Multivariable Analysis - Homework 6

Jorge Becerra, Oscar Koster and Laurens Pieper-Espada

January 15, 2021

Exercises from the notes: 4.2:1 and 4.4:1.

The maximum grade is 10 (8 points from exercise 4.2:1, and 2 points from exercise 4.4:1).

1. (8 points) Consider the 2-cubes α, β, γ in $\mathbb{R}^3$ given by $\alpha, \beta, \gamma : [0,1]^2 \to \mathbb{R}^3$ and

$$\alpha(r,t) = (r\cos(-2\pi t), r\sin(-2\pi t), -1+r),$$
$$\beta(r,t) = (r\cos 2\pi t, r\sin 2\pi t, 1-r),$$
$$\gamma(r,t) = (r\cos 2\pi t, r\sin 2\pi t, , -1+r).$$

Also imagine a differential 2-form $\omega \in \Omega^2_2(\mathbb{R}^3)$ given by

$$\omega(x,y,z) = x\,dy \wedge dz - y\,dz \wedge dx.$$

Finally introduce $\eta \in \Omega^1_2(\mathbb{R}^3)$ by

$$\eta(x,y,z) = (x^2+y^2)dz + 2xz\,dx + 2yz\,dy.$$

(a) (0.5 points) Is it true that $\omega(p) = \varepsilon^1(p)\varepsilon^{23} + \varepsilon^2(p)\varepsilon^{13}$ or is it that $\omega(p) = \varepsilon^1\varepsilon^{23}(p) + \varepsilon^2\varepsilon^{13}(p)$ for $p \in \mathbb{R}^3$?

Solution: It is true that $\omega(p) = \varepsilon^1(p)\varepsilon^{23} + \varepsilon^2(p)\varepsilon^{13}$.

(b) (2 points) Describe the boundaries $\partial(\alpha+\beta)$ and $\partial(\gamma+\beta)$ as 1-chains by explicitly writing them as linear combinations of explicit 1-cubes.

Solution: We compute the faces of the cubes α, β, γ. Notice that

$$\alpha_{2,0} = \alpha_{2,1}, \qquad \beta_{2,0} = \beta_{2,1}, \qquad \gamma_{2,0} = \gamma_{2,1},$$

so these faces cancel out and can be ignored. We have

$$\partial\alpha = -\alpha_{1,0} + \alpha_{1,1}, \qquad \partial\beta = -\beta_{1,0} + \beta_{1,1}, \qquad \partial\gamma = -\gamma_{1,0} + \gamma_{1,1},$$

where

$$\alpha_{1,0}(t) = (0,0,-1), \qquad \beta_{1,0}(t) = (0,0,1), \qquad \gamma_{1,0}(t) = (0,0,-1)$$
$$\alpha_{1,1}(t) = (\cos(-2\pi t), \sin(-2\pi t), 0), \quad \beta_{1,1}(t) = (\cos 2\pi t, \sin 2\pi t, 0), \quad \gamma_{1,1}(t) = (\cos 2\pi t, \sin 2\pi t, 0).$$

Thus,

$$\partial(\alpha+\beta) = -\alpha_{1,0} + \alpha_{1,1} - \beta_{1,0} + \beta_{1,1},$$

$$\begin{aligned}\partial(\gamma+\beta) &= -\gamma_{1,0} + \gamma_{1,1} - \beta_{1,0} + \beta_{1,1} \\ &= -\gamma_{1,0} - \beta_{1,0} + 2\beta_{1,1}.\end{aligned}$$

(c) (2 points) Compute $\int_{\partial(\alpha+\beta)} \eta$ and $\int_{\partial(\gamma+\beta)} \eta$.

Solution: Since $\partial_{e_1}\alpha_{1,0} = \partial_{e_1}\beta_{1,0} = \partial_{e_1}\gamma_{1,0} = 0$, we clearly have

$$\int_{\alpha_{1,0}} \eta = \int_{\beta_{1,0}} \eta = \int_{\gamma_{1,0}} \eta = 0.$$

Also,

$$\begin{aligned}\partial_{e_1}\alpha_{1,1}(t) &= (2\pi\sin(-2\pi t), -2\pi\cos(-2\pi t), 0), \\ \partial_{e_1}\beta_{1,1}(t) &= (-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 0), \\ \partial_{e_1}\gamma_{1,1}(t) &= (-2\pi\sin(2\pi t), 2\pi\cos(2\pi t), 0).\end{aligned}$$

