Multivariable Analysis - Homework 2

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Your grade is computed as: $10n/14$, where n is the number of correctly answered subquestions.

1. In this exercise we view $\mathbb{C}$ as a two-dimensional real vector space with basis $1, i$. Consider the subset $B = \{z \in \mathbb{C} : 1 \le |z| \le 2, \mathrm{Arg}(z) \in [0, \pi\frac{199}{100}]\}$ and $f : B \to \mathbb{C}$ defined by $f(z) = \log(z)$, more precisely, $\log\big(re^{it}\big) = \log(r) + it$.

   (a) Is B compact? Is it convex?

   Solution: B is closed and bounded in $\mathbb{C}$ and thus compact, but B is not convex since $0 \notin B$ despite 0 being on the line between 1 and -1, which are both in B.

   (b) Introduce $A = [1,2] \times [0, \pi\frac{199}{100}] \subset \mathbb{R}^2$ and show that $P : A \to B$ given by $P(r,t) = re^{it}$ is a bijection.

   Solution: It is clear that P is surjective, as every complex number can be written in the form re^{it}. For injectivity, consider two points $(r,t), (\tilde{r}, \tilde{t})$ with $P(r,t) = P(\tilde{r}, \tilde{t})$. Then $r = \tilde{r}$, since equal complex numbers have equal norms. If $t \neq \tilde{t}$, then t and $\tilde{t}$ must differ by an integer multiple of 2π, which is not possible since their distance from eachother is at most $199\pi/200$.

   (c) Compute the matrix of $P'(r,t) \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{C})$ with respect to the bases (e_2, e_1) of $\mathbb{R}^2$ and $(1, i)$ of $\mathbb{C}$.

   Solution: We may write $P = 1P^1 + iP^2$ where $P^1, P^2 : A \to \mathbb{R}$ are given by

   $$P^1(r,t) = r\cos t, \qquad P^2(r,t) = r\sin t.$$

   The matrix of $P'(r,t)$ w.r.t. the bases (e_2, e_1) of $\mathbb{R}^2$ and $(1, i)$ of $\mathbb{C}$ is then

   $$\begin{pmatrix} \partial_{e_2} P^1(r,t) & \partial_{e_1} P^1(r,t) \\ \partial_{e_2} P^2(r,t) & \partial_{e_1} P^2(r,t) \end{pmatrix} = \begin{pmatrix} -r\sin t & \cos t \\ r\cos t & \sin t \end{pmatrix}.$$

   (d) Also compute for $(r,t) \in A$ the partial derivatives of $\partial_{e_1} P(r,t)$ and $\partial_{e_2} P(r,t)$.

Solution: See part (c). We have

$$\begin{aligned}\partial_{e_1} P(r,t) &= \partial_{e_1} P^1(r,t) + i\partial_{e_1} P^2(r,t)\\ &= \cos t + i\sin t,\\ \partial_{e_2} P(r,t) &= \partial_{e_2} P^1(r,t) + i\partial_{e_2} P^2(r,t)\\ &= -r\sin t + ir\cos t.\end{aligned}$$

(e) For fixed $(r,t) \in A$ find a linear map $Q(r,t) \in \mathrm{Hom}(\mathbb{C}, \mathbb{R}^2)$ such that $P'(r,t) \circ Q(r,t) = \mathrm{id}_{\mathbb{C}}$.

Solution: We see that $\det P'(r,t) = -r$, so such a map Q exists for $r \neq 0$. By inverting the matrix in part (c), the matrix (in the bases $(1,i)$ of $\mathbb{C}$ and (e_2, e_1) of $\mathbb{R}^2$) of $Q(r,t)$ will be

$$\begin{pmatrix} -\frac{1}{r}\sin t & \frac{1}{r}\cos t \\ \cos t & \sin t \end{pmatrix}.$$

Thus, $Q(r,t)$ is the linear map satisfying

$$Q(r,t)1 = \frac{1}{r}(\cos t)e_2 + (\sin t)e_1, \qquad Q(r,t)i = -\frac{1}{r}(\sin t)e_2 + \cos te_1.$$

(f) Use the chain rule to conclude that $f'(P(r,t)) = (f \circ P)'(r,t) \circ Q(r,t)$.

Solution: The chain rule gives

$$(f \circ P)'(r,t) = f'(P(r,t)) \circ P'(r,t).$$

Right-composing with $Q(r,t)$ on both sides yields

$$(f \circ P)'(r,t) \circ Q(r,t) = f'(P(r,t)).$$

(g) Explain why $g = f \circ P = \log(r) + it$ satisfies $g'(r,t)e_1 = \frac{1}{r}$ and $g'(r,t)e_2 = i$.

Solution: We know that $g'(r,t)e_1, g'(r,t)e_2$ are the partial derivatives w.r.t. r and t in the sense of elementary calculus. The result follows from knowing the derivative of a linear map, and the fact that $\log'(r) = \frac{1}{r}$ (where we use the real map $\log : \mathbb{R}^+ \to \mathbb{R}$).

(h) Derive from the above that $D = f'(P(r,t)) \in \mathrm{Hom}(\mathbb{C}, \mathbb{C})$ is characterized by $D(w) = \frac{w}{P(r,t)}$ for all $w \in \mathbb{C}$.

