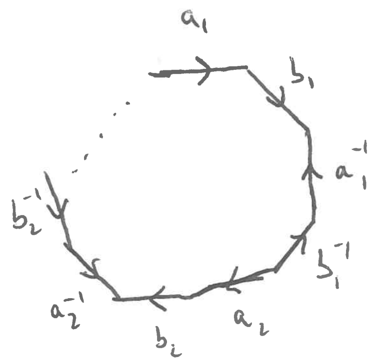
* <u>Theorem</u> (Radó) : Every surface has a triangulation.

- The theorem implies that every surface is homeomorphic to one which is obtained by identifying pairs of edges in a polygon.

• Consider the following surfaces:

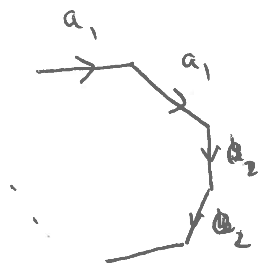
$S^2 =$ 

$\boxed{a\,a^{-1}}$

$S_g =$ 

$(g \geq 1)$

$\boxed{a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}}$

$N_g =$ 

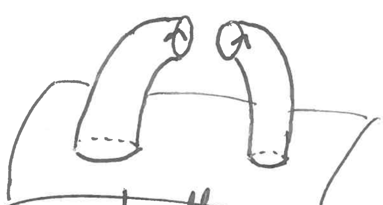
$\boxed{a_1 a_1 \cdots a_g a_g}$ $(g \geq 1)$

<u>Theorem</u> (Classification, I): Every compact surface is homeomorphic to S^2, S_g or N_g.

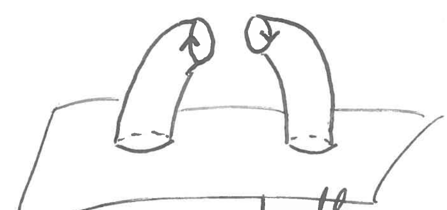
• One says that S_g has <u>genus</u> g. Sometimes $S^2 = S_0$.

• A surface is <u>non-orientable</u> if it has an open subset homeomorphic to the Möbius strip. Otherwise it is <u>orientable</u>. N_g is non-orientable, whereas S^2, S_g are orientable.

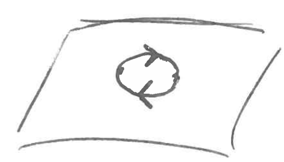
• Let S be a surface. Attaching a <u>handle</u> means to choose embeddings of two disks, remove their interiors and identify their boundaries with opposite orientation, ie $\circlearrowright \; \circlearrowleft$. A <u>cross-handle</u> is the same but orientations agree, $\circlearrowright \; \circlearrowright$. A <u>cross-cap</u> consists in choosing an embedding of a disk, remove the interior and identify the boundary antipodally.



handle



cross-handle



cross-cap

Theorem (Classification, II): Every surface S is homeomorphic to one of the following:

(a) If S is orientable, a sphere with $g \geq 0$ handles (ie, S^2 or S_g).

(b) If S is not orientable, a sphere with $g > 0$ cross-caps (ie, N_g).

- Note that $S_g = \mathbb{T} \# \overset{g}{\cdots} \# \mathbb{T}$ and $N_g = \mathbb{RP}^2 \# \overset{g}{\cdots} \# \mathbb{RP}^2$, because

attaching a handle $= \# \mathbb{T}$, cross-cap $= \# \mathbb{RP}^2$,

cross-handle $= \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

- The Euler characteristic of surfaces are:

$$\chi(S^2) = 2, \qquad \chi(S_g) = 2 - 2g, \qquad \chi(N_g) = 2 - g.$$

Theorem (Classification, III): A surface is determined by its orientability and Euler characteristic.

- $\pi_1(S^2) \cong 0$, $\quad \pi_1(S_g) = \dfrac{F(a_1 b_1 \ldots a_g b_g)}{\langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle}$, $\quad \pi_1(N_g) = \dfrac{F(a_1 \ldots a_g)}{\langle a_1 a_1 \cdots a_g a_g \rangle}$

Theorem (Classification, IV): Let S be a surface.

(a) $S \cong S^2 \iff \pi_1(S) = 0$

(b) $S \cong S_g \iff \pi_1(S)^{ab} \cong \mathbb{Z} \oplus \overset{2g}{\cdots} \oplus \mathbb{Z}$

(c) $S \cong N_g \iff \pi_1(S)^{ab} \cong \mathbb{Z} \oplus \overset{g-1}{\cdots} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

3. COVERING SPACES

- A <u>covering map</u> is a cont map $p: E \to B$ st every point in B has a neighbourhood $\mathcal{U}$ with the property that

$$p^{-1}(\mathcal{U}) \cong \coprod_{i \in I} \mathcal{U}_i \quad , \quad \mathcal{U}_i \xrightarrow[\cong]{p} \mathcal{U} .$$

- $\begin{array}{c} S^1 \to S^1 \\ z \mapsto z^n \end{array}$, $\begin{array}{c} \mathbb{R} \to S^1 \\ t \mapsto e^{2\pi i t} \end{array}$, $\begin{array}{c} S^2 \to \mathbb{RP}^2 \\ x \mapsto [x] \end{array}$ are cov. maps.