So $\partial_{e_1}\alpha_{1,1}(t), \partial_{e_1}\beta_{1,1}(t), \partial_{e_1}\gamma_{1,1}(t)$ are of the form $f(t)e_1 + g(t)e_2$ for some $f, g : [0,1] \to \mathbb{R}$. Thus, when evaluating the pullbacks

$$\eta(\alpha_{1,1}(t))(\partial_{e_1}\alpha_{1,1}(t)), \qquad \eta(\beta_{1,1}(t))(\partial_{e_1}\beta_{1,1}(t)), \qquad \eta(\gamma_{1,1}(t))(\partial_{e_1}\gamma_{1,1}(t)),$$

we need only evaluate the dx and dy terms since the dz term evaluates to zero. Since the coefficients of dx and dy in the expression $\eta(x,y,z)$ both contain z as a factor, and the third coordinates of the 1-cubes $\alpha_{1,1}(t), \beta_{1,1}(t), \gamma_{1,1}(t)$ are all zero, we get that all terms are zero, so

$$\eta(\alpha_{1,1}(t))(\partial_{e_1}\alpha_{1,1}(t)) = \eta(\beta_{1,1}(t))(\partial_{e_1}\beta_{1,1}(t)) = \eta(\gamma_{1,1}(t))(\partial_{e_1}\gamma_{1,1}(t)) = 0,$$

implying

$$\int_{\alpha_{1,1}} \eta = \int_{\beta_{1,1}} \eta = \int_{\gamma_{1,1}} \eta = 0.$$

Finally,

$$\begin{aligned}\int_{\partial(\alpha+\beta)} \eta &= -\int_{\alpha_{1,0}} \eta + \int_{\alpha_{1,1}} \eta - \int_{\beta_{1,0}} \eta + \int_{\beta_{1,1}} \eta = 0, \\ \int_{\partial(\gamma+\beta)} \eta &= -\int_{\gamma_{1,0}} \eta - \int_{\beta_{1,0}} \eta + 2\int_{\beta_{1,1}} \eta = 0.\end{aligned}$$

(d) (2 points) Compute $\int_{\alpha+\beta} \omega$.

Solution: Again, we compute the pullback in parts. We have

$$\begin{aligned}\partial_{e_1}\alpha(r,t) &= \cos(-2\pi t)e_1 + \sin(-2\pi t)e_2 + e_3,\\ \partial_{e_2}\alpha(r,t) &= 2\pi r\sin(-2\pi t)e_1 - 2\pi r\cos(-2\pi t)e_2,\end{aligned}$$

$$\begin{aligned}\partial_{e_1}\beta(r,t) &= \cos(2\pi t)e_1 + \sin(2\pi t)e_2 - e_3,\\ \partial_{e_2}\beta(r,t) &= -2\pi r\sin(2\pi t)e_1 + 2\pi r\cos(2\pi t)e_2,\end{aligned}$$

so

$$\begin{aligned}\partial_{e_1}\alpha(r,t)\wedge\partial_{e_2}\alpha(r,t) &= -2\pi r\ e_1\wedge e_2 + 2\pi r\cos(-2\pi r)\ e_2\wedge e_3 + 2\pi r\sin(-2\pi r)\ e_3\wedge e_1,\\ \partial_{e_1}\beta(r,t)\wedge\partial_{e_2}\beta(r,t) &= 2\pi r\ e_1\wedge e_2 + 2\pi r\cos(2\pi r)\ e_2\wedge e_3 + 2\pi r\sin(2\pi r)\ e_3\wedge e_1.\end{aligned}$$

Finally,

$$\begin{aligned}\omega(\alpha(r,t))(\partial_{e_1}\alpha(r,t)\wedge\partial_{e_2}\alpha(r,t)) &= 2\pi r^2(\cos^2(-2\pi t) - \sin^2(-2\pi t))\\ \omega(\beta(r,t))(\partial_{e_1}\beta(r,t)\wedge\partial_{e_2}\beta(r,t)) &= 2\pi r^2(\cos^2(2\pi t) - \sin^2(2\pi t)).\end{aligned}$$

From the symmetry of the cosine and sine functions, one sees that the Riemann integrals of the above over $(r,t)\in[0,1]^2$ will be zero:

$$\begin{aligned}\int_{\partial(\alpha+\beta)}\omega &= \int_\alpha\omega + \int_\beta\omega\\ &= \int_{(r,t)\in[0,1]^2} 2\pi r^2(\cos^2(-2\pi t) - \sin^2(-2\pi t)) + \int_{(r,t)\in[0,1]^2} 2\pi r^2(\cos^2(2\pi t) - \sin^2(2\pi t))\\ &= 0+0.\end{aligned}$$

(e) (1.5 points) Look ahead in the next sections and verify that $d\omega = 0$ and $d\eta = 0$. Can you use the fundamental theorem of calculus to explain some of the answers you found in the previous parts?