Solution: We have

$$D = g'(r,t) \circ Q(r,t).$$

In the bases (e_2, e_1) and $(1,i)$, the map $g'(r,t)$ has the matrix

$$\begin{pmatrix} 0 & \frac{1}{r} \\ 1 & 0 \end{pmatrix}.$$

Thus, D is the map whose matrix in the basis $(1, i)$ of $\mathbb{C}$ is

$$\begin{pmatrix} 0 & \frac{1}{r} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{r}\sin t & \frac{1}{r}\cos t \\ \cos t & \sin t \end{pmatrix} = \frac{1}{r} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \frac{1}{r} \begin{pmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{pmatrix},$$

which corresponds to the linear map which scales its input by a factor of $1/r$ and rotates it by an angle of $-t$, which is precisely the map

$$\omega \mapsto \frac{\omega}{P(r,t)} = \frac{\omega}{r} e^{-it}.$$

(i) For fixed $(r, t) \in A$, show that $|D| = r^{-1}$. Here we are taking the operator norm of D with respect to the standard Euclidean norm on $\mathbb{C}$.

Solution: We have

$$|D| = \max_{|w|=1} |D(w)| = \max_{|w|=1} \left| \frac{\omega}{r} e^{-it} \right| = \max_{|w|=1} \left| \frac{\omega}{r} \right| = \frac{1}{r}.$$

(j) Produce $u, v \in B$ such that $|f(u) - f(v)| > (\max_{b \in B} |f'(b)|)|u - v|$.

Solution: From the previous part, we have that $f'(b)v = v/b$ for $b \in B \subset \mathbb{C}$ and $v \in \mathbb{C}$, giving $\max_{b \in B} |f'(b)| = 1$. It suffices to find $u, v \in B$ such that $|f(u) - f(v)| > |u - v|$. Take $u = 1$ and $v = -1$. Then $|u - v| = 2$, but

$$|f(v) - f(u)| = |\log 1 + 0i - \log 1 - \pi i| = \pi > 2 = |u - v|.$$

(k) Why is there no contradiction with Lemma 2.3.2?

Solution: The proof of lemma 2.3.2 makes use of the fact that the domain of f is convex, which is not the case in this exercise.

2. (a) Explain why for any $(s, t) \in [0, 1]^2$ the vectors $\partial_{e_1}\gamma(s, y)$ and $\partial_{e_2}\gamma(s, y)$ form a basis of the tangent space $T_{\gamma(s,t)}\gamma$.

Solution: We have

$$\begin{aligned} T_{\gamma(s,t)\gamma} &= \gamma'(s,t)(\mathbb{R}^2) \\ &= \{\gamma'(s,t)v | v \in \mathbb{R}^2\} \\ &= \{v^1 \gamma'(s,t)e_1 + v^2 \gamma'(s,t)e_2 | v^1, v^2 \in \mathbb{R}\} \\ &= \operatorname{Span} \gamma'(s,t)e_1, \gamma'(s,t)e_2 \\ &= \operatorname{Span}\{\partial_{e_1}\gamma(s,t), \partial_{e_1}\gamma(s,t)\}. \end{aligned}$$

Clearly, the two partial derivatives

$$\partial_{e_1}\gamma(s,t) = (1, 0, 2s), \qquad \partial_{e_2}\gamma(s,t) = (0, 1, 0),$$

are linearly independent.

(b) For $a = e_1 + se_3$ and $b = e_2 \in \mathbb{R}^3$ find out how $\omega(p)(a \wedge b)$ depends on s and explain what this has to do with counting the number of vertical lines piercing through the parallelogram spanned by a and b using Theorem 1.7.1.

Solution: We have

$$\omega(p)(a \wedge b) = (\varepsilon^1 \wedge \varepsilon^2)((e_1 + se_3) \wedge e_2) = 1.$$

By Theorem 1.7.1, the above is equal to

$$\lim_{r\to\infty} \frac{1}{r^2} \# \left\{ (\varepsilon^1, \varepsilon^2) \left(\frac{1}{r} \mathbb{Z}^2 \right) \cap ([0,1)a + [0,1)b) \right\}.$$

For fixed r, the set inside the limit above is precisely the set of intersections between lines whose components in the e_1 and e_2 directions are respectively of the form n_1/r and n_2/r for $n_1, n_2 \in \mathbb{Z}$ (and thus varying only in the e_3 direction), and the parallelopiped spanned by a and b. Since this parallelopiped only varies in the e_3 direction with s, its projection onto the (e_1, e_2)-plane is constant, and thus the number of intersections is also constant. Thus, we can take $s = 0$, and instead count the number of intersections of lines with the parallelopiped spanned by e_1, e_2, which is a square of sidelength 1. There will be precisely r^2 such lines with integer coordinates (in the e_1, e_2 directions) scaled by $1/r$ that intersect this square. In other words,

$$\# \left\{ (\varepsilon^1, \varepsilon^2) \left(\frac{1}{r} \mathbb{Z}^2 \right) \cap ([0,1)e_1 + [0,1)e_2) \right\} = r^2.$$

Since we take the limit as $r \to \infty$ of the number of intersections divided by r^2, the limit must be 1.

(c) Compute $\int_\gamma \omega$. Can you explain intuitively why you get this answer?

Solution: We have

$$\begin{aligned}
\int_\gamma \omega &= \int_{[0,1]^2} \omega(\gamma(s,t))(\partial_{e_1}\gamma(s,t) \wedge \partial_{e_2}\gamma(s,t)) \\
&= \int_{[0,1]^2} (\varepsilon^1 \wedge \varepsilon^2)((e_1 + 2se_3) \wedge e_2) \\
&= \int_{[0,1]^2} 1 \\
&= 1.
\end{aligned}$$

For the intuition, see part (b). The difference here is that we are computing intersections with tangent vectors. Again, the parallelopiped spanned by the tangent vectors varies only in the e_3 direction. Since ω is a constant differential form, the result is the same.