Exercises for “Algebraic Topology” - 7th November 2018

In today’s lecture all homology groups are taken with coefficients in $\mathbb{Z}$.

Theorem 1 (Classification of abelian groups). *Every finitely generated abelian group G decomposes, in a unique way up to order, as a direct sum of cyclic groups,*

$$G \simeq (\mathbb{Z} \oplus \overset{r}{\cdots} \mathbb{Z}) \oplus (\mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_m\mathbb{Z}).$$

*The number r is the **rank** of G and the integers $k_1, \ldots, k_m$ (well defined up to invertible elements) are the **invariant factors**. Clearly, these values characterize G up to isomorphism.*

1. Classify the abelian group

$$G := \frac{\mathbb{Z} \oplus \mathbb{Z}}{< (5,3), (-2,2), (-11,-5) >}.$$

2. Consider the chain complex of abelian groups C_* with $C_n = 0$ for $n > 2$ and $C_n = \mathbb{Z} \oplus \mathbb{Z}$ for $n = 0, 1, 2$; with differentials given by

$$\partial_2 = \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix} \quad , \quad \partial_1 = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$

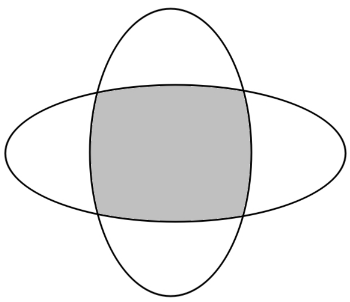
   Show that C is indeed a chain complex and compute all homology groups.

3. Compute all homology groups of the Klein bottle.

4. The projective plane $\mathbb{RP}^2$ can be obtained from a disk D^2 identifying antipodal points on its boundary. Compute all homology groups of it. (In the second part of the lecture we will compute the homology groups of $\mathbb{RP}^n$).

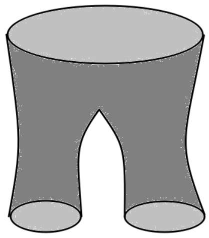
5. Compute all homology groups of the space $M_k = S^1 \cup_{\partial D^2} D^2$, where the attaching map is given by $z \in S^1 \mapsto z^k \in S^1$.

6. Compute all homology groups of the following topological space:

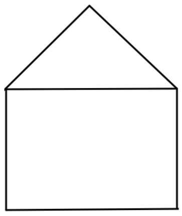


7. Compute all homology groups of a pinched torus, ie, a torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ with the subspace $\mathbb{S}^1 \times \{y\}$ collapsed.

8. Compute all homology groups of a pair of trousers.



9. Compute all homology groups of a "house":



10. Compute all homology groups of a sphere with the two poles identified.
11. Compute all homology groups of a sphere in which all points on its equator are identified antipodally.
12. Let $X = (S^1 \vee S^1) \cup_{\{u,v\} \times \partial D^2} \{u, v\} \times D^2$ with attaching maps given by the words $a^5 b^{-3}$ and $b^3(ab)^{-2}$, where a, b are indexes for the 1-cells of every 1-sphere. Compute all homology groups of X.

Remark. Every (connected) compact surface arises from a n-gon (n even) by identifying pairs of edges according to a **symbol** or **labelled scheme**.

Examples are:
–. The torus arises from a square according to the symbol $aba^{-1}b^{-1}$.
–. The Klein bottle arises from a square according to the symbol $abab^{-1}$.
–. The real projective space arises from a 2-gon according to the symbol aa.
–. The 2-sphere arises from a 2-gon according to the symbol aa^{-1}.

Theorem 2 (Classification of compact surfaces). *Every compact surface is homeomorphic to one of the following:*

(i) A sphere S^2, with symbol aa^{-1}.

*(ii) A connected sum of $g \geq 1$ tori $\Sigma_g = \mathbb{T} \# \overset{g}{\cdots} \# \mathbb{T}$, with symbol $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$. The number g is called the **genus**.*

(iii) A connected sum of p projective planes $N_p = \mathbb{RP}^2 \# \overset{p}{\cdots} \# \mathbb{RP}^2$, with symbol $a_1 a_1 \dots a_p a_p$.

*A compact surface is **orientable** if it is homeomorphic to cases (i) or (ii), and it is **non-orientable** if it is homeomorphic to case (iii). One usually says that the sphere has genus 0.*

13. Show that $\chi(\Sigma_g) = 2 - 2g$ and $\chi(N_p) = 2 - p$.
14. Compute all homology groups of all compact surfaces. (*Hint.* If you know which are their fundamental groups, you can check that their abelianizations are precisely their first homology groups).