Solution: Since

$$\begin{aligned}\omega(x,y,z) &= x\,dy\wedge dz - y\,dz\wedge dx\\ &= (y\ dx + x\ dy)\wedge dz,\end{aligned}$$

knowing the product rule allows us to see that $\omega = d\varphi$ where φ is the 1-form defined by $\varphi(x,y,z) = xy\ dz$. Stokes' theorem then implies

$$\int_{\alpha+\beta}\omega = \int_{\partial(\alpha+\beta)}\varphi = -\int_{\alpha_{1,0}}\varphi + \int_{\alpha_{1,1}}\varphi - \int_{\beta_{1,0}}\varphi + \int_{\beta_{1,1}}\varphi.$$

As in part (c), the integrals along the constant cubes $\alpha_{1,0}, \beta_{1,0}$ are zero, and the pullback of φ along the cubes $\alpha_{1,1}, \beta_{1,1}$ will be zero since the derivatives of these cubes are linear combinatons of e_1, e_2 while the form φ is some scalar multiple of dz. Thus, the entire integral is zero.
In the case of η, we have

$$\begin{aligned}d\eta(x,y,z) &= (2x\ dx + 2y\ dy)\wedge dz + (2z\ dx + 2x\ dz)\wedge dx + (2z\ dy + 2y\ dz)\wedge dz\\ &= 0,\end{aligned}$$

so

$$\int_{\partial(\alpha+\beta)} \eta = \int_{\alpha+\beta} d\eta = 0, \qquad \int_{\partial(\gamma+\beta)} \eta = \int_{\gamma+\beta} d\eta = 0.$$

2. (2 points) Suppose γ is an 8-chain on $\mathbb{R}^9$ given by $\gamma = u + v$ where $u, v : [0,1]^8 \to \mathbb{R}^9$ are determined by

$$u(t) = \cos\big(\varepsilon^4(t)\varepsilon^5(t) + \varepsilon^6(t)\big)e_8 + \sin\big(2\pi\varepsilon^3(t)\big)e_9$$

and

$$v(t) = \varepsilon^1(t)e_1 + \varepsilon^2(t)e_5 - \varepsilon^6(t)e_8.$$

Next introduce a 7-form $\omega = \eta \wedge \theta$ on $\mathbb{R}^9$, where η is a 3-form on $\mathbb{R}^9$ and θ is a 4-form on $\mathbb{R}^9$. η, θ are given by

$$\eta(p) = \exp\big(\varepsilon^1(p)\varepsilon^2(p)\varepsilon^3(p)\big)\varepsilon^1 \wedge \varepsilon^3 \wedge \varepsilon^5 - |\varepsilon^9(p)|^6 \varepsilon^2 \wedge \varepsilon^6 \wedge \varepsilon^8,$$
$$\theta(p) = \varepsilon^2(p)\varepsilon^{2578} + \varepsilon^3(p)\varepsilon^4(p)\varepsilon^{1579} + \cos\big(\varepsilon^3(p)\big)\varepsilon^{2378}$$

(a) (1 point) Argue that ω and γ are both C^1.

Solution: Notice that the component functions of η, θ are all smooth, since the basis covectors ε^i, as well as the exponential and norm maps are smooth, and that products and compositions of smooth functions are smooth. This suffices to ensure that ω is smooth. In the same way, all component functions of γ are smooth.

(b) (1 point) Suppose we have another 8-chain δ in $\mathbb{R}^9$ such that $\partial\delta = -\partial\gamma$. Prove that $\int_{\delta+\gamma} d\omega = 0$.

Solution: By Stokes' theorem,

$$\int_{\delta+\gamma} d\omega = \int_{\partial(\delta+\gamma)} \omega = \int_{\partial\delta} \omega + \int_{\partial\gamma} \omega = \int_{\partial\delta} \omega - \int_{\partial\delta} \omega = 0.$$