Multivariable Analysis - Homework 5

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1. Consider for any real number $a \in \mathbb{R}$ the following equations in N unknowns $x_1, ..., x_N$:

$$\begin{cases} x_1 + \cdots + x_N = a \\ x_1 \cdots x_N = 1 \end{cases}$$

(a) (1 pt) Introduce a function $f : \mathbb{R}^{N-2} \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(p, q) = (p_1 + \cdots + p_{N-2} + q_1 + q_2 - a, p_1 p_2 \cdots p_{N-2} q_1 q_2 - 1).$$

Explain how the inverse image $S = f^{-1}(0) \subset \mathbb{R}^{N-2} \times \mathbb{R}^2$ relates to the solution to the above equations.

Solution: Under the standard identification $\mathbb{R}^{N-2} \times \mathbb{R}^2 = \mathbb{R}^N$, the points of S, that is, the set of points such that

$$\begin{cases} p_1 + \cdots + p_{N-2} + q_1 + q_2 = a \\ p_1 \cdots p_{N-2} q_1 q_2 = 1 \end{cases}$$

are precisely the set of solutions of the above system of equations.

(b) (1 pts) Choose $a = N$ and take $s_0 = ((1, \ldots, 1), (1, 1))$. Verify that $s_0 \in S$ and there are no other positive $(p, q) \in S$ whose coordinates are all positive.

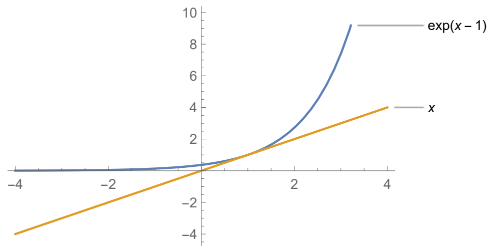
Solution: The fact that $s_0 \in S$ is readily verified. The second one is immediate using the AM-GM inequality: if $x_1, \ldots, x_n$ are positive numbers, we have

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$

and the equality holds if and only if $x_1 = \cdots = x_n$. For if $x = (x_1, \ldots, x_N) \in S$ then the equality holds (equal to one), so all the components are equal, and necessarily 1.

We also propose two extra solutions which avoid the AM-GM inequality:

Algebraic proof: The first observation is that for any $\alpha \in \mathbb{R}$, $\exp\left(\frac{x}{\alpha} - 1\right) \geq \frac{x}{\alpha}$, and the equality holds if and only if $x = \alpha$. This follows at once from the fact that $\exp(x - 1) \geq x$, the equality holding if and only if $x = 1$.



Now given $x_1, \dots, x_n$ positive numbers, let $\alpha := \frac{x_1+\cdots+x_n}{n}$ be the arithmetic mean. For every $i = 1, \dots, n$, the first observation says that $\exp\left(\frac{x_i}{\alpha} - 1\right) \geq \frac{x_i}{\alpha}$, with equality only when $x_i = \alpha$. Since this is true for every index, it is also true for the product, that is

$$\exp\left(\frac{x_1 + \cdots + x_n}{\alpha} - n\right) \geq \frac{x_1 \cdots x_n}{\alpha^n}$$

with the equality if and only if all the x_i's are equal to α. By the definition of α, the left hand side of the equation equals 1, so the equality

$$\frac{x_1 + \cdots + x_n}{n} = \alpha = \sqrt[n]{x_1 \cdots x_n}$$

holds if and only if $x_1 = \cdots = x_n = \alpha$.

Now the exercise is trivial taking $n = N$ and $(x_1, \dots, x_N) \in S$.

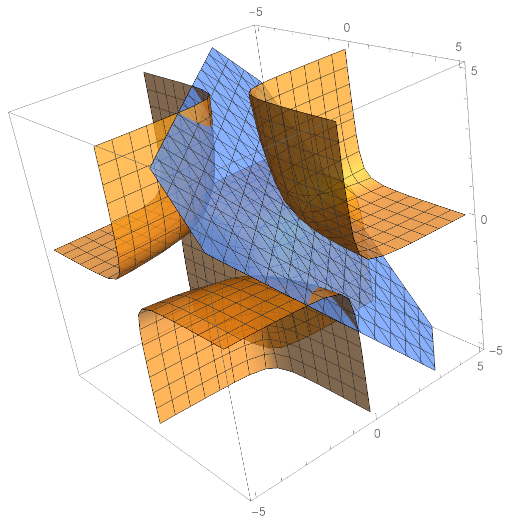
Geometric proof: Let $f_2 : \mathbb{R}^{N-2} \times \mathbb{R}^2 \to \mathbb{R}$ be the second component of f. The tangent space of this function at s_0 is

