# Multivariable Analysis - Homework 7

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Your grade is computed by adding your obtained points.

1. In this exercise we test out the proof of the Picard theorem (Theorem 3.1.1) by working out an example. We will use the same notation and terminology as used in the proof. Our vector space will be $\mathbb{R}^4$ with the Euclidean norm and our vector field will be $F : \mathbb{R}^4\ \mathbb{R}^4$ given by $F(x, y, z, w) = (xw, y, z, x)$ and the point of interest is $p = (1, 1, 1, 1)$.

   (a) (0.5 points) Is the curve $\gamma : (-1, 1) \to \mathbb{R}^4$ given by $\gamma(t) = (e^t, e^t, e^t, 1)$ an integral curve for $F$ through $p$?

   **Solution:** For $\gamma$ to be an integral curve for $F$ we need that $\gamma'(t) = F(\gamma'(t))$. We note that $F(\gamma(t)) = (e^t, e^t, e^t, e^t)$ and $\gamma'(t) = (e^t, e^t, e^t, 0)$. From this it follows that $F(\gamma(t))$ is not equal to $\gamma'(t)$, and hence $\gamma$ is not an integral curve for $F$ through $p$.

   (b) (1 point) If we take the disk $B$ to be $B = \{v \in \mathbb{R}^4 : |v - p| \leq 1\}$ then can you come up with a suitable value of $M$ as in the proof?

   **Solution:** There are many ways to solve this problem, one way is presented here. We need that $\|F(x)\| \leq M$ for $x \in B$, where $\|\cdot\|$ is the sup norm. Notice that $\|F(x)\| = \sqrt{x^2w^2 + y^2 + z^2 + x^2}$. Moreover, we have that $|v - p| \leq 1$ for $v \in B$. From this it follows by writing out absolute value and squaring both sides that:

   $$(x - 1)^2 + (y - 1)^2 + (z - 1)^2 + (w - 1)^2 \leq 1$$

   It follows that $(x - 1)^2 \leq 1$ and hence $x \leq 2$. Similarly, $y \leq 2$, $z \leq 2$ and $w \leq 2$. Therefore, we get the inequality:

   $$\|F(x)\| = \sqrt{x^2w^2 + y^2 + z^2 + x^2} \leq \sqrt{4 \cdot 4 + 4 + 4 + 4} = \sqrt{28}.$$

   Hence, an upper bound we could pick is $M = \sqrt{28} = 2\sqrt{7}$.

   (c) (1 point) Also find an upper bound $\mathcal{L} \geq L$ for the constant $L$ used in the proof using section 1.2 for bounding the operator norm.

**Solution:** Again there are many ways to solve this problem. Notice that $L = \max_{v\in B} |F'(v)|$. We have by lemma 1.2.2. that $|A| \leq \sqrt{\sum_{i,j=1}^{n} \left(A_j^i\right)^2}$.
If we compute $F'(x, y, z, w)$ w.r.t. the standard basis, we obtain the matrix:

$$F'(x, y, z, w) = \begin{pmatrix} w & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

By lemma 1.2.2. we get: $|F'(x, y, z, w)| \leq \sqrt{w^2 + x^2 + 3}$. Then again using the bounds $x \leq 2$ and $y \leq 2$ we obtain that $|F'(x, y, z, w)| \leq \sqrt{11}$. We can therefore pick $L = \sqrt{11}$. To obtain a stricter bound, one could also use that the matrix norm is bounded by the largest eigenvalue.

(d) (0.5 points) Find a value $\tau > 0$ that satisfies $\tau M < 1$ and $\tau L < 1$.

**Solution:** Take $\tau = \frac{1}{\sqrt{36}} = \frac{1}{6}$. Then $\tau M = \sqrt{\frac{28}{36}} < 1$ and $\tau L = \sqrt{\frac{11}{36}} < 1$, because $\frac{11}{36} < \frac{28}{36} < 1$.

(e) (1 point) Compute $\delta = \Phi(\gamma)$ and $\beta = \Phi(\delta)$, where $\gamma$ is the curve from part $(a)$.

**Solution:** Using the definition of $\Phi$ we compute:

$$\begin{aligned} \Phi(\gamma)(t) &= p + \int_0^t F \circ \gamma(s) ds \\ &= p + \left(e^t - 1, e^t - 1, e^t - 1, e^t - 1\right). \end{aligned}$$

And similarly, we compute:

$$\begin{aligned} \Phi(\delta)(t) &= p + \int_0^t F \circ \delta(s) ds \\ &= \left(e^{1/2} + \frac{1}{2}, e^t, e^t, e^t\right). \end{aligned}$$

(f) (1 point) Which supremum norm is bigger $\|\gamma - \beta\|$ or $\|\delta - \beta\|$? And by how much? Or are they the same?

**Solution:** We compute

$$\delta - \beta = \left(e^t - e^{2t}/2 - 1/2, 0, 0, 0\right) = \left(-1/2\left(e^t - 1\right)^2, 0, 0, 0\right)$$

Then we get:

$$\|\delta - \beta\| = \max_{|t| \leq \tau}\{\left|-1/2\left(e^\tau - 1\right)^2\right|, \left|-1/2\left(e^{-\tau} - 1\right)^2\right|\}$$

Continuing the computations for the chosen value of $\tau$.
Similarly, we get:

$$\gamma - \beta = \left(0, 0, 0, 1 - e^t\right).$$

And hence:

$$\|\gamma - \beta\| = 1 - e^\tau.$$

Now we can compute both norms for the chosen value of $\tau$ and draw a conclusion.