<u>Lemma</u> (Uniqueness of liftings) : let $p: (E, e_0) \to (B, b_0)$ a pointed covering map and let $\varphi: (T, t_0) \to (B, b_0)$ a cont map, with T <u>connected</u>. if there is a lift $\tilde{\varphi}: (T, t_0) \to (E, e_0)$ (ie, st $p \circ \tilde{\varphi} = \varphi$), then the lift is unique.

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (T, t_0) \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"] & (B, b_0) \end{tikzcd}$$

<u>Theorem</u> (Lifting criterion) : let $p: (E, e_0) \to (B, b_0)$ be a covering map, and let $f: (X, x_0) \to (B, b_0)$ a pointed map, with X connected and loc. path-connected.

There exists a unique lifting $\tilde{f}: (X, x_0) \to (E, e_0) \iff f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(E, e_0))$

$$\begin{tikzcd} & (E, e_0) \arrow[d] \\ (X, x_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (B, b_0) \end{tikzcd}$$

• In particular, for a path $\sigma : (I, 0) \longrightarrow (B, b_0)$, there is a unique lifting $\tilde{\sigma} : (I, 0) \to (E, e_0)$ since I is simply-connected. Similarly, we lift homotopies of paths $(I \times I, (0,0)) \to (B, b_0)$ in a unique way

<u>Corollary</u>: Let $p : (E, e_0) \to (B, b_0)$ be a covering map and let α, β be two paths in B with same endpoints. If $\tilde{\alpha}, \tilde{\beta}$ are their liftings, then

$$\alpha \simeq \beta \text{ rel } \{0,1\} \iff \tilde{\alpha} \simeq \tilde{\beta} \text{ rel } \{0,1\}.$$

In particular, it says that $\tilde{\alpha}, \tilde{\beta}$ also have same endpoints (<u>not true</u> if α, β not homotopic)

<u>Proposition</u>: If $p : (E, e_0) \to (B, b_0)$ is a covering map, then $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective.

<u>Proposition</u>: Let $p : E \to B$ be a covering map. If B is <u>connected</u>, all fibres have the same cardinality; and if it is finite (say n), then we say that p is <u>n-sheeted</u>. In particular, p is surjective.

• Let G be a group and suppose there is an action of G on X (ie, a group homomorphism
$$\begin{aligned} G &\longrightarrow \mathrm{Homeo}(X) \\ g &\longmapsto g : X \xrightarrow{\simeq} X \end{aligned}$$
). The action is called <u>properly discontinuous</u> if every point has an open neighbourhood U such that

$$g \neq g' \implies g(U) \cap g'(U) = \emptyset, \quad \forall g, g' \in G.$$

• If G is a group acting on X, then $X/G := X/\sim$ with $x \sim x' \iff \exists g \in G : x' = gx$.

<u>Proposition</u>: If the action $G \circlearrowright X$ is properly discontinuous, then $X \to X/G$ is a covering map.

• If $p: E \to B$ is a covering map, a <u>deck</u> or <u>covering transformation</u> is an homeomorphism $g: E \xrightarrow{\cong} E$ st $p \circ g = p$,

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

. The set of deck transformations forms a group $G(p)$ under composition. A group G acting on E and commuting with the covering map can be viewed as a subgroup of $G(p)$.

<u>Proposition</u>: let $p: E \to B$ cov map and $G \leq G(p)$. Then

1) $E \to E/G$ is a covering map
2) $E/G \to B$ is a covering map

• The condition of deck transformation means that such homeomorphism permutes elements on every fibre.

• A covering map $p: E \to B$ is <u>normal</u> if $G(p)$ acts transitively on fibres, ie, if given $e, e' \in p^{-1}(b)$, there is $g \in G(p)$ st $e' = ge$. (Here E, B are connected)

• $p: \begin{array}{l} S^1 \longrightarrow S^1 \\ z \longmapsto z^n \end{array}$ normal and $G(p) \cong \mathbb{Z}/n\mathbb{Z}$; $p: \begin{array}{l} \mathbb{R} \longrightarrow S^1 \\ t \longmapsto e^{2\pi i t} \end{array}$ normal and $G(p) \cong \mathbb{Z}$;

$G \curvearrowright X$ prop disc, then $p: X \to X/G$ normal and $G(p) \cong G$.

• A covering map $p: E \to B$ with E simply-connected is called <u>universal</u>. If there exists, then it is unique

* <u>Theorem</u>: Let $p: (E, e_0) \to (B, b_0)$ be the universal covering of B, with B locally path-connected. Then $G(p) \cong \pi_1(B, b_0)$.

• In particular, we can identify subgroups of the fundamental group with subgroups of deck transformations.