$$T_{s_0} f_2 = \text{Ker } f_2'(s_0) = \text{Ker } (1, 1, \dots, 1) = \{x \in \mathbb{R}^N : x_1 + \cdots + x_N = 0\},$$

so the affine subspace tangent to $(f_2)^{-1}(0)$ is

$$s_0 + T_{s_0} f_2 = \{x \in \mathbb{R}^N : x_1 + \cdots + x_N = N\}.$$

In other words, the first equation determines the (affine) tangent space of the zero locus of the second equation. The key point now is to note that, if $A = \{x \in \mathbb{R}^N : x_i > 0 \ \forall i\}$, then $(f_2)^{-1}(0) \cap A$ is the graph of the function $g : P \subset \mathbb{R}^{N-1} \times \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, \dots, x_{N-1}) := 1/(x_1 \cdots x_{N-1})$, where $P = \{x \in \mathbb{R}^{N-1} : x_i > 0 \ \forall i\}$.



If we show that g is convex in the subspace P, we are done, because that would imply that this is the only intersection point of the hyperplane determined by the first equation and the connected component $g(P)$. In one-variable, convexity is determined by the sign of the second derivative. In several variables, convexity is modelled by the Hessian or matrix of second partial derivatives

$$H_y g := \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(y) \right) = \begin{pmatrix} \frac{2}{x_1^3 x_2 \cdots x_{N-1}} & \frac{1}{x_1^2 x_2^2 \cdots x_{N-1}} & \cdots & \frac{1}{x_1^2 x_2 \cdots x_{N-1}^2} \\ \frac{1}{x_1^2 x_2^2 \cdots x_{N-1}} & \frac{2}{x_1 x_2^3 \cdots x_{N-1}} & \cdots & \frac{1}{x_1 x_2^2 \cdots x_{N-1}^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1^2 x_2 \cdots x_{N-1}^2} & \frac{1}{x_1 x_2^2 \cdots x_{N-1}^2} & \cdots & \frac{2}{x_1 x_2 \cdots x_{N-1}^3} \end{pmatrix} (y)$$

where $y \in \mathbb{R}^{N-1}$. The criterion says that if all minors $H_y^p g = \left(\frac{\partial^2 g}{\partial x_i \partial x_j}(y) \right)_{i,j=1,\ldots,p}$, $p = 1, \ldots, N-1$ have determinant positive at a point, then g is convex at that point. One computes (inductively) that

$$\det H_y^p g = \frac{p+1}{x_1^{p+2} \cdots x_p^{p+2} x_{p+1}^p \cdots x_{N-1}^p} > 0,$$

which concludes the proof.

(c) (2 pts) With $a = N$ and s_0 as above, is $T_{s_0} S$ the graph of a function? Does the implicit function theorem apply?

Solution: In first place we compute

$$T_{s_0} S = \operatorname{Ker} f'(s_0) = \operatorname{Ker} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \{x \in \mathbb{R}^N : x_1 + \cdots + x_N = 0\}.$$

Solving for x_N, it is obvious that $T_{s_0}S$ is the graph of the linear function $L : \mathbb{R}^{N-1} \to \mathbb{R}$, $L(x_1, \dots, x_{N-1}) := -x_1 - \cdots - x_{N-1}$.

However, the implicit function theorem does **not** apply since

$$\det\left(\frac{\partial f_i}{\partial q_j}(s_0)\right) = \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0.$$

Arguing with the tangent space, the previous statement is rephrased as follows: for $T_{s_0}S$ to be the graph of a linear map $L : \mathbb{R}^{N-2} \to \mathbb{R}^2$, $\dim T_{s_0}S$ should be $N-2$. However

$$\dim T_{s_0}S = \dim \operatorname{Ker} f'(s_0) = N - \operatorname{rank}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix} = N - 1.$$

(d) (2 pts) Now take $a = N + 1/2$ and $s_0 = (p_0, q_0)$ with $p_0 = (1, ..., 1)$ and $q_0 = (2, 1/2)$. Is $T_{s_0}S$ the graph of a function? Does the implicit function theorem apply?