2. Consider a differential 1-form $\omega : \mathbb{R}^2 \to \Lambda^1\left(\mathbb{R}^2\right)^*$ given by $\omega = gdf + dg$ where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are the functions $f(x, y) = e^x$ and $g(x, y) = x + y$.

(a) (1 point) Compute $d\omega$.

**Solution:**
Notice that $dg = dx + dy$ and $df = e^x dx$. Then we compute:

$$\begin{aligned} d\omega &= d(gdf + dg) \\ &= d(gdf) + ddf && \text{By property 4.3.1. (1)} \\ &= dg \wedge df && \text{By property 4.3.1. (4)} \\ &= (dx + dy) \wedge e^x dx \\ &= -e^x dx \wedge dy. \end{aligned}$$

(b) (1.5 points) Also compute $\int_\gamma d\omega$ where $\gamma : [0,1]^2 \to [0,1]^2$ is the identity (the standard 2-cube) without using Stokes theorem.

**Solution:**
Using the definition of the integral over a $k$-form with the pull-back we get that:

$$\int_\gamma d\omega = \int_{[0,1]^2} \gamma^*(dw).$$

If we compute $\gamma^*(dw)$, we need that $\gamma'(x,y) = \mathrm{id}_{\mathbb{R}^2}$, so $\left(\gamma'(x,y)\right)^* dx = dx$ and $\left(\gamma'(x,y)\right)^* dy = dy$. Then we obtain:

$$
\begin{aligned}
\gamma^*(dw) &= (\gamma'(x,y))^* \, dw\,(\gamma(x,y)) \\
&= e^x\,(\gamma'(x,y))^*\, dx \wedge e^x\,(\gamma'(x,y))^*\, dy \\
&= -e^x dx \wedge dy.
\end{aligned}
$$

Therefore, the integral becomes:

$$
\begin{aligned}
\int_\gamma d\omega &= \int_{[0,1]^2} \gamma^*(dw) \\
&= \int_{[0,1]^2} -e^x dxdy \\
&= 1 - e.
\end{aligned}
$$

Where the last integral is computed by using Fubini's theorem.

(c) (2.5 points) Finally compute $\int_{\partial\gamma} \omega$ using the 1-dimensional fundamental theorem of calculus.

**Solution:** Notice that the standard 2-cube is given by the curves $\gamma_1(t) = (t, 0)$, $\gamma_2(t) = (1, t)$, $\gamma_3(t) = (0, t)$ and $\gamma_4(t) = (t, 1)$.
We will use:

$$\partial\gamma = -\gamma_{1,0} + \gamma_{1,1} - \gamma_{2,1} + \gamma_{2,2} = -\gamma_3 + \gamma_2 - \gamma_4 + \gamma_1.$$

Then the integral

$$\int_{\partial\gamma} \omega = \int_{-\gamma_3+\gamma_2-\gamma_4+\gamma_1} \omega = -\int_{\gamma_3} \omega + \int_{\gamma_2} \omega + \int_{\gamma_4} \omega - \int_{\gamma_1} \omega.$$

Hence, if we compute all these separate integrals we are done. Here, we notice that by definition of the integral over a $k$-form, we obtain:

$$\int_{\gamma_i} \omega = \int_0^1 \omega\,(\gamma_i(t))\,\gamma_i(t).$$

When we write out $\omega$ we obtain, $\omega(x, y) = gdf + dg = ((x + y)e^x + 1)\,\epsilon^1 + \epsilon^2$. This yiekds:

$$\begin{aligned}
\omega\left(\gamma_1(t)\right) &= \left(te^t + 1\right)\epsilon^1 + \epsilon^2 \\
\omega\left(\gamma_2(t)\right) &= \left((t+1)e + 1\right)\epsilon^1 + \epsilon^2 \\
\omega\left(\gamma_3(t)\right) &= \left(t + 1\right)\epsilon^1 + \epsilon^2 \\
\omega\left(\gamma_4(t)\right) &= \left((t+1)e^t + 1\right)\epsilon^1 + \epsilon^2
\end{aligned}$$

Then, we find the integrals become:

$$\begin{aligned}
\int_{\gamma_1} \omega &= 2 \\
\int_{\gamma_2} \omega &= 1 \\
\int_{\gamma_3} \omega &= 1 \\
\int_{\gamma_4} \omega &= 1 + e
\end{aligned}$$

Therefore, we conclude:

$$\begin{aligned}
\int_{\partial\gamma} \omega &= -\int_{\gamma_3} \omega + \int_{\gamma_2} \omega + \int_{\gamma_4} \omega - \int_{\gamma_1} \omega \\
&= -1 + 1 - 1 - e + 2 \\
&= 1 - e.
\end{aligned}$$

Which is the same as the answer of question (b), as one expects by Stokes' theorem.