* **Theorem** (Classification of covering spaces): let (B, b_0) be a path-connected, locally path-connected space and let $p : (E, e_0) \to (B, b_0)$ be a simply connected covering space (thus universal). There is a one-to-one correspondence

$$\left\{ \begin{matrix} \text{subgroups of} \\ \pi_1(B, b_0) \end{matrix} \right\} \;=\; \left\{ \begin{matrix} \text{pointed connected} \\ \text{covering spaces} \\ (Y, y_0) \to (B, b_0) \end{matrix} \right\}$$

$$H \longmapsto (E/H, [e_0]) \to (B, b_0)$$

$$q_*(\pi_1(Y, y_0)) \longleftarrow\!\!\shortmid \; q : (Y, y_0) \to (B, b_0)$$

$$\text{normal subgroups} \longleftrightarrow \text{normal covering maps.}$$

Here two isomorphic covering spaces are considered the same, because such iso is unique.

$$\boxed{\begin{tikzcd} (E, e_0) \arrow[d, "p"'] \arrow[dr, "\pi"] & \\ (B, b_0) & (E/H, [e_0]) \arrow[l, "q"] \end{tikzcd}}$$

normal covering and $G(\pi) \cong H$

universal $\Rightarrow$ normal and $G(p) \cong \pi_1(B)$

q covering, and it is a normal covering $\Leftrightarrow$ H is normal, and in such a case $G(q) \cong G(p)/H$

- If the covering maps are not considered pointed, then the bijection is rewritten as

$$\left\{ \begin{array}{c} \text{subgroups of} \\ \pi_1(B) \\ \text{mod. conjugacy} \end{array} \right\} = \left\{ \begin{array}{c} \text{connected covering maps} \\ Y \longrightarrow B \\ \text{mod. isomorphism} \end{array} \right\}$$

In particular, if $\pi_1(B)$ is abelian, we do not have to take care of basepoints.

<u>Theorem</u> (Existence of the universal covering): A path-connected, locally path-connected space B has a universal covering space $\Longleftrightarrow$ B is <u>semi-locally simply connected</u>, ie, there is a cover $\{U_i\}$ of B st the maps induced by the inclusions $\pi_1(U_i) \longrightarrow \pi_1(B)$ are trivial.

Solutions to some of the exercises on covering theory

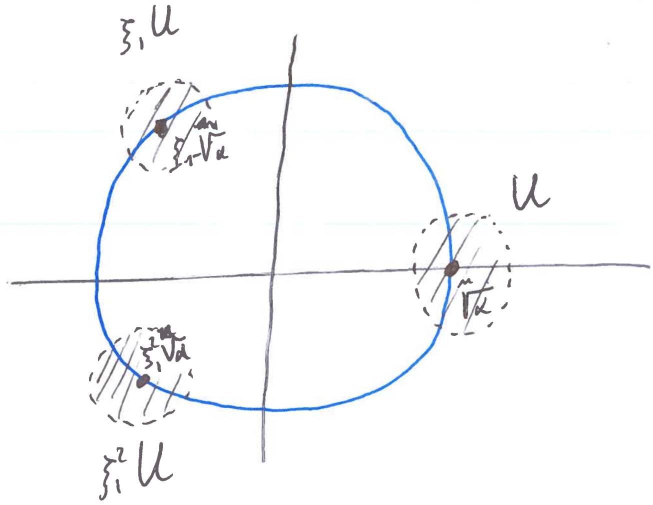
1) Show that $p: \mathbb{C}^* \longrightarrow \mathbb{C}^*$, $z \longmapsto z^n$, $\mathbb{C}^* = \mathbb{C} - 0$, is a covering map.

Sol: Let $\alpha \in \mathbb{C}^*$. Note that since p is raising to the n-th power, p^{-1} will be taking the n-th roots. If $\sqrt[n]{\alpha}$ is one of the roots (took arbitrary), then

$$p^{-1}(\alpha) = \{ \sqrt[n]{\alpha}, \xi_1 \sqrt[n]{\alpha}, \xi_1^2 \sqrt[n]{\alpha}, \dots, \xi_1^{n-1} \sqrt[n]{\alpha} \}$$

where $\xi_1, \xi_2 = \xi_1^2, \dots, \xi_{n-1} = \xi_1^{n-1}$ are the n-th roots of unity.

Let U an open nbhd of $\sqrt[n]{\alpha}$, small enough
" $\xi_1 U$ " " $\xi_1 \sqrt[n]{\alpha}$, "
⋮
$\xi_1^{n-1} U$ " " $\xi_1^{n-1} \sqrt[n]{\alpha}$ "



Obviously, by taking U small enough we can get that $U, \xi_1 U, \dots, \xi_1^{n-1} U$ are disjoint.

Let $V := p(U) \stackrel{\text{notation}}{=\!=} U^{(n)}$. Note that $\alpha \in U^{(n)}$, because $\alpha = \left(\sqrt[n]{\alpha}\right)^n \in U^{(n)}$.

Moreover, V is open, because p is holomorphic, and every holomorphic function is open.

Thus V is an open nbhd of α, which precisely is the evenly covered nbhd of α:

$$p^{-1}(V) = p^{-1}(U^{(n)}) = \sqrt[n]{U^{(n)}} = U \amalg \xi_1 U \amalg \cdots \amalg \xi_1^{n-1} U,$$

because for $u \in U$, $p^{-1}(u^n) = \{u \in U, \xi_1 u \in \xi_1 U, \ldots, \xi_1^{n-1} u \in \xi_1^{n-1} U\}$.

