Multivariable Analysis - Homework 4

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Your grade is computed as: $8 * n$ where n is the number of points obtained, with each sub-question being worth 1.25 points

1. Compute for any $n \in \mathbb{N}$ the integral $J_n = \int_{[0,1]^n} f$ where $f = \sum_{j=1}^{n} \epsilon^j \in (\mathbb{R}^n)^*$

Solution:

Take an arbitrary $n \in \mathbb{N}$, then we compute:

$$
\begin{aligned}
J_n &= \int_{[0,1]^n} f \\
&= \int_{[0,1]^n} \sum_{j=1}^{n} \epsilon^j \\
&= \sum_{j=1}^{n} \int_{[0,1]^n} \epsilon^j && \text{(By property (2) of lemma 2.2.1.)} \\
&= \sum_{j=1}^{n} \int_{[0,1]^n} x^j dx^1 \ldots dx^n && \big(\text{Replacing } \epsilon^j \text{ with } x^j\big) \\
&= \sum_{j=1}^{n} \int_{[0,1]^{n-1}} 1 dx^1 \ldots \widehat{dx^j} \ldots dx^n \int_{[0,1]} x^j dx^j && \text{(By Fubini's theorem)} \\
&= \sum_{j=1}^{n} \int_{[0,1]} x^j dx^j && \text{(Noting the first integral becomes one)} \\
&= \sum_{j=1}^{n} \frac{1}{2} \\
&= \frac{n}{2}
\end{aligned}
$$

Here the hat above a symbol means it is omitted from the list. Notice that Fubini may be applied in the fifth line because the functions x^j are continuously differentiable.

2. Consider intersecting two circles, or equivalently solving the system of equations $f(x, y) = 0$ where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x, y) = (x^2 + y^2 - 1, (x - c)^2 + y^2 + 1)$

   (a) Compute the matrix for $f'(p)$ with respect to the standard bases and write down its determinant at $p = (x, y)$ as a function of c.

   Solution: Use that $f'(p) = \partial_{e_1} f(x, y)\epsilon^1 + \partial_{e_2} f(x, y)\epsilon^2$. Working out the partial derivatives with respect to the standard basis, we find the derivative has associated matrix:

$$f'(x, y) = \begin{bmatrix} 2x & 2y \\ 2x - 2c & 2y \end{bmatrix}$$

   The determinant at $p = (x, y)$ is given by:

$$\det f'(p) = 4xy - 4xy + 4yc = 4yc.$$

   (b) To actually find a solution to $f(p) = 0$ we make a reasonable guess $z = \frac{1}{2}(c, 1)$ and assume $c \in (0, 2)$. Compute a matrix for $(f'(z))^{-1}$.

   Solution: Notice that because the determinant is non-zero, the matrix for $f'(p)$ is invertible. We can compute:

$$f'(z) = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix}$$

   and use the standard formula to invert 2×2 matrices to obtain:

$$(f'(z))^{-1} = \frac{1}{2c} \begin{bmatrix} 1 & -1 \\ c & c \end{bmatrix}.$$

   (c) Explain why we cannot apply the inverse function theorem directly to show that there exists a solution to $f(p) = 0$ near $z = 0$.

   Solution: Because f is invertible by question (a) we find there exists an open neighbourhood A around the point z such that the map that takes A to $f(A)$ is an isomorphism. However, notice that the neighbourhood A can be made arbitrarily small around z and hence might not contain the solution to $f(p) = c$. In other words, we do not know if $f(A)$ is large enough to contain the solution to the given equation.

(d) To actually find a solution we first pass to a normalized version of f called $\widetilde{f} = U \circ f \circ T$ where $U, T : \mathbb{R}^2 \to \mathbb{R}^2$ are defined by $U(p) = (f'(z))^{-1}(p - f(z))$ and $T(p) = p + z$. Show that $\widetilde{f}(0) = 0$, $\widetilde{f}'(0) = \mathrm{id}_{\mathbb{R}^2}$ and $f(p) = 0$ is equivalent to $\widetilde{f}(p - z) = \left(0, \frac{3-c^2}{4}\right)$.

Solution: First, let us compute $\widetilde{f}(0)$ by writing out the definitions.

$$\begin{aligned}\widetilde{f}(0) &= U \circ f \circ T(0)\\ &= U \circ f(z)\\ &= (f'(z))^{-1}(f(z) - f(z))\\ &= 0.\end{aligned}$$

Secondly, we will show that $\widetilde{f}'(0) = \mathrm{id}_{\mathbb{R}^2}$. Notice that by the chain rule, we find:

$$\widetilde{f}'(x, y) = U'(f \circ T(x, y)) \circ (f \circ T)'(x, y) \qquad (\clubsuit)$$

Both parts of the composition can be computed separately. We find, again by the chain rule and using the fact that $T'(p) = \mathrm{id}$, that:

$$(f \circ T)'(p) = f'(T(p)) \circ T'(p) = \begin{bmatrix} 2x + c & 2y + 1 \\ 2x - c & 2y + 1. \end{bmatrix}$$

At the point $p = 0$ we find, using part (a), that

$$(f \circ T)'(p) = f'(z). \qquad (\heartsuit)$$

Secondly, we can compute $U'(f \circ T(x, y))$. First, notice that:

$$p - f(z) = \left(x - \frac{1}{4}(c^2 + 1) - 1, y - \frac{1}{4}(c^2 + 1) - 1\right)$$

.