Solution: If s_0 is as given then

$$\det\left(\frac{\partial f_i}{\partial q_j}(s_0)\right) = \det\begin{pmatrix} 1 & 1 \\ 1/2 & 2 \end{pmatrix} = 3/2 \neq 0,$$

and by the implicit function theorem there exist a neighbourhood $X \subseteq \mathbb{R}^{N-2}$ of p_0 and a neighbourhood $Y \subseteq \mathbb{R}^2$ of q_0 such that $S \cap (X \times Y)$ is the graph of a function $G : X \to Y$.

2. Suppose $\gamma : [0,1] \to P \subset V$ is a C^1 map and $f : P \to \mathbb{R}$ is a continuously differentiable function.

(a) (1.5 pt) Take $P = V = \mathbb{R}^1 0$ and $\gamma(t) = te_1 + (1-t)e_{10}$ and $f(x) = |x|^2$ the square of the Euclidean norm. Compute $\int_\gamma \mathrm{d}f$

Solution: In coordinates, $\gamma(t) = (t, 0, \dots, 1-t)$ and $f(x_1, \dots, x_{10}) = x_1^2 + \cdots + x_{10}^2$, so $\mathrm{d}f = 2x_1\mathrm{d}x_1 + \cdots + 2x_{10}\mathrm{d}x_{10}$ and we compute

$$\begin{aligned}\int_\gamma \mathrm{d}f = \int_{[0,1]} \gamma^*\mathrm{d}f = \int_0^1 \mathrm{d}(\gamma^* f) &= \int_0^1 \mathrm{d}(t^2 + (1-t)^2) \\ &= \int_0^1 \mathrm{d}(2t^2 - 2t + 1) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(2t^2 - 2t + 1)\mathrm{d}t \\ &= (2 \cdot 1^2 - 2 \cdot 1 + 1) - (2 \cdot 0^2 - 2 \cdot 0 + 1) = 0.\end{aligned}$$

where we have used that a primitive of $\frac{\mathrm{d}}{\mathrm{d}t}(2t^2 - 2t + 1)$ is precisely $2t^2 - 2t + 1$ (fundamental theorem of calculus).

(b) (2.5 pts) Using the definition of the integral, the definition of $\mathrm{d}f$ and the fundamental theorem of calculus write down a general formula for $\int_\gamma \mathrm{d}f$ in terms of γ and f, where now γ and f are arbitrary as in the first sentence of the exercise.

Solution: Similarly we compute

$$\begin{aligned}\int_\gamma \mathrm{d}f &= \int_{[0,1]} \gamma^* \mathrm{d}f = \int_0^1 \mathrm{d}(\gamma^* f) = \int_0^1 \mathrm{d}(f \circ \gamma) \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\mathrm{d}t = f(\gamma(1)) - f(\gamma(0)).\end{aligned}$$

The last equality follows again by the fundamental theorem of calculus.

(c) (1 pt BONUS) Can you sketch an intuitive argument for why you get your result in the previous part in terms of the path γ crossing the level sets of f and thinking of those as a topographical map for the graph of f?

Solution: According to the discussion on page 54 in Roland's notes, the integral $\int_\gamma \mathrm{d}f$ is roughly the sum

$$\int_\gamma \mathrm{d}f \approx \sum_{i=1}^{M} \mathrm{d}f(\gamma(t_i))\gamma'(t_i)\Delta t = \sum_{i=1}^{M} f'(\gamma(t_i))\gamma'(t_i)\Delta t = \sum_{i=1}^{M} \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)(t_i)\Delta t$$

where $M >> 0$ and $\Delta t = t_{i+1} - t_i$ (this also follows from the second line of last exercise via Riemann integrals). Since the derivative expresses the rate of change of a function, the line integral $\int_\gamma \mathrm{d}f$ is approximately the (signed) sum of all small variations of f along γ, that is, it is the total variation of f from the start to the end of γ. If we think of γ crossing the level sets $f^{-1}((f \circ \gamma)(t_i))$, then each of the previous summands represents the difference of the values of f between two consecutive (along γ) level sets.