That is, every element of V has n n-th roots, and each of them lives either in U, or in $\xi_1 U$, or ..., or in $\xi_1^{n-1} U$, so

$$U \xrightarrow[\sim]{p} V \quad , \quad \xi_1 U \xrightarrow[\sim]{p} V \quad , \quad \ldots \quad , \xi_1^{n-1} U \xrightarrow[\sim]{p} V$$

are bijective, continuous and open, thus homeomorphisms. Therefore p is a covering map.

2) Let $p : X \to S$ a covering map. If S is Hausdorff $\Rightarrow$ X Hausdorff.

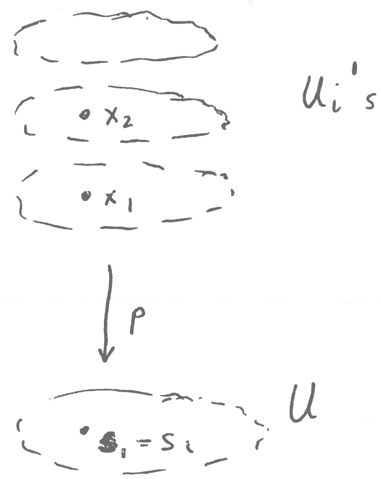
<u>Sol</u>: Let $x_1 \neq x_2 \in X$, and let $s_1 = p(x_1)$, $s_2 = p(x_2)$.

- If $s_1 \neq s_2$, since S Hausdorff, there are U_1 nbhd of s_1, U_2 nbhd of s_2, such that $U_1 \cap U_2 = \emptyset$, then $p^{-1}(U_1 \cap U_2) = \underset{\ni x_1}{p^{-1}(U_1)} \cap \underset{\ni x_2}{p^{-1}(U_2)} = \emptyset$,

and we are done.

• If $s_1 = s_2$, let U an evenly covered nbhd of such a point. Its preimage is

$p^{-1}(U) = \coprod U_i$, $U_i \xrightarrow[\sim]{p} U$. But in every sheet there is only one point of the fiber, thus x_1 and x_2 belong to diferent (and disjoint) sheets, so we are done too.



3) If $p : X \longrightarrow S$ is a finite covering map (ie, fibers are finite), and S compact $\Rightarrow X$ compact

Let's show something first:

<u>Claim</u>: If an open set $U \subset X$ contains a fiber $p^{-1}(s) \subset U$, $s \in S$; then there is a nbhd V of s st. $p^{-1}(V) \subset U$.

<u>Pf</u>: Let W be an evenly covered nbhd of s, $p^{-1}(W) = W_1 \amalg \dots \amalg W_n$. Set

$V_i := p(U \cap W_i)$ and take $V := \bigcap_{i=1}^{n} V_i$. V_i is open because every covering map is an open map (because it is a local homeomorphism), and V is open since it is a finite intersection of open sets. Let us show that $p^{-1}(V) \subset U$:

since $V \subset V_1 = p(U \cap W_1) \subset W_1$ and $W_1 \xrightarrow[\sim]{p} W$, then V is also evenly covered, $p^{-1}(V) = \widetilde{V}_1 \amalg \dots \amalg \widetilde{V}_n$, with $\widetilde{V}_i \subset U \cap W_i \subset U$. Therefore $p^{-1}(V) \subset U$. ■

<u>Proof of the exercise</u>: Let $\{U_i\}$ an open cover of X. For $s \in S$, consider $p^{-1}(s)$, which is a finite amount of points. Let $U_1, \dots, U_r$ open sets of the cover covering $p^{-1}(s)$. Set $U_s := \bigcup_{i=1}^{r} U_i$, open. Then $p^{-1}(s) \subset U_s$. By the claim, there is W_s nbhd of s s.t. $p^{-1}(W_s) \subset U_s$ as well.

Now $\{W_s : s \in S\}$ is a cover of S. Since S is compact, we can take a subcover $\{W_{s_i} : i = 1, \dots, k\}$ covering S. But we are done because

$$X = p^{-1}(S) = p^{-1}\left(\bigcup_{i=1}^{k} W_{s_i}\right) = \bigcup_{i=1}^{k} p^{-1}(W_{s_i}) \subset \bigcup_{i=1}^{k} U_{s_i}.$$

<u>Remark</u>: This proves problem 1a. The fact that M is a manifold as well follows because p is a local homeomorphism.

<u>Remark</u>: If S is connected, the converse is also true.

4) <u>(Problem 16)</u> : Given a triangulation of S given by v vertices $p_1, \dots, p_v$; e edges $E_1, \dots, E_e$ and t triangles $T_1 \dots T_t$; and $p: M \to S$ is an n-sheeted cov., S conn, comp surf; give a triangulation of M with nv vertices, ne edges and nt triangs.