Then:

$$U(p) = \frac{1}{2}\left(\frac{1}{c}(x - y), x + y - \frac{1}{2}(c^2 + 1) - 2\right).$$

Then taking the derivative w.r.t. the standard basis, we obtain:

$$U'(p) = \frac{1}{2}\begin{bmatrix} 1/c & -1/c \\ 1 & 1 \end{bmatrix} = (f'(z))^{-1} \qquad (\diamondsuit)$$

where the last equality follows from part (b).
Combining equations ($\heartsuit$) and ($\diamondsuit$) in ($\clubsuit$), we obtain:

$$\widetilde{f}'(x, y) = (f'(z))^{-1} \circ f'(z) = \mathrm{id}_{\mathbb{R}^2}$$

Lastly, we show the equivalence. Assume $f(p)=0$, then we can compute $\widetilde{f}(p-z)$

$$\begin{aligned}\widetilde{f}(p-z) &= U\circ f\circ T(p-z)\\ &= U\circ f(p)\\ &= U(0)\\ &= \left(f'(z)\right)^{-1}(-f(z))\\ &= \left(0, \frac{3-c^2}{4}\right)\end{aligned}$$

Proving the equivalence from left to right. Conversely, assume $\widetilde{f}(p-z)=\left(0, \frac{3-c^2}{4}\right)$. We now notice that:

$$\begin{aligned}U(f(p)) &= \widetilde{f}(p-z)\\ &= \left(0, \frac{3-c^2}{4}\right)\\ &= \left(f'(z)\right)^{-1}(-f(z))\\ &= U(0).\end{aligned}$$

Proving that $U(f(p))=U(0)$. Using the fact that U is an injective map (because $\left(f'(z)\right)^{-1}$ is invertible), we find that $f(p)=0$. Together with what we have shown before, we conclude that $f(p)=0$ is equivalent to $\widetilde{f}(p-z)=\left(0, \frac{3-c^2}{4}\right)$.

(e) Following the proof of the inverse function theorem, prove that $\widetilde{f}'(p)-\mathrm{id}_{\mathbb{R}^2}$ has matrix $G=\left(\begin{smallmatrix}0 & 0\\ 2x & 2y\end{smallmatrix}\right)$ for $p=(x,y)$.

Solution: We can compute:

$$\begin{aligned}\widetilde{f}'(p) &= U'(f\circ T(x,y))\circ(f\circ T)'(x,y)\\ &= \frac{1}{2}\begin{bmatrix}1/c & -1/c\\ 1 & 1\end{bmatrix}\begin{bmatrix}2x+c & 2y+1\\ 2x-c & 2y+1\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\ 2x & 2y+1\end{bmatrix}.\end{aligned}$$

From this it follows that:

$$\widetilde{f}'(p)-\mathrm{id}_{\mathbb{R}^2} = \begin{bmatrix}0 & 0\\ 2x & 2y\end{bmatrix} = G.$$

(f) Explain why $\left|\widetilde{f}'(p) - \mathrm{id}_{\mathbb{R}^2}\right| \leq 2|p|$ where $p = (x, y)$ and take the operator norm with respect to the Euclidean norm on $\mathbb{R}^2$. Conclude that the δ in the proof of the inverse function theorem may be chosen as $\delta = \frac{1}{4}$.

Solution: Using lemma 1.2.2. from the lecture notes, we find:

$$\begin{aligned}\left|\widetilde{f}'(p) - \mathrm{id}_{\mathbb{R}^2}\right| &= |G| \\ &\leq \sqrt{(2x)^2 + (2y)^2} \\ &= 2\sqrt{x^2 + y^2} \\ &= 2|(x, y)| \\ &= 2|p|.\end{aligned}$$

To conclude, we may choose $\delta = 4$, we need to show that for all x in $\overline{B}_{1/4}(0)$, we have $|G(x)| \leq \frac{1}{2}$. Take $|p| \leq \frac{1}{4}$, then by the derived equality, we find that $|G| \leq 2|p| \leq \frac{1}{2}$, proving that we can indeed take $\delta = \frac{1}{4}$.

(g) Conclude that the argument in the proof of the inverse function theorem allows us to establish there exists a unique solution of $f(p) = 0$ close to z as long as $c \in \left(\sqrt{\frac{5}{2}}, \sqrt{\frac{7}{2}}\right)$.

Solution: Using the fact that by part (f) we may pick $\delta = \frac{1}{4}$ and part (d) tells us that $\widetilde{f}(p) = (0, \frac{3-c^2}{4})$ only when $f(p) = 0$, then it follows from the proof of the Inverse function theorem that we can pick a neighbourhood such that:

$$\left|\left(0, \frac{3 - c^2}{4}\right)\right| = \left|\frac{3 - c^2}{4}\right| < \frac{\delta}{2} = \frac{1}{8}.$$

Rewriting this, gives that c should be in the interval $(\sqrt{\frac{5}{2}}, \sqrt{\frac{7}{2}})$.