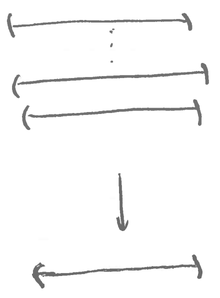
<u>Sol</u> : Set $S_0 = \{p_1, \dots, p_v\}$ a discrete space. We saw that the restriction of p is also a covering map:

$$\begin{tikzcd} p^{-1}(S_0) \arrow[r, hook] \arrow[d, "\substack{\text{covering}\\ \text{map}}"'] & M \arrow[d] \\ S_0 \arrow[r, hook] & S \end{tikzcd}$$

Since an open nbhd of p_i on S_0 is $\{p_i\}$, we see that its preimage will be n copies of p_i:

$$M_0 := p^{-1}(S_0) = \{p_1^1, \dots, p_1^n, \dots\dots, p_v^1, \dots, p_v^n\}.$$

For the edges we reason as follows: consider again the restriction to every edge



$$\begin{tikzcd} p^{-1}(E_i) \arrow[r, hook] \arrow[d, "\substack{\text{cov}\\ \text{map}}"'] & M \arrow[d] \\ E_i \arrow[r, hook] & S \end{tikzcd}$$

But since $E_i \cong (0,1)$ is simply connected (and loc. path-connected), the only covering spaces it has are the trivial ones, ie, copies of itself : $p^{-1}(E_i) \cong E_i \sqcup \overset{n}{\dots} \sqcup E_i$.

For the triangles we reason in the same way.

Immediate consequence: $\boxed{\chi(M) = n\,\chi(S)}$

5) (Problem 3): Let $p: Y \to X$ be a covering, $G(p)$ be its deck transformation group and $H \subset G(p)$ a subgroup. Show that

(a) $Y \to Y/H$

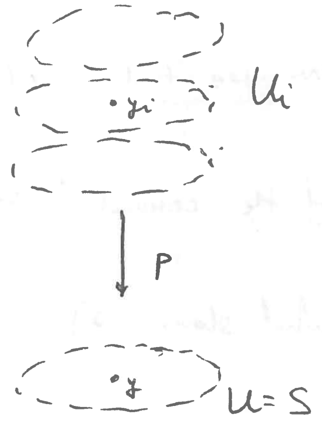
(b) $Y/H \to X$

$$\begin{tikzcd} Y \arrow[dr, "cov"] \arrow[dd, "cov"'] & \\ & Y/H \arrow[dl, "cov"] \\ X & \end{tikzcd}$$

are covering maps.

Proof: Since this is a local issue on X, we can suppose that $X = U$ is connected and the covering map is trivial, $p^{-1}(X) = Y = \coprod U_i$, with $U_i \xrightarrow[\sim]{p} U$.

Every $h \in H$ permutes the connected components of Y (it is homeomorphism thus takes connected components to con. comp.), ie, every h defines a permutation on I, call $h(U_i) =: U_{h(i)}$.



Denote $y_i \in U_i$ the unique element in the fiber $p^{-1}(y)$, $y \in U$ which lies on U_i.

Since $p \circ h = p$, we get that $h(y_i) = y_{h(i)}$ (h permutes the fibers).

Endowing I with the discrete topology, we have a homeomorphism

$$Y = \coprod U_i \xrightarrow{\sim} \coprod (U \times i) = U \times I$$
$$y_i \longmapsto (y, i)$$

Via this homeomorphism, $h : Y = \coprod U_i \to \coprod U_i = Y$ acts as

$$h : U \times I \longrightarrow U \times I$$
$$(y, i) \longmapsto (y, h(i)).$$

Therefore, $Y/H = (U \times I)/H = U \times (I/H) \longrightarrow U$ is the trivial covering map (h seen as above doesn't change U), what shows b).

Now, $Y = U \times I \longrightarrow U \times (I/H) = Y/H$ is a covering map, since we see that $Y/H = U \times (I/H)$ is a disjoint union of copies of U, namely union of the connected components $U \times [i]$, whose preimage is $\coprod_{k \in [i]} (U \times k) = \coprod U$, what shows a).

—— o ——

6) (Problem 4) Classify all (connected) coverings of S^1, $\mathbb{RP}^2$ and $\mathbb{T}$.

First recall the

<u>Theorem</u> (Classification of coverings): let $p:(X,x_0) \longrightarrow (S,s_0)$ be a covering map with X simply-connected and S path-con and loc. path-connected. There is a one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{subgroups} \\ \text{of} \\ \pi_1(S)_{s_0} \end{array} \right\} = \left\{ \begin{array}{c} \text{pointed connected} \\ \text{coverings} \\ (Y,y_0) \to (S,s_0) \end{array} \right\}$$

$$H \longmapsto (X/H, [x_0]) \longrightarrow (S, s_0)$$

$$q_*(\pi_1(Y)_{y_0}) \longleftarrow\!\!| \; q:(Y,y_0) \longrightarrow (S,s_0)$$

$$\text{normal subgroups} \longleftrightarrow \text{normal coverings}$$

where since this is the universal covering of S, $G(p) \cong \pi_1(S)_{s_0}$, and we identify $H \subset \pi_1(S)_{s_0} \cong G(p)$. Here two isomorphic pointed connected coverings $(Y,y_0) \to (S,s_0)$, $(Y',y_0') \to (S,s_0)$ (ie, $\exists\, \varphi$ homeomorph. st

$$\begin{tikzcd} (Y,y_0) \arrow[rr, "\varphi", "\sim"'] \arrow[dr] & & (Y',y_0') \arrow[dl] \\ & (S,s_0) & \end{tikzcd}$$)

are considered the same, which is reasonable since such an iso is unique.

$$
\begin{tikzcd}
(X, x_0) \arrow[dr, "\pi"] \arrow[dd, "p"'] & \\
& (X/H, [x_0]) \arrow[dl, "q"] \\
(S, s_0) &
\end{tikzcd}
$$

normal covering and $G(\pi) \cong H$

universal $\Downarrow$ normal cov. and $G(p) \cong \pi_1(S)_{s_0}$

q covering, and it is normal $\Longleftrightarrow$ H normal (and in such a case $G(q) \cong G(p)/H$.

If the connected coverings are not considered pointed, the correspondence is rewritten as

$$\left\{ \begin{array}{c} \text{subgroups of} \\ \pi_1(S) \\ \text{mod. conjugacy} \end{array} \right\} = \left\{ \begin{array}{c} \text{connected coverings} \\ Y \longrightarrow S \\ \text{mod. isomorphism} \end{array} \right\}.$$

Therefore, if $\pi_1(S)$ is abelian, we don't have to take care of base-points.

a) <u>Connected coverings of S^1</u>:

Its universal covering is $\mathbb{R} \xrightarrow{p} S^1$, $t \longmapsto e^{2\pi i t}$, because $\mathbb{R}$ is simply-con.

First we need to specify the isomorphism $G(p) \cong \pi_1(S^1) = \mathbb{Z}$, i.e., compute $G(p)$:

Homeomorphisms $g : \mathbb{R} \to \mathbb{R}$ making
$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[dr] & \mathbb{R} \arrow[d] \\ & S^1 \end{tikzcd}$$
commute?

Note that for $t, t' \in \mathbb{R}$,

$$p(t) = p(t') \iff e^{2\pi i t} = e^{2\pi i t'} \iff e^{2\pi i (t-t')} = 1 \iff t - t' = n \in \mathbb{Z}$$

Therefore, there is an evident family of deck transformations, $g_n : \mathbb{R} \to \mathbb{R}$, $g_n(t) = t + n$, since the previous line says that it acts on fibers. Are there more? Let $g \in \mathcal{G}(p)$. Since it has to act on fibers, it will be $g(0) = n$ for some $n \in \mathbb{Z}$, but $g_n(0) = n$, and since $\mathbb{R}$ is connected $g_n = g$. Thus

$$\mathcal{G}(p) \cong \mathbb{Z}$$
$$g_n \longleftrightarrow n$$

Now, we know how all subgroups of $\mathbb{Z}$ are: they are $H = m\mathbb{Z}$ for some $m \in \mathbb{Z}$.

The theorem ensures

$$\left\{ \begin{matrix} \text{subgroups of} \\ \mathbb{Z} \end{matrix} \right\} \;=\!=\; \left\{ \begin{matrix} \text{connected coverings} \\ Y \to S^1 \end{matrix} \right\}$$

$$H = m\mathbb{Z} \longrightarrow \mathbb{R}/m\mathbb{Z} \to S^1 \quad \underline{\underline{\text{doing what?}}}$$

The subgroup $m\mathbb{Z} = \{0, \pm m, \pm 2m, \dots\}$ identifies with the subgroup $\{Id, g_{\pm m}, g_{\pm 2m}, \dots\}$ of $G(p)$. So in $\mathbb{R}/m\mathbb{Z}$ we identify the points $t,\ g_{\pm m}(t) = t \pm m,\ t \pm 2m, \dots$.

and $0 \sim m$, so we obtain $\mathbb{R}/m\mathbb{Z} \cong S^1$ for $m \geq 1$ (we obtain the interval $\underset{0}{\vdash}\!\!\!\text{———}\!\!\!\underset{m}{\dashv}$ where we identify the extremes). In concrete, the iso is

$$\begin{array}{ccc} \mathbb{R}/m\mathbb{Z} & \xrightarrow{\quad\sim\quad} & S^1 \\ [t] & \longmapsto & e^{\frac{2\pi i t}{m}} \\ \| & & \\ [t+m] & \longmapsto & e^{\frac{2\pi i (t+m)}{m}} = e^{\frac{2\pi i t}{m}} \underbrace{e^{\frac{2\pi i \cancel{m}}{\cancel{m}}}}_{1} = e^{2\pi i t} \end{array}$$

✓

(ie, for t to run S^1 with $e^{2\pi i t}$ we need the interval $[0,1]$; but if we have to run through S^1 with the interval $[0,m]$, we will need to do it in a slower speed).

So what are the connected coverings? Since the diagram must commute,

$$\begin{tikzcd} \mathbb{R} \arrow[dr] \arrow[dd] & \\ & \mathbb{R}/m\mathbb{Z} = S^1 \arrow[dl] \\ S^1 & \end{tikzcd} \quad , \quad \begin{tikzcd} t \arrow[dr, mapsto] \arrow[dd, mapsto] & \\ & {[t]} = e^{\frac{2\pi i t}{m}} \arrow[dl, mapsto] \\ e^{2\pi i t} & \end{tikzcd}$$

$$z \longmapsto z^m$$

So all connected covering maps are :

- $\underline{m=0}$: $\mathbb{R}/0 = \mathbb{R} \longrightarrow S^1$, the universal one $\left(\mathbb{R} \xrightarrow{\pi = Id} \mathbb{R}/0 = \mathbb{R}\right)$

- $\underline{m \geq 1}$: $\mathbb{R}/m\mathbb{Z} = S^1 \longrightarrow S^1$, $z \longmapsto z^m$. In particular for $m=1$, we have $m\mathbb{Z} = \mathbb{Z} = \pi_1(S^1)$ and we obtain the identity.

b) Connected coverings of $\mathbb{RP}^2$

Its universal covering space is $S^2 \xrightarrow{p} \mathbb{RP}^2$, $x \longmapsto [x]$, because S^2 is simply connected. What is $G(p)$? The equivalence relation that we set on S^2 to obtain $\mathbb{RP}^2$ is $x \sim -x$, so

$$\begin{tikzcd} S^2 \arrow[rr, "g"] \arrow[dr, "p"'] & & S^2 \arrow[dl, "p"] \\ & \mathbb{RP}^2 & \end{tikzcd} \qquad \begin{array}{l} [g(x)] = p(g(x)) = p(x) = [x] \\ \text{ie, } g(x) = \pm x \end{array}$$

So $Id, \sigma \in G(p)$, $\sigma(x) = -x$ the antipodal map , and since any other deck transformation coincide with either Id or σ , and S^2 is connected , we conclude

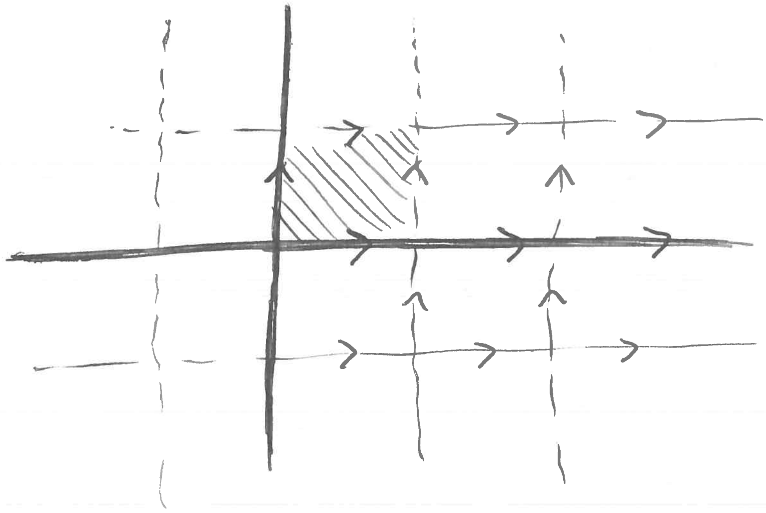
$$G(p) \cong \{Id, \sigma\} \cong \mathbb{Z}/2\mathbb{Z} \qquad (\text{generated by } \sigma)$$

(and as the theory says, it is isomorphic to the fundamental group of $\mathbb{RP}^2$).

$\mathbb{Z}/2\mathbb{Z}$ only has two subgroups ; either 0 or $\mathbb{Z}/2\mathbb{Z}$; so the only connected coverings we have are either the universal one $\mathbb{S}^2 \xrightarrow{p} \mathbb{RP}^2$ (quotient by 0), or the identity $\mathrm{Id}: \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ (quotient by $\mathbb{Z}/2\mathbb{Z} \cong G(p)$)

c) Connected coverings of $\mathbb{T}$

Recall that we obtained the torus as the quotient of $\mathbb{R}^2$ by the action of $\mathbb{Z}^2$: if we let $\mathbb{Z}^2$ act on $\mathbb{R}^2$ as $(n, m) * (x, y) := (x+n, y+m)$, the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ is precisely $\mathbb{T}$:



Such an action is properly discontinuous (by taking balls of radius $< \frac{1}{2}$), so the projection $p: \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}$ is a covering map. Note that the homeomorphisms $\mathbb{R}^2 \xrightarrow{(n,m)} \mathbb{R}^2$, $(x,y) \longmapsto (x+n, y+m)$ that the group action defines are deck transformations (by def!)

$$\begin{tikzcd} \mathbb{R}^2 \arrow[rr] \arrow[dr] & & \mathbb{R}^2 \arrow[dl] \\ & \mathbb{R}^2/\mathbb{Z}^2 & \end{tikzcd} \qquad \begin{tikzcd} (x,y) \arrow[rr, mapsto] \arrow[dr, mapsto] & & (x+n, y+m) \arrow[dl] \\ & {[(x,y)] = [(x+n, y+m)]} \checkmark & \end{tikzcd}$$

The general theory ensures also that such a covering is normal, and $\mathbb{Z}^2 = G(p)$.

Moreover, since $\mathbb{R}^2$ is simply-connected, it is the universal covering, so as expected

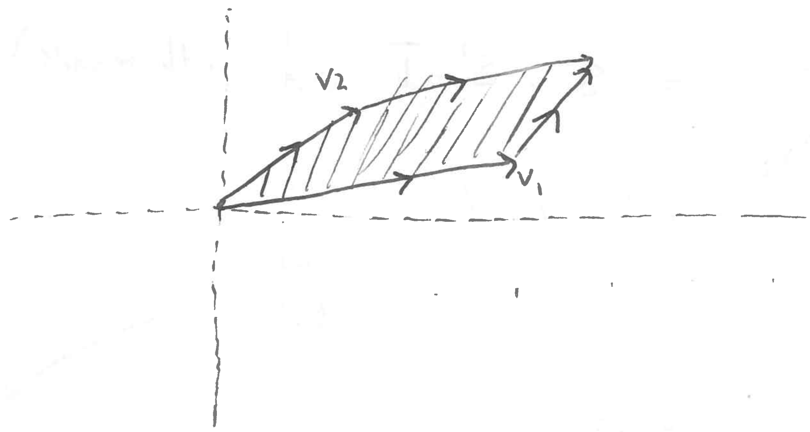
$$\pi_1(\mathbb{T}) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \oplus \pi_1(S^1) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2 = G(p).$$

<u>Remark</u>: When we let $\mathbb{Z}^2$ act on $\mathbb{R}^2$, and we obtain the net, implicitly we have chosen the standard basis $e_1 = (1,0)$, $e_2 = (0,1)$ of $\mathbb{Z}^2$ and done quotient

$$\frac{\mathbb{R}e_1 \oplus \mathbb{R}e_1}{\mathbb{Z}e_1 \oplus \mathbb{Z}e_1} = \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} = \mathbb{T},$$

but we could have also chosen another basis of $\mathbb{Z}^2$ (as $\mathbb{Z}$-module), eg. $v_1 = (3,1)$ and $v_2 = (2,1)$, (it is a $\mathbb{Z}$-basis because $\det\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = 1$ invertible in $\mathbb{Z}$), and the quotient

$\dfrac{\mathbb{R}v_1 \oplus \mathbb{R}v_2}{\mathbb{Z}v_1 \oplus \mathbb{Z}v_2}$ is the torus as well:



Let's get down to business: subgroups of $\mathbb{Z} \oplus \mathbb{Z}$? Obviously we have $n\mathbb{Z} \oplus 0$, $0 \oplus m\mathbb{Z}$, $n\mathbb{Z} \oplus m\mathbb{Z}$ as subgroups. Are they all? No! Here we have taken the standard basis, $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, but we can also have subgroups H of $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ which are of the form $H = n\mathbb{Z}v_1 \oplus m\mathbb{Z}v_2$ for another $\mathbb{Z}$-basis $\{v_1, v_2\}$ of $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ (note that this last case encodes the others).

These are all possible subgroups. Therefore,

$$\left\{ \begin{array}{c} \text{subgroups of} \\ \mathbb{Z} \oplus \mathbb{Z} \end{array} \right\} = \left\{ \begin{array}{c} \text{connected coverings} \\ Y \to \mathbb{T} \end{array} \right\}$$

$$H = n\mathbb{Z}v_1 \oplus m\mathbb{Z}v_2 \longmapsto \mathbb{R}^2/H \to \mathbb{T}$$

- $\underline{n, m \neq 0}$: Arguing similarly as for S^1 we have

$$\frac{\mathbb{R}v_1 \oplus \mathbb{R}v_2}{n\mathbb{Z}v_1 \oplus m\mathbb{Z}v_2} = S^1 \times S^1 = \mathbb{T}, \quad (\text{as in the remark}), \quad \text{and}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r] \arrow[d] & \dfrac{\mathbb{R}^2}{n\mathbb{Z}v_1 \oplus m\mathbb{Z}v_2} = S^1 \times S^1 \arrow[dl] \\ \mathbb{T} = S^1 \times S^1 & \end{tikzcd}, \qquad \begin{tikzcd} (t,s) \arrow[r, mapsto] \arrow[d, mapsto] & \left(e^{\frac{2\pi i t}{n}}, e^{\frac{2\pi i t}{m}}\right) \arrow[dl, mapsto] \\ \left(e^{2\pi i t}, e^{2\pi i s}\right) & \end{tikzcd}$$

$$(z, w) \longmapsto (z^n, w^m)$$

• $n = 0$ or $m = 0$ (but no both) :

$$\frac{\mathbb{R}^2}{n\mathbb{Z}v_1 \oplus 0} = S^1 \times \mathbb{R}, \quad \text{and}$$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r] \arrow[d] & \frac{\mathbb{R}^2}{n\mathbb{Z}v_1 \oplus 0} = S^1 \times \mathbb{R} \arrow[dl] \\
S^1 \times S^1 &
\end{tikzcd}
\quad ,
\begin{tikzcd}
(t,s) \arrow[r, mapsto] \arrow[d, mapsto] & \left(e^{\frac{2\pi i t}{n}}, s\right) \arrow[dl, mapsto] \\
\left(e^{2\pi i t}, e^{2\pi i s}\right) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(z,s) \arrow[d, mapsto] \\
\left(z^n, e^{2\pi i s}\right)
\end{tikzcd}$$

(similar for $n = 0$)

• $n = m = 0$:

$\mathbb{R}^2/0 = \mathbb{R}^2 \longrightarrow \mathbb{T}$, the universal covering.